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ALTERNATIVE THEOREMS FOR SET-VALUED MAPS BASED ON A NONLINEAR SCALARIZATION

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Dedicated to Professor T. Rockafellar on his 70th birthday.

Abstract: Based on a comparison of a set with the zero vector with respect to a given convex cone, we establish five types of alternative theorems for set-valued maps without any convexity assumption, which are proved by a nonlinear scalarization technique. As an application, we obtain optimality conditions for vector optimization problems with set-valued maps.

Key words: *alternative theorem, nonlinear scalarization, vector optimization, set-valued optimization, set-valued maps, optimality conditions*

Mathematics Subject Classification: *90C29, 90C46, 49J53*

1 Introduction

This paper is concerned with alternative theorems for set-valued maps based on a nonlinear scalarization. Alternative theorems of the Farkas and Gordan types play important roles in many applications, especially in optimization theory concerning optimality conditions for nonlinear programming problems and in their duality theory. A generalized Gordan alternative theorem was given for a vector-valued function by Jeyakumar [9] in 1986, and its generalization to set-valued maps was proved by Li [12] in 1999 and Yang et al. [22] in 2000. These results rely on certain convexity assumptions like cone-subconvexlikeness in order to adopt a separation approach; see also [2, 6] for alternative theorems of set-valued maps. If we look at this approach from a different point of view, we will know that those proofs are based on a linear scalarization like an inner product. On the one hand, a nonlinear scalarization for vector-valued functions was introduced and applied to nonconvex separation theorems by Gerth (Tammer) and Weidner [5] in 1990, and similar approaches have been taken for several applications in [1, 3, 4, 20, 21] and at the same time we have researched some fundamental properties of a specific form of those nonlinear scalarizations in [15, 16]. By using special scalarizing functions under this type of nonlinear scalarization, we establish alternative theorems for set-valued maps without any convexity assumption.

In this paper, based on comparison between a vector and a set in an ordered vector space, we show five types of alternative theorems for set-valued maps; see also [11] for a comparison method between two sets. There is a one-to-one correspondence between order structures matched with the vector structure of the space and convex cones in a real ordered

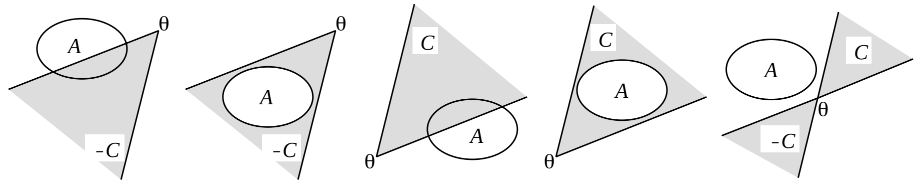

Figure 1: Five types of classification for comparison between the zero vector and a set.

vector space, and those convex cones are called positive cones; see [8] and [19]. A positive cone is often used as a dominance cone in a decision problem for a minimizer with respect to a preference structure. The set of the inverses of vectors in a positive cone is called a negative cone. When comparing the zero vector and a set with respect to a given positive cone, there are seven types of relationships; we classify nonempty sets into seven categories (not disjoint possibly) according to whether they intersect with each of a positive cone, its negative cone, and their complement: $S_1 := \{A \in 2^Y : A \cap B_1 = \emptyset, A \cap B_2 \neq \emptyset, A \cap B_3 \neq \emptyset\}$, $S_2 := \{A \in 2^Y : A \cap B_1 = \emptyset, A \cap B_2 \neq \emptyset, A \cap B_3 = \emptyset\}$, $S_3 := \{A \in 2^Y : A \cap B_1 \neq \emptyset, A \cap B_2 = \emptyset, A \cap B_3 \neq \emptyset\}$, $S_4 := \{A \in 2^Y : A \cap B_1 \neq \emptyset, A \cap B_2 = \emptyset, A \cap B_3 = \emptyset\}$, $S_5 := \{A \in 2^Y : A \cap B_1 = \emptyset, A \cap B_2 = \emptyset, A \cap B_3 \neq \emptyset\}$, $S_6 := \{A \in 2^Y : A \cap B_1 \neq \emptyset, A \cap B_2 \neq \emptyset, A \cap B_3 = \emptyset\}$, $S_7 := \{A \in 2^Y : A \cap B_1 \neq \emptyset, A \cap B_2 \neq \emptyset, A \cap B_3 \neq \emptyset\}$ where Y is a topological vector space, C is a convex cone in Y with nonempty interior and $C \neq Y$, $B_1 := C$, $B_2 := -C$, $B_3 := (C \cup (-C))^c$, not disjoint possibly but $B_1 \cup B_2 \cup B_3 = Y$. However, alternative situations which we consider can be made whether a given set intersects with each of a positive cone and its negative cone, or whether it is contained entirely in one of the two cones, so alternative situations which are considered in the cases of S_1, S_6 and S_7 (or S_3, S_6 and S_7) are the same one. For such reason, we restrict the relationships between the zero vector and a set to the five cases as illustrated in Figure 1. Under this basic policy, we establish five types of alternative theorems 3.1–3.5 with respect to the interior of a convex cone in the sense of weak efficiency. Besides, we present five types of alternative theorems 3.6–3.10 with respect to the closure of a convex cone in the sense of strong efficiency.

