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TURNPIKE RESULTS FOR A CLASS OF VARIATIONAL PROBLEMS ARISING IN CONTINUUM MECHANICS

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Dedicated to Terry Rockafellar on his seventieth birthday.

Abstract: In this paper we study the structure of optimal solutions of one-dimensional second order variational problems arising in continuum mechanics. We are interested in properties of the optimal solutions which are independent of the length of the interval, for all sufficiently large intervals. The study of these properties is based on the relation between variational problems on bounded large intervals and a limitimg problem on $[0, \infty)$.

Key words: *good function, infinite horizon, periodic minimizer, turnpike property*

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1 Introduction

In this paper we analyse the structure of optimal solutions of the variational problems

$$\int_0^T f(w(t), w'(t), w''(t))dt \to \min \tag{P}$$

$$w \in W^{2,1}([0,T]),\ (w(0), w'(0)) = x \text{ and } (w(T), w'(T)) = y,$$

where $T > 0$, $x, y \in R^2$, $W^{2,1}([0,T]) \subset C^1$ is the Sobolev space of functions possessing an integrable second derivative and f belongs to a space of functions to be described below. The interest in variational problems of the form (P) stems from the theory of thermodynamical equilibrium for second-order materials developed in [2, 4, 6-8].

In this paper we also consider the following problem on the half line:

$$\inf\left\{\liminf_{T\to\infty} T^{-1}\int_0^T f(w(t), w'(t), w''(t))dt :\ w \in W^{2,1}_{loc}([0,\infty))\right\}. \tag{P_∞}$$

Here $W^{2,1}_{loc}([0,\infty)) \subset C^1$ denotes the Sobolev space of functions possessing a locally integrable second derivative and f belongs to a space of functions to be described below.

We are interested in properties of the optimal solutions of the problem (P) which are independent of the length of the interval, for all sufficiently large intervals. The study of these properties is based on the relation between variational problems (P) on bounded large intervals and the limitimg problem (P_∞) on $[0,\infty)$.

Denote by $\mathfrak{A}$ the set of all continuous functions f: $R^3 \to R$ such that for each $N>0$ the function $|f(x,y,z)| \to \infty$ as $|z| \to \infty$ uniformly on the set $\{(x,y) \in R^2\colon |x|,|y| \le N\}$. For the set $\mathfrak{A}$ we consider the uniformity which is determined by the following base:

$$E(N,\epsilon,\Gamma) = \{(f,g) \in \mathfrak{A} \times \mathfrak{A} : |f(x_1,x_2,x_3) - g(x_1,x_2,x_3)| \le \epsilon \tag{1.1}$$

$$\text{for each } (x_1,x_2,x_3) \in R^3 \text{ such that } |x_i| \le N,\ i=1,2,3$$

$$\text{and } (|f(x_1,x_2,x_3)|+1)(|g(x_1,x_2,x_3)|+1)^{-1} \in [\Gamma^{-1},\Gamma]$$

$$\text{for each } (x_1,x_2,x_3) \in R^3 \text{ such that } |x_1|,|x_2| \le N\},$$

where $N>0$, $\epsilon>0$, $\Gamma>1$. Clearly, the uniform space $\mathfrak{A}$ is Hausdorff and has a countable base. Therefore $\mathfrak{A}$ is metrizable (by a metric ρ) [3]. It is not difficult to verify that the uniform space $\mathfrak{A}$ is complete.

Let $a=(a_1,a_2,a_3,a_4) \in R^4$, $a_i>0$, $i=1,2,3,4$ and let α, β, γ be positive numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$, $\gamma > 1$. Denote by $\mathfrak{M}(\alpha,\beta,\gamma,a)$ the set of all functions $f \in \mathfrak{A}$ such that:

$$f(w,p,r) \ge a_1|w|^\alpha - a_2|p|^\beta + a_3|r|^\gamma - a_4 \text{ for all } (w,p,r) \in R^3; \tag{1.2}$$

$$f,\ \partial f/\partial p \in C^2,\ \partial f/\partial r \in C^3,\ \partial^2 f/\partial r^2(w,p,r) > 0 \text{ for all } (w,p,r) \in R^3; \tag{1.3}$$

there is a monotone increasing function $M_f : [0,\infty) \to [0,\infty)$ such that for every $(w,p,r) \in R^3$

$$\sup\{f(w,p,r),\ |\partial f/\partial w(w,p,r)|,\ |\partial f/\partial p(w,p,r)|,\ |\partial f/\partial r(w,p,r)|\} \le$$

$$M_f(|w|+|p|)(1+|r|^\gamma). \tag{1.4}$$

Denote by $\bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$ the closure of $\mathfrak{M}(\alpha,\beta,\gamma,a)$ in $\mathfrak{A}$. We consider the topological subspace $\bar{\mathfrak{M}}(\alpha,\beta,\gamma,a) \subset \mathfrak{A}$ with the relative topology. Let $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$. Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\left\{\liminf_{T\to+\infty} T^{-1}\int_0^T f(w(t),w'(t),w''(t))dt\colon w \in A_x\right\}, \tag{1.5}$$

where

$$A_x = \{v \in W^{2,1}_{loc}([0,\infty))\colon (v(0),v'(0)) = x\}.$$

It is easy to verify that $\mu(f)$ is well defined and is independent of the initial vector x. A function $w \in W^{2,1}_{loc}([0,\infty))$ is called an (f)-*good function* if the function

$$\phi^f_w\colon T \to \int_0^T [f(w(t),w'(t),w''(t)) - \mu(f)]dt,\ T \in (0,\infty)$$

is bounded. For every $w \in W^{2,1}_{loc}([0,\infty))$ the function ϕ^f_w is either bounded or diverges to $+\infty$ as $T \to +\infty$ and moreover, if ϕ^f_w is a bounded function, then

$$\sup\{|(w(t),w'(t))|\colon t \in [0,\infty)\} < \infty$$

[14, Proposition 3.5]. Leizarowitz and Mizel [4] established that for every $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$ satisfying $\mu(f) < \inf\{f(w,0,s)\colon (w,s) \in R^2\}$ there exists a periodic (f)-good function. In [13] it was shown that this resut is valid for every $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$.

Let $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$. For each $T>0$ define a function $U_T^f\colon R^2 \times R^2 \to R$ by

$$U_T^f(x,y) = \inf\left\{\int_0^T f(w(t),w'(t),w''(t))dt\colon w \in W^{2,1}([0,T]),\right.$$
$$\left.(w(0),w'(0)) = x \text{ and } (w(T),w'(T)) = y\right\}. \tag{1.6}$$

In [4], analyzing problem (P_∞) Leizarowitz and Mizel studied the function $U_T^f\colon R^2 \times R^2 \to R$, $T>0$ and established the following representation formula

$$U_T^f(x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x,y),\ x,y \in R^2,\ T>0, \tag{1.7}$$

where $\pi^f\colon R^2 \to R$ and $(T,x,y) \to \theta_T^f(x,y),\ x,y \in R^2,\ T>0$ are continuous functions,

$$\pi^f(x) = \inf\left\{\liminf_{T\to\infty}\int_0^T [f(w(t),w'(t),w''(t)) - \mu(f)]dt\colon\right.$$
$$\left. w \in W_{loc}^{2,1}([0,\infty)) \text{ and } (w(0),w'(0)) = x\right\},\ x \in R^2, \tag{1.8}$$

$\theta_T^f(x,y) \geq 0$ for each $T>0$, and each $x,y \in R^2$, and for every $T>0$, and every $x \in R^2$ there is $y \in R^2$ satisfying $\theta_T^f(x,y) = 0$.

Leizarowitz and Mizel established the representation formula for any integrand $f \in \mathfrak{M}(\alpha,\beta,\gamma,a)$, but their result also holds for every $f \in \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a)$ without change in the proofs.

Denote by $|\cdot|$ the Euclidean norm in R^n. For $\tau>0$ and $v \in W^{2,1}([0,\tau])$ we define $X_v\colon [0,\tau] \to R^2$ as follows:

$$X_v(t) = (v(t),v'(t)),\ t \in [0,\tau].$$

We also use this definition for $v \in W_{loc}^{2,1}([0,\infty))$ and $v \in W_{loc}^{2,1}(R)$.

Put

$$\mathfrak{M} = \mathfrak{M}(\alpha,\beta,\gamma,a),\ \bar{\mathfrak{M}} = \bar{\mathfrak{M}}(\alpha,\beta,\gamma,a).$$

We consider functionals of the form

$$I^f(T_1,T_2,v) = \int_{T_1}^{T_2} f(v(t),v'(t),v''(t))dt, \tag{1.9}$$

$$\Gamma^f(T_1,T_2,v) = I^f(T_1,T_2,v) - (T_2-T_1)\mu(f) - \pi^f(X_v(T_1)) + \pi^f(X_v(T_2)), \tag{1.10}$$

where $-\infty < T_1 < T_2 < +\infty$, $v \in W^{2,1}([T_1,T_2])$ and $f \in \bar{\mathfrak{M}}$.

We denote by $\mathrm{mes}(E)$ the Lebesgue measure of a measurable set $E \subset R$ and by $\mathrm{int}(D)$ the interior of a subset D of a metric space.

If $v \in W_{loc}^{2,1}([0,\infty))$ satisfies

$$\sup\{|X_v(t)|:\ t \in [0,\infty)\} < \infty,$$

then the set of limiting points of $X_v(t)$ as $t \to \infty$ is denoted by $\Omega(v)$.

