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THE LEGENDRE TRANSFORM OF CORRESPONDENCES

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Dedicated to Terry Rockafellar on his seventieth birthday.

Abstract: We introduce a notion of conjugacy for relations between a normed vector space and the real field. When the correspondence is a Legendre function, one recovers the Legendre transform.

Key words: *conjugacy, correspondence, duality, jet, Legendre transform, multifunction, nonsmooth analysis*

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1 Introduction

In many cases of interest in mathematics, set-valued functions have to replace single-valued functions. Since a problem depending on a parameter w does not always have a unique solution, one is led to the study of the multifunction S whose value at w is the set $S(w)$ of solutions of the problem for the parameter w. Of course, such a fact makes the study more difficult and mathematicians have just started to devise tools to deal with such questions. They often play a role in practical problems. For instance, a consumer wishing to buy some good has often the choice between different proposals with different prices; also, a firm may have different processes of production whose costs may range in some interval. Therefore, multifunctions with values in $\mathbb{R}$ have an interest.

In nonlinear analysis, the search of critical (or stationary) points often replaces the search of minimizers or maximizers. In doing so, one replaces the optimization problem with an equation or an inclusion. Reciprocity or duality relationships may thus bring a precious insight on the models of many nonlinear phenomena. A systematic treatment has been proposed by Ekeland [9], [10]. It departs from other more classical generalized convexity approaches to duality ([20], [25], [30], [32], [33] and their references). Still his approach enables to treat convex duality, Toland's duality, duality in the calculus of variations in a single framework. In [26] we introduced a slight extension of the Ekeland transform in order to obtain a scheme which fully encompasses the Legendre-Fenchel conjugacy for lower semicontinuous convex functions; we applied it to the study of characteristics associated with an Hamilton-Jacobi equation, relating the recent and beautiful global convex theory of Rockafellar and Wolenski ([31]) to the classical approach.

In the present paper we extend this scheme to multifunctions from a normed vector space (n.v.s.) X to $\mathbb{R}$. While in [9], [10] Ekeland studied the case of the transform of a function or a

Lagrangian submanifold, here we start with an arbitrary multifunction. We use elementary tools from set-valued analysis and nonsmooth analysis, while most arguments of [9] rely on tools from differential topology. In [26] the attention is focused on the case the Ekeland transform of a function is again a function; in general, the Ekeland transform of a function is a multifunction. Although here we do not pursue such an aim, we give conditions ensuring that the transform of a multifunction is a function. Our motivation lies in the fact that duality is particularly useful when the dual problem is simpler than the primal problem.

As in [26], we endeavour to extend the transform in order to fully encompass the case of the Fenchel transform. Thus, instead of looking for a restricted class of well behaved convex functions as in [28, Chapter 26], we consider a framework which encompasses all lower semicontinuous convex functions and classical Legendre functions. As mentioned above, it even goes beyond functions. We also evoke some extensions of classical geometrical concepts to nonsmooth situations. These situations occur in important fields such as mathematical programming. Among these concepts is a one-sided version of the notion of jet called the subjet or hypergraph of the multifunction and a notion of contact subset. It is shown that the hypergraph of a multifunction is a contact subset, in a sense introduced in section 2 which is again a one-sided version of the notion of Lagrangian submanifold in the sense of [9].

Because the Ekeland transform, as defined in [9], [10] does not preserve the contact structure, we introduce a variant of it. However, in order to preserve the notion of conjugate, we adopt a definition which is compatible with the notions of [9], [10], [26]. The price to be paid is that the correspondence between the hypergraph of the conjugate and the transform of the hypergraph is not as natural as one would like. This surprising fact concerning the change of signs may be related to the way the Fenchel conjugate of a convex function is defined: it is a tangential way, through affine functions whose graphs are below the graph of the function or are tangent to it.

We devote some attention to the subjet of a convex function on a Hilbert space, bringing some additional information about its structure of Lipschitzian submanifold as displayed in [29].

In [9] and [10] a duality is defined in terms of perturbations for the problems of looking for critical points. It is applied to various problems, such as eigenvalue problems, Toland's duality and problems of the calculus of variations. We do not deal with such applications here. We refer to these references for that and to [27] for an application to the Clarke duality. We hope that the development of set-valued analysis will lead to some more applications.

2 Contact Subsets and Liouville Subsets

In the sequel X and Y are two n.v.s. in duality via a coupling function $\langle \cdot, \cdot \rangle$. If $F : S \rightrightarrows T$ is a correspondence (or multifunction, or multimapping or relation) between two sets we denote its graph by $G(F) := \{(s,t) \in S \times T : t \in F(s)\}$. Although the identification of F with $G(F)$ would not cause ambiguities in what we have in view, we avoid it here. The inverse of F is the correspondence F^{-1} characterized by $G(F^{-1}) := \{(t,s) \in T \times S : t \in F(s)\}$.

We use the concept of *tangent cone* $T(E,x)$ (or contingent cone) to a subset E of a n.v.s. X at $x \in \operatorname{cl} E$ (the closure of E): it is the set of $v \in X$ such that there is a sequence $((t_n, v_n)) \to (0_+, v)$ in $\mathbb{R} \times X$ satisfying $x + t_n v_n \in E$ for each $n \in \mathbb{N}$. It is known (see [21], [30]) that when $E \subset X \times \mathbb{R}$ is the epigraph $E_f := \{(x,r) \in X \times \mathbb{R} : r \geq f(x)\}$ of a function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ finite at x, the tangent cone to $E := E_f$ at $x_f := (x, f(x))$ is the epigraph of the function $df(x,\cdot)$ given by

$$df(x,u) := \liminf_{(t,v)\to(0_+,u)} \frac{1}{t}\left(f(x+tv)-f(x)\right).$$

The *normal cone* $N(E,x)$ to E at x is the polar cone to the tangent cone:

$$N(E,x) := (T(E,x))^0 := \{y \in Y : \langle v,y\rangle \leq 0 \ \forall v \in T(E,x)\}.$$

The *coderivative* at $(x,w) \in G(F)$ of a multimapping $F : X \rightrightarrows W$ between two n.v.s. paired with Y and Z respectively is the multimapping $D^*F(x,w) : Z \rightrightarrows Y$ given by

$$y \in D^*F(x,w)(z) \Leftrightarrow (y,-z) \in N(G(F),(x,w)).$$

If $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is finite at x, the (*directional* or *Dini-Hadamard, Hadamard* or *contingent) subdifferential* $\partial f(x)$ of f at x is given by

$$y \in \partial f(x) \Leftrightarrow (y,-1) \in N(E_f, x_f) \Leftrightarrow \forall u \in X \quad \langle y,u\rangle \leq df(x,u),$$

where E_f is the epigraph of f and $x_f := (x, f(x))$. Thus $\partial f(x) = D^*E_f(x_f)(1)$ with $x_f := (x, f(x))$. Other subdifferentials could be used, with the related concepts of normal cones. In [10] the Clarke subdifferential is used in the locally Lipschitzian case; several results below (in particular Lemma 5 and Proposition 6) would not be valid with that choice. The same can be said for the limiting subdifferential; however, since Definitions 1, 3 and 4 can be reformulated in terms of normal cones, the limiting subdifferential could be used as a variant of the concepts introduced below.