2 Nonlinear Scalarization

In this section, we introduce a nonlinear scalarization for sets and show some properties that a characteristic function and scalarizing functions have.

Let Y be a topological vector space, C a convex cone in Y with nonempty interior. We assume that $C \neq Y$, which is equivalent to

$$\operatorname{int} C \cap (-\operatorname{cl} C) = \emptyset \tag{2.1}$$

when C is a convex cone with nonempty interior. These assumptions are natural in infinite dimensional cases as well as finite dimensional cases. For instance, we give the following example.

Example 2.1. Let R be the real numbers and $E = L_p[0,1]$ with $0 < p < 1$, that is, the set of real-valued functions x on the interval $[0,1]$ for which $|x(t)|$ is Lebesgue integrable. If we take the product space $M = E \times R$ and the set $P = E \times R_+$ where R_+ is the set of positive real numbers, then P is a convex cone with nonempty interior and $P \neq M$. However, the Hausdorff topological vector space $L_p[0,1]$ with $0 < p < 1$ has no nonzero continuous linear functionals (see [18, pp.36–37]), hence M^*, the topological dual of M, does not separate points on M, and therefore M is not a locally convex space by [18, p.60, Corollary].

To begin with, we define a characteristic function

$$h_C(y;k) := \inf\{t : y \in tk - C\}$$

where $k \in \operatorname{int} C$ and moreover $-h_C(-y;k) = \sup\{t : y \in tk + C\}$. This function $h_C(y;k)$ has been treated in some papers ([17] for example) and it is regarded as a generalization of the Tchebyshev scalarization. Essentially, $h_C(y;k)$ is equivalent to the smallest strictly $(\operatorname{int} C)$-monotonic function defined by Luc in [13]. Note that $h_C(\cdot;k)$ is positively homogeneous and subadditive for every fixed $k \in \operatorname{int} C$, and hence it is sublinear and continuous.

Now, we give some useful properties of this function h_C.

Lemma 2.1. *Let $y \in Y$, then the following statements hold:*

(i) *If $y \in -\operatorname{int} C$, then $h_C(y;k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $h_C(y;k) < 0$, then $y \in -\operatorname{int} C$.*

Proof. First we prove the statement (i). Suppose that $y \in -\operatorname{int} C$, then there exists an absorbing neighborhood V_0 of 0 in Y such that $y + V_0 \subset -\operatorname{int} C$. Since V_0 is absorbing, for all $k \in \operatorname{int} C$, there exists $t_0 > 0$ such that $t_0 k \in V_0$. Therefore, $y + t_0 k \in y + V_0 \subset -\operatorname{int} C$. Hence, we have

$$\inf\{t : y \in tk - C\} \leq -t_0 < 0,$$

which shows that $h_C(y;k) < 0$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \operatorname{int} C$ such that $h_C(y;k) < 0$. Then, there exist $t_0 > 0$ and $c_0 \in C$ such that $y = -t_0 k - c_0 = -(t_0 k + c_0)$. Since $t_0 k \in \operatorname{int} C$ and C is a convex cone, we have $y \in -\operatorname{int} C$. □

Lemma 2.2. *Let $y \in Y$, then the following statements hold:*

(i) *If $y \in -\operatorname{cl} C$, then $h_C(y;k) \leq 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $h_C(y;k) \leq 0$, then $y \in -\operatorname{cl} C$.*

Proof. First we prove the statement (i). Suppose that $y \in -\operatorname{cl} C$. Then, there exist a net $\{y_\lambda\} \subset -C$ such that y_λ converges to y. For each y_λ, since $y_\lambda \in 0 \cdot k - C$ for all $k \in \operatorname{int} C$, $h_C(y_\lambda;k) \leq 0$ for all $k \in \operatorname{int} C$. By the continuity of $h_C(\cdot;k)$, $h_C(y;k) \leq 0$ for all $k \in \operatorname{int} C$.