Denote by $\mathrm{Card}(A)$ the cardinality of the set A. If $f \in \bar{\mathfrak{M}}$, $J = [T_1, T_2]$ with $T_2 > T_1$, $v \in W^{2,1}([T_1, T_2])$, then we set

$$\Gamma^f(J, v) = \Gamma^f(T_1, T_2, v).$$

The main results in this paper deal with the so-called turnpike properties of the variational problems (P). To have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand, and are essentially independent of the choice of interval and endpoint conditions.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [12]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, [5, 11] and the references mentioned there).

The turnpike properties of problem (P) were studied in [9, 10, 14, 15]. In [15] we established the existence of an everywhere dense G_δ-subset $\mathcal{F} \subset \bar{\mathfrak{M}}$ such that each integrand $f \in \mathcal{F}$ has the following turnpike property:

There exists a nonempty compact set $H(f) \subset R^2$ depending only on the function f such that for any $\epsilon > 0$ there exist constants $L_1, L_2 > 0$ which depend only on $|x|, |y|$ and ϵ such that for each optimal solution v of problem (P) and each $\tau \in [L_1, T - L_1]$ the set $\{(v(t), v'(t)) : t \in [\tau, \tau + L_2]\}$ is equal to the set $H(f)$ up to ϵ in the Hausdorff metric.

In [9, 15] we considered certain important subspaces of the space $\mathfrak{M}$ equipped with natural uniformities and showed that each of them contains an everywhere dense G_δ subset such that each its element f has the following two properties:

The problem (P_∞) has a unique up to translation periodic minimizer w.

Let $T_w > 0$ be a period of w. For any $\epsilon > 0$ there exists a constant $L > 0$ which depends only on $|x|, |y|$ and ϵ such that for each optimal solution v of problem (P) and each $\tau \in [L, T - L - T_w]$ there exists $s \in [0, T_w)$ such that

$$|(v(\tau + t), v'(\tau + t)) - (w(s + t), w'(s + t))| \leq \epsilon \text{ for each } t \in [0, T_w].$$

The results of [9, 15] establish that most integrands (in the sense of Baire's categories) have the turnpike properties. Since the space $\bar{\mathfrak{M}}$ and its subspaces considered in [9, 15] contain integrands which do not have the turnpike properties these results cannot be essentially improved. Nevertheless, some questions are still open. It is very important and interesting to obtain some knowledge about the structure of extremals of problem (P) with arbitrary integrand $f \in \mathfrak{M}$.

In this paper we show (see Theorem 2.1) that for each integrand $f \in \mathfrak{M}$ the following property holds: For each pair of positive numbers ϵ, l there exists a constant $L > l$ which depends only on $|x|, |y|, l$ and ϵ such that for each optimal solution v of problem (P) and each closed subinterval $D \in [0, T]$ of length L there exists a closed subinterval $D_1 \subset D$ of length l and a periodic minimizer w of problem (P_∞) such that

$$|(v(t), v'(t)) - (w(t), w'(t))| \leq \epsilon \text{ for each } t \in D_1.$$

2 Main Results

Let $f \in \mathfrak{M}$. Denote by $\sigma(f)$ the set of all $w \in W^{2,1}_{loc}(R)$ which have the following property:

There is $T_w > 0$ such that

$$w(t+T_w) = w(t) \text{ for all } t \in R \text{ and } I^f(0,T_w,w) = \mu(f)T_w. \tag{2.1}$$

In other words $\sigma(f)$ is the set of all periodic minimizers of (P_∞). By Theorem 4.1 of [13], $\sigma(f) \neq \emptyset$.

The following result established in [9, Lemma 3.1] describes the structure of periodic minimizers of (P_∞).

Proposition 2.1. *Let $f \in \mathfrak{M}$. Assume that $w \in \sigma(f)$,*

$$w(0) = \inf\{w(t) : \ t \in R\}$$

and $w'(t) \neq 0$ for some $t \in R$. Then there exist $\tau_1(w) > 0$ and $\tau(w) > \tau_1(w)$ such that the function w is strictly increasing on $[0, \tau_1(w)]$, w is strictly decreasing in $[\tau_1(w), \tau(w)]$,

$$w(\tau_1(w)) = \sup\{w(t) : \ t \in R\} \text{ and } w(t+\tau(w)) = w(t) \text{ for all } t \in R.$$

Corollary 2.1. *Let $f \in \mathfrak{M}$, $t_0 \in R$, $w \in \sigma(f)$,*

$$w'(t) \neq 0 \text{ for some } t \in R \text{ and } w(t_0) = \inf\{w(t) : \ t \in R\}.$$

Then there exist $\tau_1(w) > 0$ and $\tau(w) > \tau_1(w)$ such that the function w is strictly increasing in $[t_0, t_0+\tau_1(w)]$, w is strictly descreasing in $[t_0+\tau_1(w), t_0+\tau(w)]$,

$$w(t_0+\tau_1(w)) = \sup\{w(t) : \ t \in R\} \text{ and } w(t+\tau(w)) = w(t) \text{ for all } t \in R.$$

Let $f \in \mathfrak{M}$. By Corollary 2.1, each $w \in \sigma(f)$ which is not a constant has a minimal period which will be denoted by $\tau(w)$. Put

$$\sigma(f,0) = \{w \in \sigma(f) : \ w \text{ is a constant}\}. \tag{2.2}$$

For each $T > 0$ set

$$\sigma(f,T) = \sigma(f,0) \cup \{w \in \sigma(f) : w \text{ is not a constant and } \tau(w) \leq T\}. \tag{2.3}$$

The following theorem is our main result.

Theorem 2.1. *Let $f \in \mathfrak{M}$ and let l, M_0, M_1, ϵ be positive numbers. Then there exist $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that for each $g \in \mathcal{U}$, each $T \geq L$ and each $v \in W^{2,1}([0,T])$ which satisfies*

$$|(v(0), v'(0))|, \ |(v(T), v'(T))| \leq M_0,$$

$$I^g(0,T,v) \leq U_T^g((v(0),v'(0)),(v(T),v'(T))) + M_1$$

the following property holds: For each $s \in [0, T-L]$ there are $s_1 \in [s, s+L-l]$ and $w \in \sigma(f)$ such that

$$|(v(s_1+t), v'(s_1+t)) - (w(t), w'(t))| \leq \epsilon \text{ for all } t \in [0,l]. \tag{2.4}$$

Theorem 2.1 will be proved in Section 5.

The next theorem which will be proved in Section 6 describes the structure of good functions.

Theorem 2.2. *Let $f \in \mathfrak{M}$ and let l, ϵ be positive numbers. Then there exist $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that for each $g \in \mathcal{U}$ and each (g)-good function $v \in W^{2,1}_{loc}([0,\infty))$ there exist $T_0 \geq 0$ such that the following property holds: For each $s \geq T_0$ there are $s_1 \in [s, s+L-l]$ and $w \in \sigma(f)$ such that inequality (2.4) is valid.*

The next two results which will be proved in Section 7 describe the structure of approximate solutions of problem (P_∞).

Theorem 2.3. *Let $f \in \mathfrak{M}$, l, ϵ be positive numbers and let $v \in W^{2,1}_{loc}([0,\infty))$ satisfy*

$$\limsup_{T\to\infty} T^{-1} I^f(0,T,v) = \mu(f)$$

and

$$\sup\{|(v(t), v'(t))| : \ t \in [0,\infty)\} < \infty. \tag{2.5}$$

Then there exists $L_0 > l$ such that the following assertion holds: For each $\gamma > 0$ there is $T_\gamma > L_0$ such that for each $T \geq T_\gamma$ there are a finite number of closed intervals $J_1, \dots, J_{q_T}$ such that

$$q_T \leq \gamma T, \tag{2.6}$$

$$mes(J_i) \leq L_0, \ i = 1, \dots, q_T, \tag{2.7}$$

$$\begin{aligned} &int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers} \\ &i, p \in \{1, \dots, q_T\} \text{ such that } i \neq p, \end{aligned} \tag{2.8}$$

and if

$$s \in [0, T - L_0] \text{ and } [s, s+L_0] \cap J_i = \emptyset \text{ for all } i = 1, \dots, q_T, \tag{2.9}$$

then there are $s_1 \in [s, s+L_0-l]$ and $w \in \sigma(f)$ such that (2.4) is valid.

Theorem 2.4. *Let $f \in \mathfrak{M}$, l, ϵ be positive numbers and let $v \in W^{2,1}_{loc}([0,\infty))$ satisfy (2.5). Assume that there exists a strictly increasing sequence of positive numbers $\{T_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} T_i = \infty$ and*

$$\lim_{i\to\infty} T_i^{-1} I^f(0, T_i, v) = \mu(f). \tag{2.10}$$

Then there exists $L_0 > l$ such that the following assertion holds: For each $\gamma > 0$ there is a natural number j_γ with $T_{j_\gamma} > L_0$ such that for each integer $j \geq j_\gamma$ the inequality $T_j \geq L_0$ holds and there are a finite number of closed intervals $J_1, \dots, J_{q_j}$ such that

$$q_j \leq \gamma T_j, \ mes(J_i) \leq L_0 \text{ for all } i = 1, \dots, q_j,$$

$$\begin{aligned} &int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers} \\ &i, p \in \{1, \dots, q_j\} \text{ such that } i \neq p, \end{aligned}$$

and if

$$s \in [0, T_j - L_0] \text{ and } [s, s+L_0] \cap J_i = \emptyset \text{ for all } i = 1, \dots, q_j,$$

then there are $s_1 \in [s, s+L_0-l]$ and $w \in \sigma(f)$ such that (2.4) is valid.

$\boxed{3}$ Concretization of the Main Results

In [9, Lemma 3.2] it was proved the following result.