The following concept seems to be new, but it is closely related to classical notions. Recall that if $f : U \to V$ is a differentiable mapping between two open subsets of n.v.s. its first order jet is the set

$$J^1 f := \{(x, f'(x), f(x)) : x \in U\}.$$

Such a notion can be extended to manifolds and to higher order jets (see [6, § 12], [19, Chapter 4] for instance). In nonsmooth analysis, a one-sided version is in order (see [14], [23] for instance).

Definition 1 *The subjet J^-F (or hypergraph) of a multifunction $F : X \rightrightarrows \mathbb{R}$ is the correspondence $J^-F : X \times Y \rightrightarrows \mathbb{R}$ given by*

$$J^-F := \{(x,y,z) \in X \times Y \times \mathbb{R} : (x,z) \in G(F),\ y \in D^*F(x,z)(1)\}.$$

Let us relate the subjet J^-f of a function $f : X \to \overline{\mathbb{R}}$ to the subjet of its associated graph $G_f := G(f)$ or epigraph E_f. Although the first equivalence below is simple and worth of notice, we have not been able to find it in the literature.

Proposition 2 *Given a lower semicontinuous (l.s.c.) function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty,\infty\}$ with graph G_f and epigraph E_f and $x \in X$ such that $z := f(x) \in \mathbb{R}$, one has*

$$(x,y,z) \in J^-G_f \Leftrightarrow (x,y,z) \in J^-E_f \Leftrightarrow y \in \partial f(x).$$

Proof. The last equivalence is just a rewriting of the equivalence of the relations $(y,-1) \in N(E_f,(x,z))$ and $y \in \partial f(x)$. The first equivalence amounts to $(y,-1) \in N(G_f,(x,z))$ if, and only if, $(y,-1) \in N(E_f,(x,z))$.

Since G_f is contained in E_f, one has $N(E_f,(x,z)) \subset N(G_f,(x,z))$. Thus it remains to prove that if $y \in X^*$ is such that $(y,-1) \in N(G_f,(x,z))$ then one has $(y,-1) \in N(E_f,(x,z))$. Let $(u,r) \in T(E_f,(x,z))$. Let us first observe that setting $p := df(x,u) := \inf\{s \in \mathbb{R} : (u,s) \in T(E_f,(x,z))\}$, we cannot have $p = -\infty$. Otherwise we can find a sequence $((t_n,u_n)) \to (0_+,u)$ with

$$p_n := \frac{1}{t_n}\left(f(x+t_nu_n) - f(x)\right) \to -\infty.$$

Without loss of generality we may assume that $q_n := -p_n > 0$ for each $n \in \mathbb{N}$ and since f is l.s.c. and $(x+t_nu_n) \to x$ we have $0 \geq \limsup_n \left(f(x+t_nu_n) - f(x)\right) \geq \liminf_n \left(f(x+t_nu_n) - f(x)\right) \geq 0$, hence $(t_np_n) \to 0$. Then $(q_n^{-1}u_n) \to 0$ and

$$(0,-1) = \lim_n \frac{1}{t_nq_n}\left[(x+t_nq_n(q_n^{-1}u_n), f(x+t_nu_n)) - (x, f(x))\right] \in T(G_f,(x,z))$$

and we get the contradiction $1 = \langle (y,-1),(0,-1)\rangle \leq 0$. Thus $p > -\infty$ and since $p \leq r$, p is finite. Since $(u,p) \in T(G_f,(x,z))$ as $\left(t_n^{-1}\left[(x+t_nu_n, f(x+t_nu_n)) - (x,f(x))\right]\right) \to (u,p)$, we have

$$\langle (y,-1),(u,r)\rangle = \langle y,u\rangle - r \leq \langle y,u\rangle - p = \langle (y,-1),(u,p)\rangle \leq 0$$

and, since (u,r) is arbitrary in $T(E_f,(x,z))$, we get $(y,-1) \in N(E_f,(x,z))$. □

The following concepts are nonsmooth and unilateral versions of the notion of Lagrangian submanifold in the sense of [9, Def. 1.1]. As for the latter notion, they use the differential 1-form ω defined as follows:

$$\omega : ((x,y,z);(u,v,w)) \mapsto w - \langle y,u\rangle.$$

Such a differential form could be defined on the product $T^*V \times \mathbb{R}$ of the cotangent bundle T^*V of a differentiable manifold V with $\mathbb{R}$ by using the Liouville 1-form θ on T^*V (given by $\theta_{(x,y)}(u,v) = \langle y,u\rangle$ when V is an open subset of a n.v.s.) and the 1-form dt on $\mathbb{R}$ and setting $\omega = dt - \theta$, with an obvious abuse of notation consisting in identifying one-forms with their pull-backs on $T^*V \times \mathbb{R}$. Our choice of the terminology is justified in Example 1 below.

Definition 3 *A subset M of $X \times Y \times \mathbb{R}$ is said to be a contact subset (resp. a co-contact subset) of $X \times Y \times \mathbb{R}$ if for any $(x,y,z) \in M$ and for any $(u,v,w) \in T(M,(x,y,z))$ one has $w - \langle y,u\rangle \geq 0$ (resp. $w - \langle y,u\rangle \leq 0$).*

Definition 4 *Let M be a subset of $X \times Y \times \mathbb{R}$ and let P be its projection on $X \times \mathbb{R}$. The set M will be called a Liouville subset of $X \times Y \times \mathbb{R}$ if for any $(x,y,z) \in M$ and for any $(u,w) \in T(P,(x,z))$ one has $w - \langle y,u\rangle \geq 0$.*

Note that M is a contact subset of $X \times Y \times \mathbb{R}$ if, and only if, for any $(x,y,z) \in M$ one has $(y,0,-1) \in N(M,(x,y,z))$ and that M is a Liouville subset of $X \times Y \times \mathbb{R}$ if, and only if, for any $(x,y,z) \in M$ one has $(y,-1) \in N(P,(x,z))$. The following statement contributes to the clarification of the relationships between the two notions.