Next we prove the statement (ii). Let $y \in Y$. Suppose that there exists $k \in \operatorname{int} C$ such that $h_C(y;k) \leq 0$. In the case $h_C(y;k) < 0$, from (ii) of Lemma 2.1, it is clear that $y \in -\operatorname{cl} C$. So we assume that $h_C(y;k) = 0$ and show that $y \in -\operatorname{cl} C$. By the definition of h_C, for each $n = 1, 2, \ldots$, there exists $t_n \in R$ such that

$$h_C(y;k) \leq t_n < h_C(y;k) + \frac{1}{n} \tag{2.2}$$

and

$$y \in t_n k - C. \tag{2.3}$$

From (2.2), $\lim_{n\to\infty} t_n = 0$. From (2.3), there exists $c_n \in C$ such that $y = t_n k - c_n$, that is, $c_n = t_n k - y$. Since $c_n \to -y$ as $n \to \infty$, we have $y \in -\operatorname{cl} C$. □

Lemma 2.3. *Let $y \in Y$, then the following statements hold:*

(i) *If $y \in \operatorname{int} C$, then $h_C(y;k) > 0$ for all $k \in \operatorname{int} C$;*

(ii) *If $y \in \operatorname{cl} C$, then $h_C(y;k) \geq 0$ for all $k \in \operatorname{int} C$.*

Proof. To show the statement (i), let $y \in \operatorname{int} C$, and then $y \notin -\operatorname{cl} C$ by (2.1). By the contraposition of (ii) in Lemma 2.2, $h_C(y;k) > 0$ for all $k \in \operatorname{int} C$.

Next, to show the statement (ii), let $y \in \operatorname{cl} C$, and then $y \notin -\operatorname{int} C$ by (2.1). By the contraposition of (ii) in Lemma 2.1, $h_C(y;k) \geq 0$ for all $k \in \operatorname{int} C$. □

The following lemma shows (strictly) monotone property on $h_C(\cdot;k)$, which has been investigated in [5] and [1].

Lemma 2.4. *Let $y, \bar{y} \in Y$, then the following statements hold:*

(i) *If $y \in \bar{y} + \operatorname{int} C$, then $h_C(y;k) > h_C(\bar{y};k)$ for all $k \in \operatorname{int} C$;*

(ii) *If $y \in \bar{y} + \operatorname{cl} C$, then $h_C(y;k) \geq h_C(\bar{y};k)$ for all $k \in \operatorname{int} C$.*

Proof. Let $y, \bar{y} \in Y$. First, we prove the statement (i). Suppose that $y \in \bar{y} + \operatorname{int} C$, that is, there exists $c \in \operatorname{int} C$ such that $\bar{y} = y - c$. Then, for each $k \in \operatorname{int} C$, we have

$$\begin{aligned} h_C(\bar{y};k) &= h_C(y-c;k) \\ &\leq h_C(y;k) + h_C(-c;k) \quad \text{(by the subadditivity of } h_C(\cdot;k)) \\ &< h_C(y;k) \quad \text{(by (i) of Lemma 2.1).} \end{aligned}$$

Analogously, we prove the statement (ii) by using (i) of Lemma 2.2. □

Lemma 2.5. *Let $y, \bar{y} \in Y$ and $k \in \operatorname{int} C$, then the following statements hold:*

(i) *If $h_C(y;k) > h_C(\bar{y};k)$, then $h_C(y-\bar{y};k) > 0$;*

(ii) *If $h_C(y;k) \geq h_C(\bar{y};k)$, then $h_C(y-\bar{y};k) \geq 0$.*

Proof. Let $y, \bar{y} \in Y$ and $k \in \operatorname{int} C$. To show the statement (i), suppose that $h_C(y;k) > h_C(\bar{y};k)$. By the contraposition of (ii) in Lemma 2.4, we have $y - \bar{y} \notin -\operatorname{cl} C$. Therefore, the statement (i) follows from the contraposition of (ii) in Lemma 2.2. Analogously, we prove the statement (ii) by using the contrapositions of (i) in Lemma 2.4 and (ii) in Lemma 2.1. □

Remark 2.1. In the above lemma, we note that each converse does not hold.

In the following, we present main tools to prove alternative theorems for set-valued maps. They are some properties that scalarizing functions for sets have.