Proposition 3.1. *Let $f \in \mathfrak{M}$ satisfy*

$$\mu(f) < \inf\{f(t,0,0) : \ t \in R\}.$$

Then no element of $\sigma(f)$ is a constant and $\sup\{\tau(w) : \ w \in \sigma(f)\} < \infty$.

Let $f \in \mathfrak{M}$ satisfy

$$\mu(f) < \inf\{f(t,0,0) : \ t \in R\}.$$

We can choose l in Theorems 2.1-2.4 as

$$l = k \sup\{\tau(w) : \ w \in \sigma(f)\}$$

where k is a large natural number. Let $L > l$ be as guaranteed by Theorem 2.1. If an approximate solution v of problem (P) satisfies conditions of Theorem 2.1, then for each closed subinterval $D \in [0,T]$ of length L there exists a closed subinterval $D_1 \subset D$ of length $k \sup\{\tau(w) : \ w \in \sigma(f)\}$ and a periodic minimizer w of problem (P) such that

$$|(v(t), v'(t)) - (w(t), w'(t))| \le \epsilon \text{ for each } t \in D_1.$$

Clearly, the restriction of v to interval D_1 is a good approximation of the periodic minimizer w.

If $\mu(f) = \inf\{f(t,0,0) : \ t \in R\}$, then there is a periodic minimizer $w \in \sigma(f)$ which is a constant and Proposition 3.1 does not hold. Namely, the set $\{\tau(w) : \ w \in \sigma(f)\}$ can be unbounded. In this case the turnpike property in Theorems 2.1-2.4 (see inequality (2.4)) does not provide sufficient information about the periodic minimizer w if its period is larger than l.

Now we state the main result of this section which will be proved in Section 9. This result is a concretization of Theorem 2.1.

Theorem 3.1. *Let $f \in \mathfrak{M}$ and let M_0, M_1, ϵ, l_0 be positive numbers. Then there exists $h > l_0$ such that the following assertion holds: For each $l > h$ there are $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that for each $g \in \mathcal{U}$, each $T \ge L$, each $v \in W^{2,1}([0,T])$ satisfying*

$$|(v(0), v'(0))|, \ |(v(T), v'(T))| \le M_0,$$
$$I^g(0,T,v) \le U_T^g((v(0), v'(0)), (v(T), v'(T))) + M_1 \tag{3.1}$$

and each $s \in [0, T-L]$ there is $s_1 \in [s, s+L-l]$ such that at least one of the following properties holds:

(i) *there exists $w \in \sigma(f,h)$ such that*

$$|(v(s_1+t), v'(s_1+t)) - (w(t), w'(t))| \le \epsilon \text{ for all } t \in [0,l]; \tag{3.2}$$

(ii) *for each $\tau \in [s_1, s_1+l-h]$ there are $\tau_1 \in [\tau, \tau+h-l_0]$ and $\xi \in \sigma(f,0)$ such that*

$$|(v(\tau_1+t), v'(\tau_1+t)) - (\xi(0), 0)| \le \epsilon \text{ for all } t \in [0,l_0]. \tag{3.3}$$

The next theorem is a concretization of Theorem 2.2. It will be proved in Section 10.

Theorem 3.2. *Let $f \in \mathfrak{M}$ and let l_0, ϵ be positive numbers. Then there exists $h > l_0$ such that the following assertion holds: For each $l > h$ there are $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that for each $g \in \mathcal{U}$, each (g)-good function $v \in W^{2,1}_{loc}([0,\infty))$ and each sufficiently large number s there is $s_1 \in [s, s+L-l]$ such that at least one of the properties (i) and (ii) of Theorem 3.1 holds.*

The next two theorems describe the structure of approximate solutions of problem (P_∞). They will be proved in Section 11.

Theorem 3.3. *Let $f \in \mathfrak{M}$ and $v \in W^{2,1}_{loc}([0,\infty))$ satisfy*

$$\sup\{|(v(t), v'(t))| : \ t \in [0,\infty)\} < \infty, \tag{3.4}$$

$$\limsup_{T\to\infty} T^{-1} I^f(0,T,v) = \mu(f). \tag{3.5}$$

Assume that ϵ, l_0 are positive numbers. Then there exists $h > l_0$ such that for each $l > h$ there is $L > l$ for which the following assertion holds: For each $\gamma > 0$ there is $T_\gamma > L$ such that for each $T \ge T_\gamma$ there are a finite number of closed intervals $J_1, \dots, J_q$ such that

$$q \le \gamma T, \tag{3.6}$$

$$mes(J_i) \le L, \ i = 1, \dots, q, \tag{3.7}$$

$$\begin{aligned} &int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers} \\ &i, p \in \{1, \dots, q\} \text{ satisfying } i \ne p \end{aligned} \tag{3.8}$$

and if

$$s \in [0, T-L], \ [s, s+L] \cap J_i = \emptyset, \ i = 1, \dots, q, \tag{3.9}$$

then there is $s_1 \in [s, s+L-l]$ such that at least one of the properties (i), (ii) of Theorem 3.1 holds.

Theorem 3.4. *Let $f \in \mathfrak{M}$, $v \in W^{2,1}_{loc}([0,\infty))$ satisfy (3.4) and let $\{T_i\}_{i=1}^\infty$ be a strictly increasing sequence of positive numbers such that $\lim_{i\to\infty} T_i = \infty$ and*

$$\lim_{i\to\infty} T_i^{-1} I^f(0, T_i, v) = \mu(f). \tag{3.10}$$

Assume that $\epsilon > 0$, $l_0 > 0$. Then there exists $h > l_0$ such that for each $l > h$ there is $L > l$ such that the following assertion holds: For each $\gamma > 0$ there is a natural number i_γ such that $T_{i_\gamma} > L$ and that for each integer $i \ge i_\gamma$ there are a finite number of closed intervals $J_1, \dots, J_q$ such that

$$q \le \gamma T_i, \tag{3.11}$$

(3.7), (3.8) hold and that for each number s satisfying (3.9) there is $s_1 \in [s, s+L-l]$ for which at least one of the properties (i), (ii) of Theorem 3.1 holds.

4 Preliminary Results

In the sequel we use the following result [15, Proposition 5.1].

Proposition 4.1. *Let $f \in \bar{\mathfrak{M}}$. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ and a number $S > 0$ such that for every $g \in \mathcal{U}$ and every (g)-good function v,*

$$|X_v(t)| \leq S \text{ for all large enough } t.$$

Proposition 4.2. *Let $f \in \bar{\mathfrak{M}}$ and let M_1, M_2, c be positive numbers. Then there exist a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ and $S > 0$ such that the following assertion holds: If $g \in \mathcal{U}$, $T_1 \geq 0$, $T_2 \geq T_1 + c$ and if $v \in W^{2,1}([T_1, T_2])$ satisfies*

$$|X_v(T_1)|,\ |X_v(T_2)| \leq M_1 \text{ and } I^g(T_1, T_2, v) \leq U^g_{T_2 - T_1}(X_v(T_1), X_v(T_2)) + M_2,$$

then

$$|X_v(t)| \leq S \text{ for all } t \in [T_1, T_2].$$

For this result we refer the reader to [4] (see the proof of Proposition 4.4).

The next two results were obtained in [15, Propositions 3.1 and 3.2].

Proposition 4.3. *Let $f \in \bar{\mathfrak{M}}$ and let $0 < c_1 < c_2 < \infty$, $c_3 > 0$, $\epsilon \in (0,1)$. Then there exists a neighborhood V of f in $\bar{\mathfrak{M}}$ such that for every $g \in V$, every $T \in [c_1, c_2]$, and each $x, y \in R^2$ satisfying $|x|, |y| \leq c_3$, the inequality $|U^f_T(x,y) - U^g_T(x,y)| \leq \epsilon$ holds.*

Proposition 4.4. *Let $f \in \bar{\mathfrak{M}}$ and let $0 < c_1 < c_2 < \infty$, $\epsilon > 0$, $D > 0$. Then there exists a neighborhood V of f in $\bar{\mathfrak{M}}$ such that for every $g \in V$, every $T \in [c_1, c_2]$, and every $w \in W^{2,1}[0,T]$ satisfying*

$$\min\{I^f(0,T,w),\ I^g(0,T,w)\} \leq D$$

the inequality $|I^f(0,T,w) - I^g(0,T,w)| \leq \epsilon$ holds.

The next useful result was proved in [9, Lemma 2.6].

Proposition 4.5. *Let $f \in \mathfrak{M}$. Then for every compact set $E \subset R^2$ there exists a constant $M > 0$ such that for every $T \geq 1$*

$$U^f_T(x,y) \leq T\mu(f) + M \text{ for all } x, y \in E.$$

Proposition 4.5 implies the following proposition.

Proposition 4.6. *Assume that $f \in \mathfrak{M}$, $v \in W^{2,1}_{loc}([0,\infty))$, the set $\{X_v(t) :\ t \in [0,\infty)\}$ is bounded and that there is $M_0 > 0$ for which*

$$I^f(0,T,v) \leq U^f_T(X_v(0), X_v(T)) + M_0 \text{ for all } T > 0.$$

Then v is (f)-good.

In the sequal we also use the following auxiliary results.