Lemma 5 (a) *Any Liouville subset of $X \times Y \times \mathbb{R}$ is a contact subset.*

(b) *M is a Liouville subset if, and only if, M is contained in the subjet J^-P of P.*

(c) *If M is a contact subset of $X \times Y \times \mathbb{R}$ and if for every $(x, y, z) \in M$ the projection mapping $D\pi(x, y, z) : (u, v, w) \mapsto (u, w)$ maps $T(M, (x, y, z))$ onto $T(P, (x, z))$, then M is a Liouville subset of $X \times Y \times \mathbb{R}$.*

(d) *X is finite dimensional, if M is a contact subset and if for any $(x, y, z) \in M$ there exist $c \geq 0$, a neighborhood V of (x, z) and a mapping $h : V \cap P \to Y$ such that $\|h(x', z') - y\| \leq c \|(x', z') - (x, z)\|$ and $(x', h(x', z'), z') \in M$ for each $(x', z') \in V \cap P$, then M is a Liouville subset.*

Proof. (a) Let $\pi : M \to P$ be the projection given by $\pi(x, y, z) := (x, z)$ and let $P := \pi(M)$. Since for any $(x, y, z) \in M$ and for any $(u, v, w) \in T(M, (x, y, z))$ one has $(u, w) = \pi'(x, y, z)(u, v, w) \in T(P, (x, z))$, we see that any Liouville subset is a contact subset.

(b) The observation preceding the statement shows that M is a Liouville subset if, and only if, for every $(x, y, z) \in M$ one has $(y, -1) \in N(P, (x, z))$, or, in other terms, $y \in D^*P(x, z)(1)$ or, by Definition 1, $(x, y, z) \in J^- P$.

Assertion (c) is an obvious consequence of the definitions.

It remains to prove the last assertion. In fact, using the assumption, one can show that for any $(x, y, z) \in M$, $(u, w) \in T(P, (x, z))$ there exists some $v \in Y$ such that $(u, v, w) \in T(M, (x, y, z))$: it suffices to take for v a limit point of a sequence $(t_n^{-1}(h(x + t_n u_n, z + t_n w_n) - y))$ where $(t_n) \to 0_+$, $(u_n) \to u$, $(w_n) \to w$ with $(x + t_n u_n, z + t_n w_n) \in P$ for each n. Such a limit point exist by the compactness of balls in Y. $\square$

Because

$$T(A, x) \subset T(B, x) \quad \text{for } A \subset B,\ x \in \text{cl}(A), \tag{2.1}$$

$$T(A \cup B, x) = T(A, x) \cup T(B, x) \text{ for } x \in \text{cl}(A \cup B), \tag{2.2}$$

we observe that the intersection of a family of Liouville subsets is a Liouville subset and the union of a finite family of Liouville subsets is a Liouville subset. Similar results hold for contact subsets. Such simple assertions would not be valid in the smaller class of Liouville submanifolds without transversality or regularity conditions.

A fundamental example is given in the following statement. Note that here we cannot apply Lemma 5 b) because we do not know whether $G(F)$ is the projection of $J^- F$ (in general this is not the case, even when F is a function).

Proposition 6 *The subjet $J^- F$ of a correspondence $F : X \rightrightarrows \mathbb{R}$ is a Liouville subset of $X \times Y \times \mathbb{R}$.*

Proof. Clearly, the projection P of $M := J^- F$ is contained in $G(F)$. Let $(x, y, z) \in M$ and let $(u, w) \in T(P, (x, z)) \subset T(G(F), (x, z))$. The construction of M ensures that $(y, -1) \in N(G(F), (x, z))$. Thus we have

$$\langle u, y \rangle - w = \langle (u, w), (y, -1) \rangle \leq 0,$$

so that M is a Liouville subset of $X \times Y \times \mathbb{R}$. $\square$

Another example of contact subset deals with a relatively smooth situation, close to the one considered in [9].

Example 1. Let M be a subset of $X \times Y \times \mathbb{R}$ which is smooth, i.e. such that for each $m \in M$ the tangent cone $T(M, m)$ is a vector subspace (this obviously occurs when M is a differentiable submanifold). Then M is a contact subset iff for each $m \in M$ the linear form $\omega_m(\cdot)$ is null on $T(M, m)$. In [9] submanifolds satisfying that property are called Lagrangian submanifolds; however this terminology is not in accordance with the usual one in symplectic geometry (see [1]-[4], [8], [11], [18], [34]-[38] for instance), so that we substitute a new one for which no ambiguity may (hopefully) occur. □

The following example deals with a situation familiar in mathematical programming in which constraints are defined by equalities and inequalities. It is inspired by [9, Lemma 3.1] in which a finite number of equalities are considered and a qualification condition (linear independence of the derivatives) is assumed. Here M is a nonsmooth subset.

Example 2. Let X_0 be an open subset of X, let $Y = X^*$ and let $f : X_0 \to \mathbb{R}$, $g : X_0 \to Z$ be mappings of class C^1, where Z is some n.v.s. Given a convex subset C of Z, let

$$M := \{(x, f'(x) + z^* \circ g'(x), f(x)) : x \in g^{-1}(C),\ z^* \in N(C, g(x))\}.$$

Let us check that M is a Liouville subset of $X \times Y \times \mathbb{R}$. We first observe that the projection P of M on $X \times \mathbb{R}$ is just the graph of the restriction of f to $g^{-1}(C)$. Given $m := (x, y, z) \in M$ and $(u, w) \in T(P, (x, z))$ we have, for some $z^* \in N(C, g(x))$, $y = f'(x) + z^* \circ g'(x)$, hence

$$w - \langle y, u\rangle = f'(x)u - \langle f'(x) + z^* \circ g'(x), u\rangle = -\langle z^*, g'(x)u\rangle \geq 0$$

since $g'(x)u \in T(C, g(x))$ as $u \in T(g^{-1}(C), x)$. In the case of a classical mathematical programming problem, one has $X = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $C = -\mathbb{R}^m_+$ and $z^* \in N(C, g(x))$ if, and only if its components z_i^* are nonnegative and such that $z_i^* g_i(x) = 0$. □

Example 3. Let X, W be n.v.s., let X_0 be an open subset of X and let $h : X_0 \to W$ be a differentiable mapping. Given a Liouville subset N of $W \times W^* \times \mathbb{R}$, let

$$M := \{(x, x^*, r) \in X \times X^* \times \mathbb{R} : \exists w^* \in W^*,\ x^* = w^* \circ h'(x),\ (h(x), w^*, r) \in N\}.$$

Then M is a Liouville subset of $X \times X^* \times \mathbb{R}$. In fact, given $(u, s) \in T(P, (x, r))$, where P is the projection of M on $X \times \mathbb{R}$, we have $(h'(x)u, s) \in T(Q, (h(x), r))$, where Q is the projection of N on $W \times \mathbb{R}$, as easily checked. Therefore $s - \langle x^*, u\rangle = s - \langle h'(x)^T(w^*), u\rangle = s - \langle w^*, h'(x)u\rangle \geq 0$.

This example encompasses the preceding one, as one can see by taking $W := X \times Z$, $h(x) := (x, g(x))$,

$$N := \{(x, z, x^*, z^*, r) :\ x^* = f'(x),\ r = f(x),\ z \in C,\ z^* \in N(C, z)\},$$

which is a Liouville subset of $X \times Z \times X^* \times Z^* \times \mathbb{R}$. □

Example 4. Let X be a Hilbert space and let $f : X \to \mathbb{R}$ be a closed proper convex function. Then, the subjet $J^- f$ in the sense of convex analysis is a Liouville subset of $X \times Y \times \mathbb{R}$ since the Fenchel-Moreau subdifferential coincides with the subdifferential we use here, so that Proposition 6 applies. We will study this example in more details in the next section.