Proposition 2.1. *Let $A \subset Y$, then the following statements hold:*

(i) *If $A \cap (-\operatorname{int} C) \neq \emptyset$, then $\inf_{y \in A} h_C(y;k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $\inf_{y \in A} h_C(y;k) < 0$, then $A \cap (-\operatorname{int} C) \neq \emptyset$.*

Proof. First we prove the statement (i). Suppose that $A \cap (-\operatorname{int} C) \neq \emptyset$. Then, there exists $y \in A \cap (-\operatorname{int} C)$. By (i) of Lemma 2.1, for all $k \in \operatorname{int} C$, $h_C(y;k) < 0$, and hence, $\inf_{y \in A} h_C(y;k) < 0$.

Next we prove the statement (ii). Suppose that there exists $k \in \operatorname{int} C$ such that $\inf_{y \in A} h_C(y;k) < 0$. Then, there exist $\varepsilon_0 > 0$ and $y_0 \in A$ such that

$$h_C(y_0;k) \leq \inf_{y \in A} h_C(y;k) + \varepsilon_0 < 0.$$

By (ii) of Lemma 2.1, we have $y_0 \in -\operatorname{int} C$, which implies that $A \cap (-\operatorname{int} C) \neq \emptyset$. □

Proposition 2.2. *Let $A \subset Y$, then the following statements hold:*

(i) *If $A \subset -\operatorname{int} C$ and A is a compact set, then $\sup_{y \in A} h_C(y; k) < 0$ for all $k \in \operatorname{int} C$;*

(ii) *If there exists $k \in \operatorname{int} C$ with $\sup_{y \in A} h_C(y; k) < 0$, then $A \subset -\operatorname{int} C$.*

Proof. First we prove the statement (i). Assume that A is a compact set and suppose that $A \subset -\operatorname{int} C$. Then, for all $k \in \operatorname{int} C$,

$$A \subset \bigcup_{t>0} (-tk - \operatorname{int} C).$$

By the compactness of A, there exist $t_1, \ldots, t_m > 0$ such that

$$A \subset \bigcup_{i=1}^{m} (-t_i k - \operatorname{int} C).$$

Since $-t_q k - \operatorname{int} C \subset -t_p k - \operatorname{int} C$ for $t_p < t_q$, there exists $t_0 := \min\{t_1, \ldots, t_m\} > 0$ such that $A \subset -t_0 k - \operatorname{int} C$. For each $y \in A$, we have

$$h_C(y; k) = \inf\{t : y \in tk - C\} \leq -t_0.$$

Hence,

$$\sup_{y \in A} h_C(y; k) \leq -t_0 < 0.$$

Next, we prove the statement (ii). Suppose that there exists $k \in \operatorname{int} C$ such that $\sup_{y \in A} h_C(y; k) < 0$. Then, for all $y \in A$, $h_C(y; k) < 0$. By (ii) of Lemma 2.1, we have $y \in -\operatorname{int} C$, and hence $A \subset -\operatorname{int} C$. □

Remark 2.2. When we replace A by $\operatorname{cl} A$ in (i) of Proposition 2.2, the assertion still remains.

Moreover, we can replace (i) in Proposition 2.2 by another relaxed form.

Corollary 2.1. *Let $A \subset Y$ and assume that there exists a compact set B such that $B \subset -\operatorname{int} C$. If $A \subset B - C$, then $\sup_{y \in A} h_C(y; k) < 0$ for all $k \in \operatorname{int} C$.*

Proof. Assume that there exists a compact set B such that $B \subset -\operatorname{int} C$ and $A \subset B - C$. By applying (i) of Proposition 2.2 to B instead of A, for all $k \in \operatorname{int} C$,

$$\sup_{y \in B} h_C(y; k) < 0.$$

Since $A \subset B - C$, it follows from (i) of Lemma 2.1 and the subadditivity of $h_C(\cdot; k)$ that

$$h_C(y; k) \leq \sup_{z \in B} h_C(z; k)$$

for each $y \in A$. Therefore, $\sup_{y \in A} h_C(y; k) < 0$ for all $k \in \operatorname{int} C$. □

Proposition 2.3. *Let $A \subset Y$, then the following statements hold:*

(i) *If $A \cap (-\operatorname{cl} C) \neq \emptyset$, then $\inf_{y \in A} h_C(y; k) \leq 0$ for all $k \in \operatorname{int} C$;*

(ii) *If A is a compact set and there exists $k \in \operatorname{int} C$ with $\inf_{y \in A} h_C(y; k) \leq 0$, then $A \cap (-\operatorname{cl} C) \neq \emptyset$.*

Proof. First we prove the statement (i). Suppose that $A \cap (-\mathrm{cl}\, C) \neq \emptyset$. Then, there exists $y \in A \cap (-\mathrm{cl}\, C)$. By (i) of Lemma 2.2, for all $k \in \mathrm{int}\, C$, $h_C(y; k) \leq 0$, and hence $\inf_{y \in A} h_C(y; k) \leq 0$.