Proposition 4.7. *[10, Proposition 2.3] Let $f \in \mathfrak{M}$ and let $w \in W^{2,1}_{loc}([0,\infty))$ be an (f)-good function. Let $\{\xi_k\}_{k=1}^{\infty} \subset (0,\infty)$ be a sequence such that $\xi_k \to \infty$ as $k \to \infty$ and let u_k, $k = 1, 2, \dots$ be the functions given by*

$$u_k(t) = w(t + \xi_k), \ (-\xi_k \le t < \infty).$$

Then there exists a subsequence $\{u_{k_i}\}_{i=1}^{\infty}$ of $\{u_k\}_{k=1}^{\infty}$ and $u \in W^{2,\gamma}_{loc}(R^1)$ such that

$$(u_{k_i}, u'_{k_i}) \to (u, u') \text{ as } i \to \infty \text{ uniformly on } [-T,T] \text{ for all } T > 0,$$

$$u''_{k_i} \to u'' \text{ as } i \to \infty \text{ weakly in } L^{\gamma}[-T,T] \text{ for all } T > 0,$$

$$\Gamma^f(T_1, T_2, u) = 0 \text{ for each } T_1 \in R, \ T_2 > T_1;$$

$$\{(u(t), u'(t)) : \ t \in R\} \subset \Omega(v).$$

Note that the parameter γ was introduced in Introduction and $L^{\gamma}[-T,T]$ is the space of all Lebesgue measurable functions $x(t)$, $t \in [-T,T]$ for which there exists a finite integral $\int_{-T}^{T} |x(t)|^{\gamma} dt$.

Proposition 4.8. *[9, Lemma 2.8] Let $f \in \mathfrak{M}$ and $T_2 > T_1$ and let $w_1, w_2 \in W^{2,1}([T_1, T_2])$ satisfy*

$$\Gamma^f(T_1, T_2, w_1) = \Gamma^f(T_1, T_2, w_2) = 0.$$

If there is $\tau \in (T_1, T_2)$ such that $X_{w_1}(\tau) = X_{w_2}(\tau)$, then $w_1 = w_2$ in $[T_1, T_2]$.

Proposition 4.9. *[9, Lemma 2.5] Let $f \in \bar{\mathfrak{M}}$, $v \in W^{2,1}_{loc}([0,\infty))$ be an (f)-good function and let $\xi \in \Omega(v)$. Then there is $u \in W^{2,1}_{loc}(R^1)$ such that*

$$X_u(t) \in \Omega(v) \text{ for all } t \in R, \ X_u(0) = \xi, \ \Gamma^f(-T, T, u) = 0 \text{ for all } T > 0.$$

Proposition 4.10. *[10, Lemma 3.6] Assume that $f \in \mathfrak{M}$ and $v \in W^{2,1}_{loc}([0,\infty))$ satisfies*

$$\sup\{|X_v(t)| : \ t \in [0,\infty)\} < \infty,$$

$$I^f(0, T, v) = U^f_T(X_v(0), X_v(T)) \text{ for all } T > 0.$$

Then there is $w \in \sigma(f)$ such that $\Omega(w) \subset \Omega(v)$.

Proposition 4.11. *Assume that $f \in \mathfrak{M}$, $v \in W^{2,1}_{loc}([0,\infty))$,*

$$\sup\{|X_v(t)| : \ t \in [0,\infty)\} < \infty \tag{4.1}$$

and that there is $M_0 > 0$ such that

$$I^f(0, T, v) \le U^f_T(X_v(0), X_v(T)) + M_0 \text{ for all } T > 0. \tag{4.2}$$

Then there are a sequence of positive numbers $\{\xi_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \xi_i = \infty$ and $u \in \sigma(f)$ such that the sequence of functions $\{u_i\}_{i=1}^{\infty}$ defined by

$$u_i(t) = v(t + \xi_i), \ (-\xi_i \le t < \infty) \tag{4.3}$$

has the following property:

$$(u_i, u'_i) \to (u, u') \text{ as } i \to \infty \text{ uniformly on } [-T,T],$$

$$u''_i \to u'' \text{ as } i \to \infty \text{ weakly in } L^{\gamma}[-T,T] \tag{4.4}$$

for all $T > 0$.

Proof. By Proposition 4.6, (4.1) and (4.2) the function v is (f)-good. In view of Proposition 4.9 there is $v_1 \in W^{2,1}_{loc}(R)$ such that

$$\Gamma^f(T_1, T_2, v_1) = 0 \text{ for each } T_1 \in R,\ T_2 > T_1 \tag{4.5}$$

and

$$X_{v_1}(t) \in \Omega(v) \text{ for all } t \in R.$$

Clearly, $\Omega(v_1) \subset \Omega(v)$. Proposition 4.10 and (4.5) imply that there is $u \in \sigma(f)$ for which

$$\Omega(u) \subset \Omega(v_1).$$

It is not difficult to see that $\Omega(u) \subset \Omega(v)$. There exists a sequence $\{\xi_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that $\lim_{i\to\infty} \xi_i = \infty$ and

$$\lim_{i\to\infty} X_v(\xi_i) = X_u(0). \tag{4.6}$$

Consider a sequence of functions $\{u_i\}_{i=1}^{\infty}$ defined by (4.3). It follows from Proposition 4.7 that there are a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ of $\{u_i\}_{i=1}^{\infty}$ and $\tilde{u} \in W^{2,1}_{loc}(R)$ such that

$$(u_{i_k}, u'_{i_k}) \to (\tilde{u}, \tilde{u}') \text{ as } k \to \infty \text{ uniformly on } (-T, T) \text{ for all } T > 0, \tag{4.7}$$
$$u''_{i_k} \to \tilde{u}'' \text{ as } k \to \infty \text{ weakly in } L^{\gamma}[-T, T] \text{ for all } T > 0$$

and

$$\Gamma^f(-T, T, \tilde{u}) = 0 \text{ for all } T > 0. \tag{4.8}$$

By (4.7), (4.6) and (4.3),

$$X_{\tilde{u}}(0) = \lim_{i\to\infty} X_v(\xi_i) = X_u(0). \tag{4.9}$$

Since $u \in \sigma(f)$ it follows from (4.8), (4.9) and Proposition 4.8 that $u = \tilde{u}$ on R. The proposition is proved. □

5 Proof of Theorem 2.1

Assume that Theorem 2.1 does not hold. Then for each natural number n there exist $f_n \in \bar{\mathfrak{M}}$ such that

$$\rho(f, f_n) \le 1/n, \tag{5.1}$$

a number

$$T_n \ge n + l, \tag{5.2}$$

a function $v_n \in W^{2,1}([0, T_n])$ which satisfies

$$|X_{v_n}(0)|, |X_{v_n}(T_n)| \le M_0, \tag{5.3}$$

$$I^{f_n}(0, T_n, v_n) \le U^{f_n}_{T_n}(X_{v_n}(0), X_{v_n}(T_n)) + M_1 \tag{5.4}$$

and

$$s_n \in [0, T_n - n - l] \tag{5.5}$$

such that

$$\sup\{|X_{v_n}(s+t) - X_w(t)| :\ t \in [0,l]\} > \epsilon \tag{5.6}$$

for each $w \in \sigma(f)$ and each $s \in [s_n, s_n + n]$.

By Proposition 4.2 and (5.1)-(5.4) there is a constant $M_2 > 0$ such that

$$|X_{v_n}(t)| \le M_2 \text{ for all } t \in [0, T_n] \text{ and all natural numbers } n. \tag{5.7}$$

For each natural number n define $u_n \in W^{2,1}[0, n+l]$ by

$$u_n(t) = v_n(s_n + t),\ t \in [0, n+l] \tag{5.8}$$

(see (5.5)). Relations (5.7) and (5.8) imply that

$$|X_{u_n}(t)| \le M_2,\ t \in [0, n+l] \text{ and all natural numbers } n. \tag{5.9}$$

In view of (5.8) and (5.4)

$$I^{f_n}(0, n+l_n, u_n) \le U^{f_n}_{n+l_n}(X_{u_n}(0), X_{u_n}(n+l)) + M_1. \tag{5.10}$$

It follows from (5.8) and from the inequality (5.6) which holds for each $w \in \sigma(f)$, each $s \in [s_n, s_n + n]$ and all natural numbers n that

$$\sup\{|X_{u_n}(\tau + t) - X_w(t)| :\ t \in [0,l]\} > \epsilon \tag{5.11}$$

for each $w \in \sigma(f)$, each $\tau \in [0,n]$ and all natural numbers n.

Let k be a natural number. By (5.9) and the continuity of U^f_k there is $m_k > 0$ such that

$$|U^f_k(X_{u_n}(0), X_{u_n}(k))| \le m_k \text{ for all integers } n \ge k. \tag{5.12}$$

In view of (5.9), (5.1) and Proposition 4.3

$$U^f_k(X_{u_n}(0), X_{u_n}(k)) - U^{f_n}_k(X_{u_n}(0), X_{u_n}(k)) \to 0 \text{ as } n \to \infty. \tag{5.13}$$

Relations (5.12) and (5.13) imply that the sequence $\{U^{f_n}_k(X_{u_n}(0), X_{u_n}(k))\}_{n=k}^\infty$ is bounded. Together with (5.10) this implies that

$$\sup\{I^{f_n}(0,k,u_n) :\ n \text{ is an integer and } n \ge k\} < \infty. \tag{5.14}$$

It follows from (5.14), (5.1) and Proposition 4.4 that

$$\lim_{n\to\infty}[I^f(0,k,u_n) - I^{f_n}(0,k,u_n)] = 0. \tag{5.15}$$

Relations (5.15) and (5.14) imply that the sequence $\{I^f(0,k,u_n)\}_{n=1}^\infty$ is bounded from above. Combined with (5.9) and the growth condition (1.2) this implies that the sequence $\{\int_0^k |u_n''(t)|^\gamma dt\}_{n=k}^\infty$ is bounded. Since this fact holds for any natural number k it follows from (5.9) that there exist a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ and $u_* \in W^{2,1}_{loc}([0,\infty))$ such that for each natural number k

$$(u_{n_i}, u'_{n_i}) \to (u_*, u'_*) \text{ as } i \to \infty \text{ uniformly on } [0,k], \tag{5.16}$$

$$u''_{n_i} \to u''_* \text{ as } i \to \infty \text{ weakly in } L^{\gamma}[0,k]. \tag{5.17}$$