3 The Subjet of a Convex Function

The subjet of a nonsmooth function is not a smooth submanifold in general. However, in some special cases it enjoys a pleasant structure. We devote the present section to the case of the subjet $M := J^- f$ of a lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{\infty\}$ which is proper (i.e. which takes at least one finite value). Some other special cases, such as the class of piecewise smooth functions, would deserve some attention. We refer to [9, Sections 1,2] for a study of conditions ensuring that conversely a Lagrangian submanifold is (at least locally) a subjet. In the present section X is a Hilbert space identified with its dual space. When X is finite dimensional, it is shown in [31] (see also [29]) that this subjet has a Lipschitzian bijective parametrization whose inverse is also Lipschitzian. Our purpose here is to show that M can be given the structure of a Lipschitzian submanifold of $X \times X \times \mathbb{R}$ in the following (stronger) sense which extends the finite dimensional situation studied in [12] and [17].

Definition 7 *A subset M of a normed vector space Z is a Lipschitzian submanifold of Z if for each $m \in M$ there exists a decomposition of Z into a direct sum $Z = Z_1 \times Z_2$ of closed vector subspaces, neighborhoods U of m in Z, $V = V_1 \times V_2$ of $(0,0)$ in $Z_1 \times Z_2$ and a lipeomorphism (i.e. a Lipschitzian bijection with Lipschitzian inverse called a chart) $\varphi : U \to V$ such that $\varphi(U \cap M) = V_1 \times \{0\}$.*

The mapping $v_1 \to \varphi^{-1}(v_1, 0)$ then yields a local Lipschitzian parametrization of M with Lipschitzian inverse. On the other hand, one cannot assert that a Lipschitzian parametrization yields a chart φ as above since the inverse mapping theorem is not available in this context (or hardly available through some substitutes). Thus, the lack of an inverse mapping theorem for Lipschitzian mappings justifies the more precise definition given above. It also gives an easy way to extend a locally Lipschitzian function on M to a neighborhood of M. It may also be helpful for dealing with notions of measure zero subsets.

The following example shows that if M is locally the graph of a Lipschitzian mapping, then M is a Lipschitzian submanifold and not just the image of a parametrization.

Example 5. Let G be the graph of a Lipschitzian mapping g from an open subset X_0 of X to some normed space W. Then G is a Lipschitzian submanifold of $X \times W$, the lipeomorphism $\varphi : X_0 \times W \to X_0 \times W$ being given by $\varphi(u, w) = (u, w - g(u))$. Its inverse is given by $\varphi^{-1}(u', w') = (u', w' + g(u'))$ and φ maps G onto $X_0 \times \{0\}$. □

Example 6. By the preceding example, the one-jet $J^1 f := \{(x, f'(x), f(x)) : x \in X\}$ of a $C^{1,1}$ map $f : X \to W$ between two n.v.s. is also a Lipschitzian submanifold since by definition of a $C^{1,1}$ map as a differentiable function whose derivative is Lipschitzian, $g := (f', f)$ is Lipschitzian and $J^1 f$ is just the graph of g. □

This example is extended to the subjet of a convex function in the next proposition.

Note that in the preceding definition it suffices to require that the mappings φ and φ^{-1} are locally Lipschitzian since the condition is local. In the case of the subjet of a convex function we have an intermediate situation: φ and φ^{-1} are boundedly Lipschitzian, i.e. their restrictions to bounded sets are Lipschitzian; moreover a single chart suffices. Since X is not assumed to be finite dimensional, the proof of [31, 6.5] which relies on the Rademacher theorem is no more valid.

Proposition 8 *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous convex function on a Hilbert space X and let M be its subjet (or characteristic manifold). Then M has a structure of Lipschitzian submanifold of $X \times X \times \mathbb{R}$. Moreover, a single chart suffices.*

Proof. For a convex function, the contingent subdifferential coincides with the Fenchel-Moreau subdifferential so that

$$M = \{(x, y, f(x)) : y \in \partial f(x)\}.$$

Set $T := \partial f$, and let $\phi : X \times X \to X \times X$ and $\psi : X \times X \times \mathbb{R} \to X \times X \times \mathbb{R}$ be the linear isomorphisms given by $\phi(x, y) := (x + y, -x + y)$, $\psi := \phi \times I$. Then one shows as in [30, 12.15] that $\phi(G(T))$ is the graph of the mapping $u \mapsto (P(u), Q(u))$ from X into $X \times X \times \mathbb{R}$, where $P := (I + \partial f)^{-1}$ and $Q := I - P$; in fact

$$\begin{aligned}(u,v) \in \phi(G(T)) &\Leftrightarrow \exists (x,y) \in G(T): \ u = x + y, \ v = -x + y \\ &\Leftrightarrow \exists (x,y) \in G(T): \ u \in (I+T)(x), \ v = u - 2x \in (I - 2(I+T)^{-1})(u)\end{aligned}$$

The maps P,Q are known to be nonexpansive, $P(u) = \arg\min_x \left(f(x) + \frac{1}{2} \|x - u\|^2 \right)$ and thus $\phi(G(T))$ is the graph of the Lipschitzian map $S := I - 2(I+T)^{-1}$. Setting $R(u) := f(P(u))$ we get as in [31] that $\psi(M)$ is the graph of the mapping $g : u \mapsto (S(u), R(u))$ from X to $X \times \mathbb{R}$. Let us show that $R : u \mapsto f(P(u))$ is boundedly Lipschitzian. Given u, u' in the ball $B(0,r)$ with center 0 and radius r, we have $\|P(u)\| \leq a + r$, $\|P(u')\| \leq a + r$ with $a := \|P(0)\|$, hence

$$\begin{aligned}R(u') - R(u) &\leq \left(f(P(u)) + \frac{1}{2} \|P(u) - u'\|^2 \right) - \frac{1}{2} \|P(u') - u'\|^2 - f(P(u)) \\ &\leq \frac{1}{2} (P(u) - P(u') \mid P(u) - u' + P(u') - u') \\ &\leq \|u - u'\| \left(\|P(u)\| + \|P(u')\| + 2\|u'\| \right) \leq (4r + 2a) \|u - u'\| .\end{aligned}$$

Interchanging the roles of u and u', we get that R is Lipschitzian on $B(0,r)$ with rate $2(2r + a)$.

Since the graph G of the Lipschitzian mapping $g := (S, R)$ from X to $W := X \times \mathbb{R}$ is a Lipschitzian submanifold of $X \times W$, by the preceding example it is mapped by the lipeomorphism $\theta : X \times W \to X \times W$ given by $\theta(u,w) = (u, w - g(u))$ onto $X \times \{0\}$. Then, taking $\varphi := \theta \circ \psi$, we get the required chart from $X \times X \times \mathbb{R}$ onto $X \times W$ sending M onto $X \times \{0\}$. □

The preceding result can be extended to *paraconvex* (or semiconvex) functions, i.e. functions $f : X \to \mathbb{R} \cup \{\infty\}$ such that there exists some $c \in \mathbb{R}_+$ for which $g := f + \frac{1}{2} c \|\cdot\|^2$ is convex.