Next, we prove the statement (ii). Suppose that there exists $k \in \mathrm{int}\, C$ such that $\inf_{y \in A} h_C(y; k) \leq 0$. In the case $\inf_{y \in A} h_C(y; k) < 0$, from (ii) of Proposition 2.1, it is clear that $A \cap (-\mathrm{cl}\, C) \neq \emptyset$. So we assume that $\inf_{y \in A} h_C(y; k) = 0$ and show that $A \cap (-\mathrm{cl}\, C) \neq \emptyset$. By the definition of infimum, for each $n = 1, 2, \ldots$, there exist $t_n \in R$ and $y_n \in A$ such that $y_n \in t_n k - C$ and

$$\inf_{y \in A} h_C(y; k) \leq t_n < \inf_{y \in A} h_C(y; k) + \frac{1}{n}. \tag{2.4}$$

From (2.4), $\lim_{n \to \infty} t_n = 0$. Since A is compact, we may suppose that $y_n \to y_0$ for some $y_0 \in A$ without loss of generality (taking subsequence). Therefore, $y_n - t_n k \to y_0$ and then $y_0 \in -\mathrm{cl}\, C$, which shows that $A \cap (-\mathrm{cl}\, C) \neq \emptyset$. □

Remark 2.3. If A is not compact, then there are counter-examples violating the statement (ii) such as an unbounded set approaching $-\mathrm{cl}\, C$ asymptotically or an open set whose boundary intersects $-\mathrm{cl}\, C$.

Proposition 2.4. *Let $A \subset Y$, then the following statements hold:*

(i) *If $A \subset -\mathrm{cl}\, C$, then $\sup_{y \in A} h_C(y; k) \leq 0$ for all $k \in \mathrm{int}\, C$;*

(ii) *If there exists $k \in \mathrm{int}\, C$ with $\sup_{y \in A} h_C(y; k) \leq 0$, then $A \subset -\mathrm{cl}\, C$.*

Proof. First we prove the statement (i). Suppose that $A \subset -\mathrm{cl}\, C$. Then, for each $y \in A$, it follows from (i) of Lemma 2.2 that $h_C(y; k) \leq 0$ for all $k \in \mathrm{int}\, C$, and hence $\sup_{y \in A} h_C(y; k) \leq 0$ for all $k \in \mathrm{int}\, C$.

Next, we prove the statement (ii). Suppose that there exists $k \in \mathrm{int}\, C$ such that $\sup_{y \in A} h_C(y; k) \leq 0$. Then, for all $y \in A$, $h_C(y; k) \leq 0$. By (ii) of Lemma 2.2, we have $y \in -\mathrm{cl}\, C$, and hence $A \subset -\mathrm{cl}\, C$. □

3 Alternative Theorems

In this section, we present various types of alternative theorems for set-valued maps without any convexity. These alternative theorems are fundamental tools to derive optimality conditions for vector optimization problems with set-valued maps. As stated in Introduction, there are five types of relationships between the zero vector and each image of a set-valued map with respect to a given positive cone.

Now, we consider several characterizations for images of a set-valued map by the nonlinear and strictly monotone characteristic function h_C. Let X and Y be a nonempty set and a topological vector space, $F : X \to 2^Y$ a set-valued map, respectively. We observe the following four types of scalarizing functions:

$$\begin{aligned}
&(1) \quad \psi_C^F(x; k) := \sup \{h_C(y; k) : y \in F(x)\}, \\
&(2) \quad \varphi_C^F(x; k) := \inf \{h_C(y; k) : y \in F(x)\}, \\
&(3) \quad -\varphi_C^{-F}(x; k) = \sup \{-h_C(-y; k) : y \in F(x)\}, \\
&(4) \quad -\psi_C^{-F}(x; k) = \inf \{-h_C(-y; k) : y \in F(x)\}.
\end{aligned}$$

Functions (1) and (4) have symmetric properties and then results for function (4) $-\psi_C^{-F}$ can be easily proved by those for function (1) ψ_C^F. Similarly, the results for function (3)

$-\varphi_C^{-F}$ can be deduced by those for function (2) φ_C^F. By using these four functions we measure each image of set-valued map F with respect to its 4-tuple of scalars, which can be regarded as standpoints for the evaluation of the image with respect to convex cone C.