By (5.16), (5.17) and the lower semicontinuity of integral functionals [1] for each natural number k

$$I^f(0,k,u_*) \le \liminf_{i\to\infty} I^f(0,k,u_{n_i}). \tag{5.18}$$

In view of (5.16) and (5.9)

$$|X_{u_*}(t)| \le M_2 \text{ for all } t \in [0,\infty). \tag{5.19}$$

It follows from (5.18), (5.15), (5.10),(5.13), (5.16) and the continuity of U_k^f that for each natural number k

$$\begin{aligned} I^f(0,k,u_*) &\le \liminf_{i\to\infty} I^f(0,k,u_{n_i}) = \liminf_{i\to\infty} I^{f_{n_i}}(0,k,u_{n_i}) \\ &\le \liminf_{i\to\infty} [U_k^{f_{n_i}}((u_{n_i}(0), u'_{n_i}(0)), (u_{n_i}(k), u'_{n_i}(k))) + M_1] \\ &= \liminf_{i\to\infty} [U_k^f((u_{n_i}(0), u'_{n_i}(0)), (u_{n_i}(k), u'_{n_i}(k)))] + M_1 \\ &= U_k^f(X_{u_*}(0), X_{u_*}(k)) + M_1. \end{aligned}$$

Thus

$$I^f(0,k,u_*) \le U_k^f(X_{u_*}(0), X_{u_*}(k)) + M_1 \quad \text{for each natural number } k. \tag{5.20}$$

By (5.16) and (5.11) which holds for each $w \in \sigma(f)$, each $\tau \in [0,n]$ and all natural numbers n,

$$\sup\{|X_{u_*}(\tau+t) - X_w(t)| : \; t \in [0,l]\} > \epsilon/2 \tag{5.21}$$

for each $\tau \in [0,\infty)$ and each $w \in \sigma(f)$.

On the other hand it follows from (5.19), (5.20) and Proposition 4.11 that there exist a sequence of positive numbers $\{\xi_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} \xi_i = \infty$ and $w \in \sigma(f)$ such that the sequence of functions

$$h_i(t) = u_*(t+\xi_i) \; (-\xi_i \le t < \infty)$$

satisfies

$$(h_i, h'_i) \to (w, w') \text{ as } i \to \infty \text{ uniformly on } [-T,T] \text{ for all } T > 0,$$

$$h''_i \to w'' \text{ as } i \to \infty \text{ weakly in } L^{\gamma}[-T,T] \text{ for all } T > 0.$$

This implies that there is a natural number j such that for all $t \in [0,l]$

$$\epsilon/4 \ge |X_{h_j}(t) - X_w(t)| = |X_{u_*}(t+\xi_j) - X_w(t)|.$$

This contradicts (5.21). The contradiction we have reached proves the theorem.

$\boxed{6}$ Proof of Theorem 2.2

By Proposition 4.1 there exist a neighborhood $\mathcal{U}_\infty$ of f in $\bar{\mathfrak{M}}$ and a number M_0 such that for every $g \in \mathcal{U}_\infty$ and every (g)-good function v

$$|X_v(t)| \le M_0 \text{ for all large enough } t. \tag{6.1}$$

Set $M_1 = 1$. Let $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ be as guaranteed by Theorem 2.1. We may assume that

$$\mathcal{U} \subset \mathcal{U}_\infty. \tag{6.2}$$

Let $g \in \mathcal{U}$ and $v \in W^{2,1}_{loc}([0,\infty))$ be a (g)-good function. By (6.2) and (6.1) there is $t_0 > 0$ such that

$$|X_v(t)| \le M_0 \text{ for all } t \ge t_0. \tag{6.3}$$

Since v is (g)-good there is $T_0 > t_0$ such that for each $S_1 \ge T_0$, $S_2 > S_1$

$$I^g(S_1, S_2, v) \le U^g_{S_2 - S_1}(X_v(S_1), X_v(S_2)) + 1. \tag{6.4}$$

Let $s \ge T_0$ and consider the function $v : [s, s+L] \to R$. Inequalities (6.3) and (6.4) imply that

$$|X_v(s)|, |X_v(s+L)| \le M_0 \text{ and } I^g(s, s+L, v) \le U^g_L(X_v(s), X_v(s+L)) + 1.$$

By these inequalities, the choice of L, $\mathcal{U}$ and Theorem 2.1, there is $s_1 \in [s, s+L-l]$ and $w \in \sigma(f)$ such that

$$|X_v(s_1 + t) - X_w(t)| \le \epsilon \text{ for all } t \in [0, l].$$

This completes the proof of Theorem 2.2.

$\boxed{7}$ Proofs of Theorems 2.3 and 2.4

It is not difficult to prove the following auxiliary result.

Proposition 7.1. *Assume that $f \in \mathfrak{M}$ and $v \in W^{2,1}_{loc}([0,\infty))$ satisfies*

$$\sup\{|X_v(t)| : t \in [0,\infty)\} < \infty, \tag{7.1}$$

$$I^f(0, n, v) < \infty \textit{ for any natural number } n. \tag{7.2}$$

Then there exist a strictly increasing sequence t_i, $i \in J$ where $J = \{0, 1, 2, \dots\}$ or $J = \{0, \dots, q\}$ with an integer $q \ge 0$ such that:

$$t_0 = 0 \textit{ and } \Gamma^f(t_i, t_{i+1}, v) = 1 \textit{ for each integer } i \ge 0 \textit{ satisfying } i + 1 \in J;$$

$$\textit{if } J = \{0, \dots, q\}, \textit{ then } \Gamma^f(t_q, t, v) < 1 \textit{ for all } t > t_q;$$

for each $T > 0$ the set $[0, T] \cap \{t_i : i \in J\}$ is finite.

Note that the sequence t_i, $i \in J$ in Proposition 7.1 is constructed by induction.

Proposition 7.2. *Let $f \in \mathfrak{M}$, l, ϵ be positive numbers and let $v \in W^{2,1}_{loc}([0,\infty))$ satisfy (7.1) and (7.2). Then there exists $L_0 > l$ such that the following assertion holds: Assume that $\gamma > 0$, $T > L_0$ and*

$$T^{-1}\Gamma^f(0,T,v) \le \gamma/4. \tag{7.3}$$

Then there are a finite number of closed intervals $J_1, \dots, J_{q_T}$ such that

$$q_T \le \gamma T, \tag{7.4}$$

$$mes(J_i) \le L_0,\ i = 1, \dots, q_T, \tag{7.5}$$

$$int(J_i) \cap int(J_p) = \emptyset \text{ for each pair of integers}$$

$$i, p \in \{1, \dots, q_T\} \text{ such that } i \ne p, \tag{7.6}$$

and if

$$s \in [0, T - L_0] \text{ and } [s, s + L_0] \cap J_i = \emptyset \text{ for all } i = 1, \dots, q_T, \tag{7.7}$$

then there are $s_1 \in [s, s + L_0 - l]$ and $w \in \sigma(f)$ such that

$$|X_v(s_1 + t) - X_w(t)| \le \epsilon \text{ for all } t \in [0, l]. \tag{7.8}$$

Proof. Let a sequence t_i, $i \in J$ be as guaranteed by Proposition 7.1. Set $M_1 = 1$. By (7.1) there is M_0 such that

$$\sup\{|X_v(t)| :\ t \in [0, \infty)\} < M_0. \tag{7.9}$$

Let $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ be as guaranteed by Theorem 2.1. Set

$$L_0 = 2L. \tag{7.10}$$

Assume that $\gamma > 0$, $T > L_0$ and (7.3) holds. There is an integer $q \ge 0$ such that

$$t_q \le T,\ \{t_i :\ i \in J\} \cap (t_q, T) = \emptyset. \tag{7.11}$$

By (7.3) and the definition of t_i, $i \in J$ (see Proposition 7.1),

$$T\gamma/4 \ge \Gamma^f(0,T,v) \ge \sum_{i=0}^{q-1} \Gamma^f(t_i, t_{i+1}, v) = q. \tag{7.12}$$

Set

$$A = \{i \in J :\ i + 1 \le q,\ t_{i+1} - t_i \le L_0\} \tag{7.13}$$

and consider the collection of closed intervals

$$[t_i, t_{i+1}],\ i \in A. \tag{7.14}$$

Note that this collection may be empty. In view of (7.13) and (7.12)

$$t_{i+1} - t_i \le L_0 \text{ for all } i \in A, \ \mathrm{Card}(A) \le q \le T\gamma/4,$$

$$(t_{i_1}, t_{i_1+1}) \cap (t_{i_2}, t_{i_2+1}) = \emptyset \text{ for each } i_1, i_2 \in A \text{ such that } i_1 \ne i_2. \tag{7.15}$$

Now assume that

$$s \in [0, T - L_0], \ [s, s + L_0] \cap [t_i, t_{i+1}] = \emptyset \text{ for each } i \in A. \tag{7.16}$$

There is $j \in \{0, \dots, q\}$ such that

$$t_j \le s \text{ and } (t_j, s) \cap \{t_i : \ i \in J\} = \emptyset. \tag{7.17}$$

We have the following cases:

(1) $j = q$;

(2) $j < q$, $[s, s + L_0] \subset [t_j, t_{j+1}]$;

(3) $j < q$, $[s, s + L_0] \setminus [t_j, t_{j+1}] \ne \emptyset$, $j + 1 = q$;

(4) $j + 1 < q$, $[s, s + L_0] \setminus [t_j, t_{j+1}] \ne \emptyset$.