Proposition 9 *The subjet $M := J^- f$ of a proper lower semicontinuous paraconvex function $f : X \to \mathbb{R} \cup \{\infty\}$ on a Hilbert space X is a Lipschitzian submanifold of $X \times X \times \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}_+$ be such that $g := f + \frac{1}{2} c \|\cdot\|^2$ is convex. Then, for each $x \in X$, one has $\partial f(x) = \partial g(x) - cx$. Therefore, the mapping $\eta : X \times X \times \mathbb{R} \to X \times X \times \mathbb{R}$ given by $\eta(x,y,z) = (x, y + cx, z + \frac{1}{2} c \|x\|^2)$ applies $J^- f$ onto $J^- g$ and has for inverse the mapping given by $\eta^{-1}(x', y', z') = (x', y' - cx', z' - \frac{1}{2} c \|x'\|^2)$ which is also boundedly Lipschitzian. It follows that $J^- f$ is also a Lipschitzian submanifold. □

4 The Ekeland Transform of a Relation

We first define the Ekeland mapping as in [9] and a variant of it. We will apply it to correspondences and justify the choice of signs later on.

Definition 10 *The Ekeland mapping is the mapping $E : X \times Y \times \mathbb{R} \to Y \times X \times \mathbb{R}$ given by*

$$E(x,y,z) := (y, x, \langle x, y \rangle - z).$$

The tilted Ekeland mapping is the mapping $\widetilde{E} : X \times Y \times \mathbb{R} \to Y \times X \times \mathbb{R}$ given by

$$\widetilde{E}(x,y,z) := (y, -x, z - \langle x, y \rangle).$$

Note that this last mapping involves the symplectic operator

$$J := \begin{bmatrix} 0 & I_Y \\ -I_X & 0 \end{bmatrix}.$$

Remark. The preceding definition can be extended to the case of a partial Ekeland transform when two variables are present but the transform is performed on one of them only. In fact it can be set in the framework of vector bundles. Given a vector bundle $p : V \to B$ with base B and the dual bundle $p^* : V^* \to B$, one can define E as a vector bundle morphism from $V \times_B V^* \times \mathbb{R}$ to $V^* \times_B V \times \mathbb{R}$, assuming that the fibers are reflexive Banach spaces or n.v.s. in duality (here $V \times_B V^*$ stands for $\{(v, v^*) \in V \times V^* : p(v) = p^*(v^*)\}$); in a bundle trivialization it reads as

$$(b, x, y, z) \mapsto (b, y, x, \langle x, y\rangle - z).$$

However, it is not our purpose to deal with such extensions here. □

On the difference with E, the tilted Ekeland mapping $\widetilde{E}$ is no more an involution; however (with an obvious abuse of notation) $\widetilde{E} \circ \widetilde{E} = (-I_{X\times Y}) \times I_{\mathbb{R}}$ and the inverse $\widetilde{E}^{-1}$ of $\widetilde{E}$ has a similar expression: $\widetilde{E}^{-1}(y', x', z') = (-x', y', z' - \langle x', y'\rangle)$. The presence of the minus signs from E to $\widetilde{E}$ evokes the passage from the concave conjugate of a function to its convex conjugate.

The Ekeland mapping satisfies the following striking property (Proposition 11) in which ω and ω' are the differential 1-forms on $X \times Y \times \mathbb{R}$ and $Y \times X \times \mathbb{R}$ respectively given by

$$\omega : ((x, y, z); (u, v, w)) \mapsto w - \langle y, u\rangle,$$
$$\omega' : ((y', x', z'); (v', u', w')) \mapsto w' - \langle x', v'\rangle.$$

This property is an analogue of [9, Thm 2.2] in which E is used. By analogy with the notion of contact manifold, we call contact space a n.v.s. endowed with a differential 1-form such as the pair $(X \times Y \times \mathbb{R}, \omega)$.

Proposition 11 *The tilted Ekeland transform is an isomorphism between the contact spaces $(X \times Y \times \mathbb{R}, \omega)$ and $(Y \times X \times \mathbb{R}, \omega')$ in the sense that it is a C^∞-diffeomorphism such that the pull-back $\widetilde{E}^*(\omega')$ of ω' by $\widetilde{E}$ is ω. Moreover $E^*(\omega') = -\omega$.*

Proof. Clearly $\widetilde{E}$ is a C^∞-diffeomorphism. Moreover, for any $((x, y, z), (u, v, w)) \in (X \times Y \times \mathbb{R}) \times (X \times Y \times \mathbb{R})$ the pull-back $\widetilde{E}^*(\omega')$ of ω' by $\widetilde{E}$ is given by

$$\begin{aligned}
\widetilde{E}^*(\omega')\left((x, y, z); (u, v, w)\right) &:= \omega'\left(\widetilde{E}(x, y, z); \widetilde{E}'(x, y, z)(u, v, w)\right) \\
&= \omega'\left((y, -x, z - \langle x, y\rangle); (v, -u, w - \langle u, y\rangle - \langle x, v\rangle)\right) \\
&= (w - \langle u, y\rangle - \langle x, v\rangle) - \langle -x, v\rangle \\
&= w - \langle u, y\rangle = \omega\left((x, y, z); (u, v, w)\right).
\end{aligned}$$

The proof of the second assertion is similar and is given in [9, Thm 2.2]. □

The next statement reveals a troubling fact.

Proposition 12 *If M is a contact subset of $X \times Y \times \mathbb{R}$, then its tilted Ekeland transform $\widetilde{E}(M)$ is a contact subset of $Y \times X \times \mathbb{R}$ and its Ekeland transform $E(M)$ is a co-contact subset of $Y \times X \times \mathbb{R}$.*

Proof. Let $(y', x', z') \in \widetilde{E}(M)$, so that $(x, y, z) \in M$ for $x := -x'$, $y := y'$, $z := z' - \langle x', y' \rangle$. Let $(v', u', w') \in T(\widetilde{E}(M), (y', x', z'))$. By definition of the tangent cone, there exist sequences $(t_n) \to 0_+$, $((u'_n, v'_n, w'_n)) \to (u', v', w')$ such that

$$(y', x', z') + t_n(v'_n, u'_n, w'_n) \in \widetilde{E}(M) \text{ for each } n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, one has

$$(x_n, y_n, z_n) := (-x' - t_n u'_n, y' + t_n v'_n, z' + t_n w'_n - \langle x' + t_n u'_n, y' + t_n v'_n \rangle) \in M$$

so that

$$(-u', v', w' - \langle x', v' \rangle - \langle u', y' \rangle) = \lim_n t_n^{-1}(x_n - x, y_n - y, z_n - z) \in T(M, (x, y, z)).$$

Since M is a contact subset, one has

$$(w' - \langle x', v' \rangle - \langle u', y' \rangle) - \langle -u', y \rangle \geq 0$$

or $w' - \langle x', v' \rangle \geq 0$. This shows that $\widetilde{E}(M)$ is a contact subset of $Y \times X \times \mathbb{R}$. The proof of the second assertion is similar. □

The preceding result gives some incentive to adopt the following definition. It may appear as intricate. In fact we do not know whether $E(J^- F)$ is an hypergraph nor a Liouville subset, so that in general we cannot find a multifunction $G : Y \rightrightarrows \mathbb{R}$ whose hypergraph is $E(J^- F)$.