Consider the case (1) with $j = q$. By (7.17) and (7.16), $[s, s + L_0] \subset [t_q, T]$. Combined with (7.11) and the definition of t_i, $i \in J$ (see Proposition 7.1) this inclusion implies that

$$\Gamma^f(s, s + L_0, v) \le \Gamma^f(t_q, T, v) \le 1.$$

This inequality implies that

$$I^f(s, s + L_0, v) \le U^f_{L_0}(X_v(s), X_v(s + L_0)) + 1. \tag{7.18}$$

It follows from (7.18), (7.9), (7.10), the choice of L,$\mathcal{U}$ and Theorem 2.1 that there are

$$s_1 \in [s, s + L - l] \text{ and } w \in \sigma(f)$$

such that (7.8) holds.

Consider the case (2). Then by the definition of t_i, $i \in J$ (see Proposition 7.1)

$$\Gamma^f(s, s + L_0, v) \le \Gamma^f(t_j, t_{j+1}, v) = 1.$$

This inequality implies (7.18). It follows from (7.18), (7.9), (7.10), the choice of L and $\mathcal{U}$ and Theorem 2.1 that there are $s_1 \in [s, s + L - l]$ and $w \in \sigma(f)$ such that (7.8) holds.

Consider the case (3). Then by (7.17), (7.11) and (7.16),

$$t_j \le s < t_{j+1} = t_q < s + L_0 \le T. \tag{7.19}$$

It follows from (7.19) and (7.10) that there is $s_0 \in [s, s + L]$ such that

$$\text{either } [s_0, s_0 + L] \subset [t_j, t_{j+1}] \text{ or } [s_0, s_0 + L] \subset [t_{j+1}, T] = [t_q, T].$$

In both cases it follows from (7.11) and the definition of t_i, $i \in J$ (see Proposition 7.1) that

$$\Gamma^f(s_0, s_0 + L, v) \le \max\{\Gamma^f(t_j, t_{j+1}, v), \Gamma^f(t_q, T, v)\} \le 1.$$

This inequality implies that

$$I^f(s_0, s_0 + L, v) \le U_L^f(X_v(s_0), X_v(s_0 + L)) + 1. \tag{7.20}$$

It follows from (7.20), (7.9), the choice of L and $\mathcal{U}$ and Theorem 2.1 that there are $s_1 \in [s_0, s_0 + L - l]$ and $w \in \sigma(f)$ such that (7.8) holds.

Consider the case (4). By (7.17)

$$t_j \le s < t_{j+1} < s + L_0. \tag{7.21}$$

Inequality (7.21) implies that

$$[s, s + L_0] \cap [t_j, t_{j+1}] \ne \emptyset, \; [s, s + L_0] \cap [t_{j+1}, t_{j+2}] \ne \emptyset.$$

Combined with (7.16) these relations imply that

$$j, j + 1 \notin A.$$

Together with (7.13) this relation implies that

$$t_{j+1} - t_j > L_0, \; t_{j+2} - t_{j+1} > L_0. \tag{7.22}$$

In view of (7.22) and (7.21)

$$[s, s + L_0] \subset [t_j, t_{j+2}].$$

Since $L_0 = 2L$ (see (7.10)) there is $s_0 \in [s, s + L]$ such that

$$\text{either } [s_0, s_0 + L] \subset [t_j, t_{j+1}] \text{ or } [s_0, s_0 + L] \subset [t_{j+1}, t_{j+2}].$$

In both cases it follows from the definition of t_i, $i \in J$ (see Proposition 7.1) that

$$\Gamma^f(s_0, s_0 + L, v) \le \max\{\Gamma^f(t_j, t_{j+1}, v), \; \Gamma^f(t_{j+1}, t_{j+2}, v)\} = 1.$$

This inequality implies (7.20). It follows from (7.20), (7.9), the choice of L and $\mathcal{U}$ and Theorem 2.1 that there are $s_1 \in [s_0, s_0 + L - l]$ and $w \in \sigma(f)$ such that (7.8) holds. This completes the proof of the proposition. □

Proof of Theorem 2.3. It follows from (2.5) that

$$\begin{aligned}\limsup_{T\to\infty} T^{-1}\Gamma^f(0, T, v) &\le \limsup_{T\to\infty} T^{-1}[I^f(0, T, v) - T\mu(f) - \pi^f(X_v(0)) + \pi^f(X_v(T))] \\ &\le \limsup_{T\to\infty} T^{-1}[I^f(0, T, v) - T\mu(f)] = 0\end{aligned}$$

and

$$\lim_{T\to\infty} T^{-1}\Gamma^f(0, T, v) = 0. \tag{7.23}$$

Let $L_0 > l$ be as guaranteed by Proposition 7.2. Let $\gamma > 0$. By (7.23) there is $T_\gamma > L_0$ such that for each $T \ge T_\gamma$,

$$\Gamma^f(0, T, v) \le T\gamma/4. \tag{7.24}$$

Let $T \ge T_\gamma$. Then (7.24) holds and the assertion of Theorem 2.3 follows from the choice of L_0 and Proposition 7.2. □

Proof of Theorem 2.4. It follows from (2.10) and (2.5) that

$$\lim_{i\to\infty} T_i^{-1}\Gamma^f(0,T_i,v) = \lim_{i\to\infty} T_i^{-1}[I^f(0,T_i,v) - \mu(f)T_i - \pi^f(X_v(0)) + \pi^f(X_v(T_i))]$$
$$= \lim_{i\to\infty} T_i^{-1}[I^f(0,T_w,w) - \mu(f)T_i] = 0.$$

Thus

$$\lim_{i\to\infty} T_i^{-1}\Gamma^f(0,T_i,v) = 0. \tag{7.25}$$

Let $L_0 > l$ be as guaranteed by Proposition 7.2. Let $\gamma > 0$. By (7.25) there exists a natural number j_γ such that $T_{j_\gamma} > L_0$ and that for each integer $j \geq j_\gamma$,

$$T_j > L_0,\ \Gamma^f(0,T_j,v) \leq \gamma T_j/4. \tag{7.26}$$

Let an integer $j \geq j_\gamma$. Then (7.26) holds and the assertion of Theorem 2.4 follows from the choice of L_0 and Proposition 7.2. □

8 Auxiliary Results for Theorem 3.1

Lemma 8.1. *Let $f \in \mathfrak{M}$ and let ϵ, h, M be positive numbers. Then there is $h_0 > h$ such that the following assertion holds: Assume that $v \in W^{2,1}([0,h_0])$,*

$$|X_v(t)| \leq M,\ t \in [0,h_0], \tag{8.1}$$

$$\Gamma^f(0,h_0,v) = 0, \tag{8.2}$$

$$\text{either } v'(t) \geq 0 \text{ for all } t \in [0,h_0] \text{ or } v'(t) \leq 0 \text{ for all } t \in [0,h_0]. \tag{8.3}$$

Then there are $s \in [0,h_0-h]$ and $\xi \in \sigma(f,0)$ such that

$$|X_v(s+t) - (\xi(0),0)| \leq \epsilon \text{ for all } t \in [0,h]. \tag{8.4}$$

Proof. Let us assume the converse. Then for each natural number n there is $v_n \in W^{2,1}([0,n+h])$ such that

$$|X_{v_n}(t)| \leq M,\ t \in [0,h+n], \tag{8.5}$$

$$\Gamma^f(0,n+h,v_n) = 0, \tag{8.6}$$

$$\text{either } v'(t) \geq 0 \text{ for all } t \in [0,h+n] \text{ or } v'(t) \leq 0 \text{ for all } t \in [0,n+h] \tag{8.7}$$

and that for each $s \in [0,n]$ and each $\xi \in \sigma(f,0)$

$$\sup\{|X_{v_n}(s+t) - (\xi(0),0)| :\ t \in [0,h]\} > \epsilon. \tag{8.8}$$

Let k be a natural number. By (8.6) for each natural number $n \geq k$

$$I^f(0,k,v_n) = U_k^f(X_{v_n}(0), X_{v_n}(k)). \tag{8.9}$$

Since the function U_k^f is continuous it follows from (8.9) and (8.5) that the sequence $\{I^f(0,k,v_n)\}_{n=k}^\infty$ is bounded. Combined with (8.5) and (1.2) this implies that the sequence

$\{\int_0^k |v_n''(t)|^\gamma dt\}_{n=k}^\infty$ is bounded. Since this fact holds for all natural numbers k it follows from (8.5) that there are a subsequence $\{v_{n_i}\}_{i=1}^\infty$ of $\{v_n\}_{n=1}^\infty$ and $v_* \in W_{loc}^{2,1}([0,\infty))$ such that for each natural number k

$$(v_{n_i}, v'_{n_i}) \to (v_*, v'_*) \text{ as } i \to \infty \text{ uniformly on } [0,k],$$

$$v''_{n_i} \to v''_* \text{ as } i \to \infty \text{ weakly in } L^\gamma[0,k]. \tag{8.10}$$

By (8.10) and the lower semicontinuity of integral functionals [1] for each natural number k

$$I^f(0,k,v_*) \le \liminf_{i\to\infty} I^f(0,k,v_{n_i}). \tag{8.11}$$

It follows from (8.11), (8.10) and (8.6) that for each natural number k

$$\begin{aligned}\Gamma^f(0,k,v_*) &= I^f(0,k,v_*) - k\mu(f) - \pi^f(X_{v_*}(0)) + \pi^f(X_{v_*}(0)) \\ &\le \liminf_{i\to\infty} I^f(0,k,v_{n_i}) - k\mu(f) - \lim_{i\to\infty} \pi^f(X_{v_{n_i}}(0)) \\ &\quad + \lim_{i\to\infty} \pi^f(X_{v_{n_i}}(k)) \le \liminf_{i\to\infty} \Gamma^f(0,k,v_{n_i}) = 0\end{aligned}$$

and

$$\Gamma^f(0,k,v_*) = 0 \text{ for all natural numbers } k. \tag{8.12}$$

Relations (8.10) and (8.5) imply that

$$|X_{v_*}(t)| \le M,\ t \in [0,\infty). \tag{8.13}$$

In view of (8.10) and (8.8) for each $s \in [0,\infty)$ and each $\xi \in \sigma(f,0)$,

$$\sup\{|X_{v_*}(s+t) - (\xi(0),0)| :\ t \in [0,h]\} \ge \epsilon/2. \tag{8.14}$$

By (8.10) and (8.7),

$$\text{either } v'_*(t) \ge 0 \text{ for all } t \in [0,\infty) \text{ or } v'_*(t) \le 0 \text{ for all } t \in [0,\infty). \tag{8.15}$$

It follows from (8.13) and (8.15) that there exists

$$d_0 = \lim_{t\to\infty} v_*(t). \tag{8.16}$$

In view of (8.16) and Proposition 4.7 $\Omega(v_*) = (d_0, 0)$ and $\mu(f) = f(d_0,0,0)$. This contradicts (8.14). The contradiction we have reached proves Lemma 8.1. □

Lemma 8.2. *Let $f \in \mathfrak{M}$ and let ϵ, h be positive numbers. Then there is $h_1 > h$ such that for each $w \in \sigma(f) \setminus \sigma(f,0)$ satisfying $\tau(w) \ge h_1$ the following property holds:*

(C1) *For each $s \in R$ there are $s_1 \in [s, s+h_1-h]$ and $\xi \in \sigma(f,0)$ such that*

$$|X_w(s_1+t) - (\xi(0),0)| \le \epsilon \text{ for all } t \in [0,h].$$

Proof. By Proposition 4.1 there is $M > 0$ such that

$$|X_w(t)| \le M \text{ for all } t \in R \text{ and all } w \in \sigma(f). \tag{8.17}$$

Let $h_0 > h$ be as guaranteed by Lemma 8.1. Put

$$h_1 = 8h_0. \tag{8.18}$$

Assume that

$$w \in \sigma(f) \setminus \sigma(f, 0) \text{ and } \tau(w) \geq h_1. \tag{8.19}$$

We show that (C1) hods for any $s \in R$. Evidently, it is sufficient to show that (C1) holds for any $s \in [0, \tau(w)]$.

Let

$$s \in [0, \tau(w)]. \tag{8.20}$$

There is $t_0 \in [0, \tau(w))$ such that

$$w(t_0) = \inf\{w(t) : \ t \in R\}. \tag{8.21}$$

By Corollary 2.1 there is $\tau_1 \in (0, \tau(w))$ such that

$$w \text{ is strictly increasing on } [t_0, t_0 + \tau_1] \text{ and strictly decreasing on}$$

$$[t_0 + \tau_1, t_0 + \tau(w)]. \tag{8.22}$$

Relations (8.20) and (8.19) imply that

$$[s, s + h_1] \subset [t_0 - \tau(w), t_0 + 2\tau(w)]. \tag{8.23}$$

Set

$$\begin{aligned} I_1 &= [s, s + h_1] \cap [t_0 - \tau(w), t_0 - \tau(w) + \tau_1], \\ I_2 &= [s, s + h_1] \cap [t_0 - \tau(w) + \tau_1, t_0], \\ I_3 &= [s, s + h_1] \cap [t_0, t_0 + \tau_1], \\ I_4 &= [s, s + h_1] \cap [t_0 + \tau_1, t_0 + \tau(w)], \\ I_5 &= [s, s + h_1] \cap [t_0 + \tau(w), t_0 + \tau(w) + \tau_1], \\ I_6 &= [s, s + h_1] \cap [t_0 + \tau(w) + \tau_1, t_0 + 2\tau(w)]. \end{aligned} \tag{8.24}$$

By (8.23) and (8.24),

$$[s, s + h_1] \subset \cup_{j=1}^{6} I_j$$

and there is $p \in \{1, 2, 3, 4, 5, 6\}$ such that

$$\text{mes}(I_p) \geq h_1/6 = 4h_0/3. \tag{8.25}$$

In view of (8.24) and (8.22)

$$\text{either } w'(t) \geq 0 \text{ for all } t \in I_p \text{ or } w'(t) \leq 0 \text{ for all } t \in I_p. \tag{8.26}$$

Inequality (8.17) implies that

$$|X_w(t)| \leq M \text{ for all } t \in I_p. \tag{8.27}$$

Since $w \in \sigma(f)$ we have

$$\Gamma^f(I_p, w) = 0. \tag{8.28}$$

It follows from (8.25), (8.26), (8.27), (8.28), the choice of h_0 and Lemma 8.1 that there are a number s_1 and $\xi \in \sigma(f, 0)$ such that

$$[s_1, s_1 + h] \subset I_p,$$

$$|X_w(s_1 + t) - (\xi(0), 0)| \leq \epsilon \text{ for all } t \in [0, h].$$

Lemma 8.2 is proved. □

$\boxed{9}$ Prooof of Theorem 3.1

By Lemma 8.2 there is $h > l_0$ such that the following property holds:

(C2) For each $w \in \sigma(f) \setminus \sigma(f, 0)$ satisfying $\tau(w) \geq h$ and each $s \in R$ there are $s_1 \in [s, s + h - l_0]$ and $\xi \in \sigma(f, 0)$ such that

$$|X_w(s_1 + t) - (\xi(0), 0)| \leq \epsilon/4 \text{ for all } t \in [0, l_0]. \tag{9.1}$$

Let $l > h$. By Theorem 2.1 there exist $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that the following property holds:

(C3) For each $g \in \mathcal{U}$, each $T \geq L$, each $v \in W^{2,1}([0, T])$ which satisfies

$$|X_v(0)|, |X_v(T)| \leq M_0, \tag{9.2}$$

$$I^g(0, T, v) \leq U_T^g(X_v(0), X_v(T)) + M_1 \tag{9.3}$$

and each $s \in [0, T - L]$ there are $s_1 \in [s, s + L - l]$ and $w \in \sigma(f)$ such that

$$|X_v(s_1 + t) - X_w(t)| \leq \epsilon/4 \text{ for all } t \in [0, l]. \tag{9.4}$$

Assume that $g \in \mathcal{U}$, $T \geq L$, $v \in W^{2,1}([0, T])$ satisfies (9.2), (9.3) and $s \in [0, T - L]$. By property (C3) there are

$$s_1 \in [s, s + L - l] \text{ and } w \in \sigma(f) \tag{9.5}$$

such that (9.4) is valid. If $w \in \sigma(f, h)$, then property (i) of Theorem 3.1 holds.

Assume that $w \notin \sigma(f, h)$. Then w is not a constant and

$$\tau(w) > h. \tag{9.6}$$

Let

$$\tau \in [s_1, s_1 + l - h]. \tag{9.7}$$

In view of (9.5), (9.6) and property (C2) there exist

$$t_1 \in [\tau - s_1, \tau - s_1 + h - l_0] \tag{9.8}$$

and

$$\xi \in \sigma(f, 0) \tag{9.9}$$

such that

$$|X_w(t_1+t)-(\xi(0),0)| \le \epsilon/4 \text{ for all } t \in [0,l_0]. \tag{9.10}$$

Set

$$\tau_1 = t_1 + s_1. \tag{9.11}$$

Relations (9.8) and (9.11) imply that

$$\tau = \tau - s_1 + s_1 \le t_1 + s_1 = \tau_1 \le \tau - s_1 + h - l_0 + s_1 = \tau + h - l_0,$$

$$\tau_1 \in [\tau, \tau + h - l_0]. \tag{9.12}$$

Let

$$t \in [0, l_0]. \tag{9.13}$$

We show that

$$|X_v(\tau_1+t)-(\xi(0),0)| \le \epsilon.$$

It follows from (9.13), (9.12) and (9.7) that

$$\tau_1 + t \in [\tau, \tau+h] \subset [s_1, s_1+l]. \tag{9.14}$$

Combined with (9.4) this inclusion implies that

$$|X_v(\tau_1+t) - X_w(\tau_1+t-s_1)| \le \epsilon/4. \tag{9.15}$$

By (9.11)

$$\tau_1 + t - s_1 = t_1 + t.$$

Combined with (9.13) and (9.10) this equality implies that

$$|X_w(\tau_1+t-s_1)-(\xi(0),0)| \le \epsilon/4.$$

Together with (9.15) this inequality implies that

$$|X_v(\tau_1+t)-(\xi(0),0)| \le \epsilon/2. \tag{9.16}$$

We have shown that for each τ satisfying (9.7) there are τ_1 satisfying (9.12) and $\xi \in \sigma(f,0)$ such that (9.16) holds for all $t \in [0,l_0]$.

Therefore property (ii) of Theorem 3.1 holds. Theorem 3.1 is proved.

10 Proof of Theorem 3.2

By Proposition 4.1 there exist a neighborhood $\mathcal{U}_\infty$ of $f \in \bar{\mathfrak{M}}$ and a number M_0 such that for every $g \in \mathcal{U}_\infty$ and every (g)-good function v

$$|X_v(t)| \le M_0 \text{ for all large enough } t. \tag{10.1}$$

Set $M_1 = 1$. Let $h > l_0$ be as guaranteed by Theorem 3.1.