Definition 13 *The Ekeland transform of a multifunction $F : X \rightrightarrows \mathbb{R}$ is the multifunction $F^E : Y \rightrightarrows \mathbb{R}$ given by*

$$F^E(y) := \{\langle x, y \rangle - z : (x, y, z) \in J^- F\}.$$

Thus, in spite of the fact that $E(J^- F)$ is not a Liouville subset, nor a contact subset, we treat it as if it were an hypergraph:

$$\begin{aligned} z' \in F^E(y) \quad &\Leftrightarrow \quad \exists x \in X : \quad z := \langle x, y \rangle - z' \in F(x),\ y \in D^* F(x, z)(1) \\ &\Leftrightarrow \quad \exists x \in X : (y, x, z') \in E(J^- F). \end{aligned}$$

Let us first check the compatibility of this definition with the one for an Ekeland function given in [26]. There, a function $f : X \to \mathbb{R} \cup \{\infty\}$ is said to be an *Ekeland function* if for any $y \in Y$ and any $x \in (\partial f)^{-1}(y)$ the number $\langle x, y \rangle - f(x)$ does not depend on the choice of x in $(\partial f)^{-1}(y)$; then $f^E(y)$ is defined as that value.

Lemma 14 *If f is an Ekeland function, and if F is the multifunction whose graph is the graph of f, i.e. $F(\cdot) = \{f(\cdot)\}$, then $F^E(\cdot) = \{f^E(\cdot)\}$.*

If F is the epigraph of a function f, i.e. $F(\cdot) = \{f(\cdot) + \mathbb{R}_+\}$, then $F^E(y) = \{\langle x, y \rangle - f(x) : x \in (\partial f)^{-1}(y)\}$.

Proof. Given $y \in Y$, we have $z' \in F^E(y)$ if, and only if, there exists some $x \in X$ such that $(y, -1) \in N(F, (x, z))$ and $z' = \langle x, y \rangle - z$, with $z = f(x)$. In view of Proposition 2 this means that $y \in \partial f(x)$ and $z' = \langle x, y \rangle - f(x)$ for some $x \in X$, hence, that $z' = f^E(y)$.

The second assertion stems from the fact that when $y \in D^* F(x, z)(1)$, i.e. $(y, -1) \in N(F, (x, z))$, then $z = f(x)$ since otherwise $(0, -1) \in T(F, (x, z))$ and one gets the contradiction $\langle (0, -1), (y, -1) \rangle > 0$. Thus one has $z' \in F^E(y)$ if, and only if, there exists $x \in X$ such that $y \in \partial f(x)$, $z' = \langle x, y \rangle - f(x)$. □

Example 7. Let F be given by $F(x) := [-\frac{1}{2}x^2, +\infty)$ for $x \in \mathbb{R}$. Then $F^E(y) = \{-\frac{1}{2}y^2\}$ for $y \in \mathbb{R}$. □

Next, we give a reversibility property. In the statement, we say that a mapping $g : U \to V$ between two metric spaces (U, d_U), (V, d_V) is *open at a linear rate* at $x \in U$ *onto its image* if there exist some neighborhood W of $g(x)$ and some $c > 0$ such that for each $w \in W \cap g(U)$ there exists some $u \in U$ such that $g(u) = w$ and $d_U(u, x) \leq c d_V(w, g(x))$. Equivalently, g is open at a linear rate at $x \in U$ onto its image if there exist some $c, r_0 > 0$ such that for each $r \in (0, r_0)$ one has

$$B(g(x), r) \cap g(U) \subset g(B(x, cr)).$$

This is a variant of a property which has been widely studied; in particular, it has been shown that it amounts to a pseudo-Lipschitz property of the inverse g^{-1} of g (see [13], [22] for instance).

Proposition 15 *Let $f : X_0 \to \mathbb{R}$ be a Fréchet differentiable function on an open subset X_0 of X. Under one of the following two conditions f is a selection of $f^{EE} := (f^E)^E$:*

(a) *for each $x_0 \in X_0$ the mapping $f' : X_0 \to Y := X^*$ is injective and open at a linear rate at x_0 onto its image;*

(b) *for each $x_0 \in X_0$ the mapping $x \mapsto (f'(x), \langle f'(x), x\rangle - f(x))$ from X_0 into $Y \times \mathbb{R}$ is open at a linear rate at x_0 onto its image.*

Proof. We first observe that assumption (a) implies assumption (b), so that we suppose assumption (b) holds. We have to show that $(x, f(x)) \in G(F^E)$ for each $x \in X_0$, where $F := f^E : Y \rightrightarrows \mathbb{R}$. In other terms, given $x \in X_0$ we have to find some $(y, z) \in F$ such that $(x, -1) \in N(F, (y, z))$ and $f(x) = \langle x, y\rangle - z$. It suffices to show that $(y, z) := (f'(x), \langle y, x\rangle - f(x))$ is such that $(x, -1) \in N(F, (y, z))$. In order to do so, we pick some $(y', z') \in T(F, (y, z))$. There exist sequences $(t_n) \to 0_+$, $(y'_n) \to y'$, $(z'_n) \to z'$ such that

$$(y, z) + t_n(y'_n, z'_n) \in F \qquad \forall n \in \mathbb{N}.$$

In view of assumption (b) there exist some $c > 0$ and a sequence (x_n) in X_0 such that $\|x_n - x\| \leq ct_n$ and

$$(y + t_n y'_n, z + t_n z'_n) = (f'(x_n), \langle f'(x_n), x_n\rangle - f(x_n))$$

Let $x'_n := t_n^{-1}(x_n - x)$, $\varepsilon_n := t_n^{-1}\left(f(x_n) - f(x) - t_n f'(x)x'_n\right)$, so that $\|x'_n\| \leq c$ for each n, $(\varepsilon_n) \to 0$ and

$$\begin{aligned} t_n z'_n &= \langle f'(x_n), x_n\rangle - f(x_n) - (\langle y, x\rangle - f(x)) \\ &= \langle y + t_n y'_n, x\rangle + \langle y + t_n y'_n, t_n x'_n\rangle - (t_n\langle f'(x), x'_n\rangle + t_n\varepsilon_n) - \langle y, x\rangle \end{aligned}$$

Thus $z'_n = \langle y'_n, x\rangle + t_n\langle y'_n, x'_n\rangle - \varepsilon_n$ and taking limits we get $z' - \langle y', x\rangle = 0$, so that $(x, -1) \in N(F, (y, z))$. □

The following local consequence incorporates [9, Prop. 2.6]. It makes a connection with the classical theory of Legendre transform. Here we say that a function $f : X_0 \to \mathbb{R}$ on an open subset X_0 of a n.v.s. X is a *classical Legendre function* if it is of class C^2 on X_0 and if its derivative f' is a diffeomorphism from X_0 onto an open subset Y_0 of Y. The assumption of the following statement are slightly less restrictive.