Let $l > h$. By the choice of h and Theorem 3.1 there exist $L > l$ and a neighborhood $\mathcal{U}$ of f in $\bar{\mathfrak{M}}$ such that for each $g \in \mathcal{U}$, each $T \geq L$, each $v \in W^{2,1}([0,T])$ satisfying

$$|X_v(0)|, |X_v(T)| \leq M_0,$$
$$I^g(0,T,v) \leq U_T^g(X_v(0), X_v(T)) + 1 \tag{10.2}$$

and each $s \in [0, T-L]$ there is $s_1 \in [s, s+L-l]$ such that at least one of properties (i) and (ii) of Theorem 3.1 holds.

We may assume that

$$\mathcal{U} \subset \mathcal{U}_\infty. \tag{10.3}$$

Let $g \in \mathcal{U}$ and let $v \in W^{2,1}_{loc}([0,\infty))$ be a (g)-good function. By (10.3) and (10.1) there is $t_0 > 0$ such that

$$|X_v(t)| \leq M_0 \text{ for all } t \geq t_0. \tag{10.4}$$

Since v is (g)-good there is $T_0 > t_0$ such that for each $s_1 \geq T_0$, $s_2 > s_1$

$$I^g(s_1, s_2, v) \leq U^g(X_v(s_1), X_v(s_2)) + 1. \tag{10.5}$$

Let $s \geq T_0$ and consider the function $v : [s, s+L] \to R$. Inequalities (10.4) and (10.5) imply that

$$|X_v(s)|,\ |X_v(s+L)| \leq M_0 \text{ and } I^g(s, s+L, v) \leq U_L^g(X_v(s), X_v(s+L)) + 1.$$

It follows from these inequalities and the choice of L and $\mathcal{U}$ that there is $s_1 \in [s, s+L-l]$ such that at least one of properties (i) and (ii) of Theorem 3.1 holds. Theorem 3.2 is proved.

$\boxed{11}$ Proofs of Theorems 3.3 and 3.4

Proposition 11.1. *Let $f \in \mathfrak{M}$ and let ϵ, l_0 be positive numbers. Then there is $h > l_0$ such that for each $v \in W^{2,1}_{loc}([0,\infty))$ satisfying*

$$\sup\{|X_v(t)| :\ t \in [0,\infty)\} < \infty, \tag{11.1}$$

$$I^f(0,n,v) < \infty \textit{ for any natural number } n \tag{11.2}$$

and each $l > h$ there is $L > l$ such that the following assertion holds:

(A1) *Assume that $\gamma > 0$, $T > L$ and*

$$T^{-1}\Gamma^f(0,T,v) \leq \gamma/4. \tag{11.3}$$

Then there are a finite number of closed intervals $J_1, \dots, J_q$ such that

$$q \leq \gamma T, \tag{11.4}$$

$$mes(J_i) \leq L,\ i = 1, \dots, q, \tag{11.5}$$

$$int(J_i) \cap int(J_p) = \emptyset \textit{ for each pair of} \tag{11.1}$$

$$\textit{integers } i, p \in \{1, \dots, q\} \textit{ such that } i \neq p \tag{11.6}$$

and if

$$s \in [0, T - L] \text{ and } [s, s + L] \cap J_i = \emptyset, \ i = 1, \dots, q, \tag{11.7}$$

then there is $s_1 \in [s, s + L - l]$ *such that at least one of properties (i), (ii) of Theorem 3.1 holds.*

Proof. By Lemma 8.2 there is $h > l_0$ such that property (C2) holds (see Section 9). Let $v \in W^{2,1}_{loc}([0, \infty))$ satisfy (11.1) and (11.2) and let $l > h$. By Proposition 7.2 there exists $L > l$ such that the following property holds:

(C4) For each $\gamma > 0$, $T > L$ satisfying (11.3) there are a finite number of closed intervals $J_1, \dots, J_q$ such that (11.4), (11.5) and (11.6) hold and if a number s satisfies (11.7), then there are $s_1 \in [s, s + L - l]$ and $w \in \sigma(f)$ such that

$$|X_v(s_1 + t) - X_w(t)| \le \epsilon/4 \text{ for all } t \in [0, l]. \tag{11.8}$$

Now assume that $\gamma > 0$ and $T > L$ satisfy (11.3). Let $J_1, \dots, J_q$ be a finite number of closed intervals as guaranteed by property (C4).

To complete the proof of the proposition we only need to show that for each s satisfying (11.7) there is $s_1 \in [s, s + L - l]$ such that at least one of properties (i) and (ii) of Theorem 3.1 holds.

Assume that s satisfies (11.7). By property (C4) there are

$$s_1 \in [s, s + L - l] \text{ and } w \in \sigma(f) \tag{11.9}$$

such that (11.8) is valid.

If $w \in \sigma(f, h)$, then property (i) of Theorem 3.1 holds.

Assume that $w \notin \sigma(f, h)$. Then w is not a constant and

$$\tau(w) > h. \tag{11.10}$$

Let

$$\tau \in [s_1, s_1 + l - h]. \tag{11.11}$$

By (11.9), (11.10) and property (C2) (see section 9) there exist

$$t_1 \in [\tau - s_1, \tau - s_1 + h - l_0] \tag{11.12}$$

and

$$\xi \in \sigma(f, 0) \tag{11.13}$$

such that

$$|X_w(t_1 + t) - (\xi(0), 0)| \le \epsilon/4, \ t \in [0, l_0]. \tag{11.14}$$

Set

$$\tau_1 = t_1 + s_1. \tag{11.15}$$

Relations (11.15) and (11.12) imply that

$$\begin{aligned}\tau &= \tau - s_1 + s_1 \le t_1 + s_1 \le \tau_1 \le \tau - s_1 + h - l_0 + s_1 \\ &= \tau + h - l_0,\end{aligned} \tag{11.2}$$

$$\tau_1 \in [\tau, \tau + h - l_0]. \tag{11.16}$$

Let

$$t \in [0, l_0]. \tag{11.17}$$

We show that $|X_v(\tau_1 + t) - (\xi(0), 0)| \le \epsilon$. In view of (11.16) and (11.11)

$$\tau_1 + t \in [\tau, \tau + h] \subset [s_1, s_1 + l]. \tag{11.18}$$

Combined with (11.8) this inclusion implies that

$$|X_v(\tau_1 + t) - X_w(\tau_1 + t - s_1)| \le \epsilon/4. \tag{11.19}$$

By (11.15)

$$\tau_1 + t - s_1 = t_1 + t.$$

Combined with (11.14) and (11.17) this equality implies that

$$|X_w(\tau_1 + t - s_1) - (\xi(0), 0)| \le \epsilon/4.$$

Together with (11.19) this inequality implies that

$$|X_v(\tau_1 + t) - (\xi(0), 0)| \le \epsilon/2. \tag{11.20}$$

We have shown that for each τ satisfying (11.11) there are τ_1 satisfying (11.16) and $\xi \in \sigma(f, 0)$ such that (11.20) holds for all $t \in [0, l_0]$. Thus property (ii) of Theorem 3.1 holds. Proposition 11.1 is proved. □

Proof of Theorem 3.3. By (3.4) and (3.5)

$$\begin{aligned}\limsup_{T\to\infty} T^{-1}\Gamma^f(0, T, v) &= \limsup_{T\to\infty} T^{-1}[I^f(0, T, v) - \mu(f)T - \pi^f(X_v(0)) + \pi^f(X_f(T))] \\ &= \limsup_{T\to\infty} T^{-1} I^f(0, T, v) - \mu(f) = 0\end{aligned}$$

and

$$\limsup_{T\to\infty} T^{-1}\Gamma^f(0, T, v) = 0. \tag{11.21}$$

Let $h > l_0$ be as guaranteed by Proposition 11.1. Let $l > h$. By the choice of h and Proposition 11.1 there is $L > l$ such that Assertion (A1) of Proposition 11.1 holds.

Let $\gamma > 0$. By (11.21) there is $T_\gamma > L$ such that for each $T \ge T_\gamma$,

$$T^{-1}\Gamma^f(0, T, v) \le \gamma/4. \tag{11.22}$$

Let $T \ge T_\gamma$. Then (11.22) holds and the assertion of the theorem follows from assertion (A1) of Proposition 11.1. □

Proof of Theorem 3.4. By (3.4) and (3.10)

$$\lim_{i\to\infty} T_i^{-1}\Gamma^f(0,T_i,v) = \lim_{i\to\infty} T_i^{-1}[I^f(0,T_i,v) - \mu(f)T_i - \pi^f(X_v(0)) + \pi^f(X_v(T_i))]$$
$$= \lim_{i\to\infty} T_i^{-1}[I^f(0,T_i,v) - \mu(f)T_i] = 0. \tag{11.23}$$

Let $h > l_0$ be as guaranteed by Proposition 11.1. Let $l > h$. By the choice of h and Proposition 11.1 there is $L > h$ such that Assertion (A1) of Proposition 11.1 holds.

Let $\gamma > 0$. By (11.23) there is a natural number i_γ such that $T_{i_\gamma} > L$ and

$$T_i^{-1}\Gamma^f(0,T_i,v) \le \gamma/4 \text{ for all integers } i \ge i_\gamma. \tag{11.24}$$

Let $i \ge i_\gamma$ be an integer. Then (11.24) is valid and the assertion of the theorem follows from assertion (A1) of Proposition 11.1. □

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