Corollary 16 *Let X_0 be an open subset of a reflexive Banach space X and let $f : X_0 \to \mathbb{R}$ be a function of class C^2 on X_0. Suppose that for some $x_0 \in X_0$ the Hessian $f''(x_0)$ of f at x_0 is an isomorphism from X onto $Y := X^*$. Then, there exist neighborhoods U, V of x_0 and $y_0 := f'(x_0)$ respectively such that, for $W := U \times V \times \mathbb{R}$, one has $E(J^- f \cap W) = J^- g$ for some function g of class C^2 on V called the classical Legendre transform f^L of f. Moreover one has $E(J^- g) \cap W = J^- f \cap W$ and, for each $(x, y) \in U \times V$,*

$$y = f'(x) \Leftrightarrow x = g'(y),$$
$$g''(y)v = -g'(y)^T \left(f''(x) \left(g'(y)v \right) \right) \text{ if } x = g'(y), \ v \in Y.$$

Proof. The inverse mapping theorem yields neighborhoods U, V of x_0 and $y_0 := f'(x_0)$ respectively such that $f' \mid U$ is a C^1 diffeomorphism from U to V. Thus $f \mid U$ is a classical Legendre function of class C^2. It can be shown that g given by $g(y) = \langle y, (f' \mid U)^{-1}(y) \rangle - f((f' \mid U)^{-1}(y))$ is in fact of class C^2 on V (see [26] for instance) and $g' = (f' \mid U)^{-1}$. The last formula is obtained by differentiating the left-hand side of the relation $g'(y) \circ f'(g(y)) = I_X$ for $y \in V$. □

When F is a convex multifunction (i.e. when its graph is convex) one has a surprising property: F^E is a function. Of course, this property is reminiscent of the definition of the usual Fenchel conjugate; but it is in contrast with the fact that the Ekeland transform of a function is usually a multifunction.

Lemma 17 *If F is a convex multifunction, and if $z' \in F^E(y)$ then one has $z' = \sup\{\langle x, y \rangle - z : x \in X, \ z \in F(x)\}$. Thus F^E is a function on $\operatorname{dom} F^E = \{y \in Y : \exists (x, z) \in F, (y, -1) \in N(F, (x, z))\}$.*

Moreover, if for each $x \in X$ the set $F(x)$ is closed and if f given by $f(x) := \inf F(x)$ is proper, then, for each $y \in \operatorname{dom} F^E$ one has $F^E(y) = \{f^*(y)\}$.

Proof. By what precedes, when $z' \in F^E(y)$ there exists some $x_0 \in X$ such that $z_0 := \langle x_0, y \rangle - z' \in F(x_0)$, $y \in D^* F(x_0, z_0)(1)$. This last relation means that $(y, -1) \in N(F, (x_0, z_0))$ or, in view of the convexity of F,

$$\langle x - x_0, y \rangle - (z - z_0) \leq 0 \qquad \forall (x, z) \in F,$$

or in turn $\langle x, y \rangle - z \leq \langle x_0, y \rangle - z_0 = z'$ for any $(x, z) \in F$. Taking the supremum over F, and observing that it is attained for $(x, z) = (x_0, z_0)$, we get the announced equality.

The proof of the second assertion is similar to the proof of Lemma 14. □

Example 8. Suppose X is reflexive and $Y = X^*$ (or, more generally, that X and Y are in metric duality). Let F be given by $G(F) := \{(x, z) \in X \times \mathbb{R} : 2z \geq \|x\|^2\}$. Then $F^E(y) = \{\frac{1}{2}\|y\|^2\}$. □

A general explanation can be given by using the concepts of Ekeland set and Ekeland multifunction introduced in [26]. Recall that a subset E of a Banach space is called an *Ekeland set* if its indicator function ι_E is an Ekeland function, i.e. if for any $x_1, x_2 \in E$ and any $y \in N(E, x_1) \cap N(E, x_2)$ one has $\langle x_1, y \rangle = \langle x_2, y \rangle$. A multimapping F between two n.v.s. is said to be an *Ekeland multimapping* if its graph $G(F)$ is an Ekeland subset of the product space. In particular, any convex multimapping is an Ekeland multimapping.

Proposition 18 *If $F : X \rightrightarrows \mathbb{R}$ is an Ekeland multifunction, then its Ekeland transform is a function.*

Proof. By definition, $F : X \rightrightarrows \mathbb{R}$ is an Ekeland multifunction if, and only if, for any $(y, w) \in Y \times \mathbb{R}$ and any $(x_i, z_i) \in G(F)$ for $i = 1, 2$ such that $y \in D^*F(x_i, z_i)(w)$ one has $\langle x_1, y\rangle - wz_1 = \langle x_2, y\rangle - wz_2$. In particular when $(x_i, z_i) \in G(F)$ for $i = 1, 2$ are such that $y \in D^*F(x_i, z_i)(1)$ for $i = 1, 2$, one gets $\langle x_1, y\rangle - z_1 = \langle x_2, y\rangle - z_2$. Therefore F^E is single-valued. □

Such a simple situation may appear in nonconvex cases.

Lemma 19 *Suppose X is reflexive. Let $g : Y = X^* \to \mathbb{R}$ be a twice differentiable function and let $F : X \rightrightarrows \mathbb{R}$ be given by*

$$F(x) := \{z \in \mathbb{R} : \exists p \in Y,\ x = g'(p), z = h(p) := \langle g'(p), p\rangle - g(p)\} \qquad x \in X.$$

If for each $p \in Y$ the mapping $g''(p) : Y \to X$ is surjective, then the Ekeland transform F^E of F is a mapping which coincides with g on its domain $\operatorname{dom} F^E$.

Proof. Let $(y', z') \in F^E$. By definition, there exists some $(x, y, z) \in J^-F$ such that $y' = y$ and $z' = \langle x, y\rangle - z$. Now $(x, y, z) \in J^-F$ means that $(y, -1) \in N(F, (x, z))$ and $z \in F(x)$. Let $p \in Y$ be such that $x = g'(p)$, $z = h(p) = \langle g'(p), p\rangle - g(p)$. Since for each $v \in Y$ one easily checks that

$$\frac{1}{t}\left((g'(p + tv), h(p + tv) - (g'(p), h(p))\right) \to (g''(p)v, \langle g''(p)v, p\rangle),$$

one has $(g''(p)v, \langle g''(p)v, p\rangle) \in T(F, (x, z))$. Thus

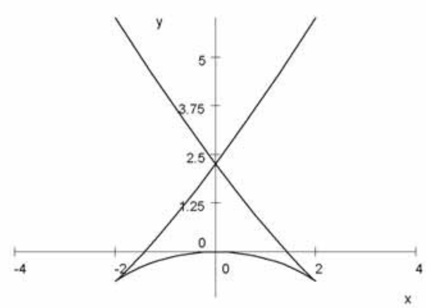
$$\langle g''(p)v, y\rangle - \langle g''(p)v, p\rangle = \langle (g''(p)v, \langle g''(p)v, p\rangle), (y, -1)\rangle \leq 0.$$

Since $g''(p)$ is surjective, we get $p = y (= y')$. Therefore $z' = \langle x, y\rangle - z = \langle g'(p), y\rangle - (\langle g'(p), p\rangle - g(p)) = g(p) = g(y')$. □

When the surjectivity assumption of the second derivative does not hold, the conclusion may fail.

Example 9. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by:

$$F(x) := \{z \in \mathbb{R} : \exists p \in \mathbb{R},\ x = p^3 - 3p,\ z = \frac{3}{4}p^2(p^2 - 2)\}.$$



Then F has the shape of a swallow tail which is symmetric with respect to $\{0\} \times \mathbb{R}$ and has singular points $(-2, -\frac{3}{4}), (2, -\frac{3}{4}), (0, \frac{9}{4})$. For $x \in \mathbb{R}$ the set $F(x)$ consists in one point for $|x| > 2$, two points for $x \in \{-2, 0, 2\}$ and three points for $x \in]-2, 0[\cup]0, 2[$. One has $N(F, (-2, -\frac{3}{4})) = \{(r, s) : r + s \leq 0\}$, $N(F, (2, -\frac{3}{4})) = \{(r, s) : -r + s \leq 0\}$ and $N(F, (0, \frac{9}{4})) = \{0\}$. Therefore

$$F^E = \{(y, -2y + \frac{3}{4}) : y \leq 1\} \cup \{(y, 2y + \frac{3}{4}) : y \geq -1\} \cup \{(y, \frac{1}{4}y^2(y^2 - 6)) : y \neq -1, 0, 1\}.$$

Note that the two half lines are tangent to the main curve at the points $(-1, -\frac{5}{4}), (1, -\frac{5}{4})$ respectively. Thus, while the two cusps of F no more appear in its transform F^E, two smooth bifurcation points show up. □

Example 10. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by:

$$F(x) := \{z \in \mathbb{R} : \exists p \in \mathbb{R},\ x = 2p(p+1)(2p+1),\ z = p(p+1)(3p^2 + p)\}.$$

Its graph is the semi-algebraic set appearing in relation (2.55) of [9]; it is the projection on $\mathbb{R}^2$ of a smooth submanifold of $\mathbb{R}^3$. It is represented in [9] and a similar analysis can be done. □

Question: Under which conditions does the following implication hold?

$$y \in D^*F(x, z)(1) \Rightarrow x \in D^*F^E(y, z')(1) \text{ with } z' := \langle x, y\rangle - z.$$

Such a property is the analogue of the fundamental property of the Legendre transform.

5 Legendre Transform

As in the case of functions, we introduce an extension of the Ekeland transform. Our purpose is to recover the usual convex conjugacy. For that reason, we use a closure process.

Definition 20 *The Legendre transform of a multifunction $F : X \rightrightarrows \mathbb{R}$ is the multifunction $F^L : Y \rightrightarrows \mathbb{R}$ given by*

$$z \in F^L(y) \Leftrightarrow \exists (x_n, y_n, r_n) \in J^-F,\ z_n = \langle x_n, y_n\rangle - r_n,\ (y_n, z_n) \to (y, z), (\langle x_n, y_n - y\rangle) \to 0.$$

The conditions we impose aim at restricting the graph of F^L as much as possible. Clearly, the graph of F^L contains the graph of F^E (take $(y_n, z_n) = (y, z)$) and is contained in its closure $\operatorname{cl} F^E$.

Proposition 21 *If f is a classical Legendre function considered as a multifunction F, then F^L coincides with the graph of the Legendre transform f^L of f as defined in Corollary 16.*

Proof. Since the graph of f^L is F^E, it is contained in F^L. Conversely, given (y, z) in the graph of F^L, taking a sequence (x_n, y_n, r_n) in J^-F and (z_n) as in Definition 20, we have $x_n = (f^L)'(y_n)$, $z_n = f^L(y_n)$ and since f^L is continuously differentiable, we get $z = \lim f^L(y_n) = f^L(y)$. □

Proposition 22 *If F is the epigraph of a convex function f, then F^L is contained in the epigraph of the conjugate function f^* of f and contains the graph of f^*.*

Proof. Let (y, z) be in the graph of F^L and let (x_n, y_n, r_n) be a sequence of J^-F as in Definition 20. In particular, as observed before, $r_n = f(x_n)$, so that $z_n = \langle x_n, y_n\rangle - f(x_n) = f^*(y_n)$, and $z = \lim_n z_n \geq f^*(y)$ since f^* is lower semicontinuous. Thus (y, z) is in the epigraph of f^*.

Now, for each (y,z) in the graph of f^*, using the consequence of the Bronsted-Rockafellar theorem given in [24, Prop. 1.1] applied to f^*, one can find some sequence $((y_n,x_n))$ in the graph of ∂f^* such that $(f^*(y_n)) \to f^*(y)$ and $(\langle x_n, y_n - y\rangle) \to 0$. Setting $z_n := f^*(y_n)$, $r_n := \langle x_n, y_n\rangle - z_n$ we have $(x_n,y_n,r_n) \in J^-F$ and $(y_n,z_n) \to (y,z)$. Therefore $z \in F^L(y)$. □

Final remarks. 1) A Lagrangian subset of $X \times Y$ could be defined as a subset L of $X \times Y$ such that $d\theta((x,y);(u,v),(u',v')) = 0$ for any $(x,y) \in L$ and $(u,v),(u',v') \in T(L,(x,y))$, where $d\theta$ is the differential 2-form defined by

$$d\theta((x,y);(u,v),(u',v')) = \langle u,v'\rangle - \langle u',v\rangle,$$

dropping the condition that $T(L,(x,y))$ is a vector subspace isomorphic to X. To the best of the author's knowledge, such a condition has not been used much in nonsmooth analysis (see [16] however). It would be interesting to explore the possibility of its uses.

2) Some attempts have been devised in order to detect whether a multifunction $F : X \rightrightarrows X^*$ is the subdifferential of a function (see [5], [7], for instance). Using such criteria, one could introduce a transform $g : X^* \to \mathbb{R}\cup\{\infty\}$ of a function $f : X \to \mathbb{R}\cup\{\infty\}$ by integrating the multifunction $(\partial f)^{-1} : X^* \rightrightarrows X$. Then a comparison with the Ekeland transform would be in order.

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