First, we present five types of alternative theorems for set-valued maps when we compare each image of set-valued map with the zero vector with respect to the interior of a convex cone.

Theorem 3.1. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. Then, exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \cap (-\operatorname{int} C) \neq \emptyset$;*

(II) *There exists $k \in \operatorname{int} C$ such that $\varphi_C^F(x; k) \geq 0$ for all $x \in X$.*

Proof. First, we assume that system (I) is consistent. Then, there exists $x \in X$ such that $F(x) \cap (-\operatorname{int} C) \neq \emptyset$. By (i) of Proposition 2.1, $\varphi_C^F(x; k) < 0$ for all $k \in \operatorname{int} C$, which shows that system (II) is not consistent.

Next, we assume that system (II) is not consistent. Then, for all $k \in \operatorname{int} C$, there exists $x \in X$ such that $\varphi_C^F(x; k) < 0$. By (ii) of Proposition 2.1, system (I) is consistent. □

Theorem 3.2. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. If F is compact-valued on X, then exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \subset -\operatorname{int} C$;*

(II) *There exists $k \in \operatorname{int} C$ such that $\psi_C^F(x; k) \geq 0$ for all $x \in X$.*

Proof. First, we assume that system (I) is consistent. Then, there exists $x \in X$ such that $F(x) \subset -\operatorname{int} C$. By (i) of Proposition 2.2, $\psi_C^F(x; k) < 0$ for all $k \in \operatorname{int} C$, which shows that system (II) is not consistent.

Next, we assume that system (II) is not consistent. Then, for all $k \in \operatorname{int} C$, there exists $x \in X$ such that $\psi_C^F(x; k) < 0$. By (ii) of Proposition 2.2, system (I) is consistent. □

Corollary 3.1. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. Assume that if $F(x) \subset -\operatorname{int} C$, then there exists a compact subset $B \subset -\operatorname{int} C$ such that $F(x) \subset B - C$. Then, exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \subset -\operatorname{int} C$;*

(II) *There exists $k \in \operatorname{int} C$ such that $\psi_C^F(x; k) \geq 0$ for all $x \in X$.*

Proof. First, we assume that system (I) is consistent. Then, there exists $x \in X$ such that $F(x) \subset -\operatorname{int} C$. By Corollary 2.1, $\psi_C^F(x; k) < 0$ for all $k \in \operatorname{int} C$, which shows that system (II) is not consistent.

Next, we assume that system (II) is not consistent. Then, for all $k \in \operatorname{int} C$, there exists $x \in X$ such that $\psi_C^F(x; k) < 0$. By (ii) of Proposition 2.2, system (I) is consistent. □

Theorem 3.3. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. Then, exactly one of the following two systems is consistent:*

(I) *There exists* $x \in X$ *such that* $F(x) \cap \operatorname{int} C \neq \emptyset$*;*

(II) *There exists* $k \in \operatorname{int} C$ *such that* $-\varphi_C^{-F}(x;k) \leq 0$ *for all* $x \in X$*.*

Proof. The proof is completed simply by replacing F by $-F$ in the proof of Theorem 3.1. □

Theorem 3.4. *Let* X *and* Y *be a nonempty set and a topological vector space,* C *a convex cone in* Y *with nonempty interior, and* $F : X \to 2^Y$ *a set-valued map, respectively. If* F *is compact-valued on* X*, then exactly one of the following two systems is consistent:*

(I) *There exists* $x \in X$ *such that* $F(x) \subset \operatorname{int} C$*;*

(II) *There exists* $k \in \operatorname{int} C$ *such that* $-\psi_C^{-F}(x;k) \leq 0$ *for all* $x \in X$*.*

Proof. The proof is completed simply by replacing F by $-F$ in the proof of Theorem 3.2. □

Corollary 3.2. *Let* X *and* Y *be a nonempty set and a topological vector space,* C *a convex cone in* Y *with nonempty interior, and* $F : X \to 2^Y$ *a set-valued map, respectively. Assume that if* $F(x) \subset \operatorname{int} C$*, then there exists a compact subset* $B \subset \operatorname{int} C$ *such that* $F(x) \subset B + C$*. Then, exactly one of the following two systems is consistent:*

(I) *There exists* $x \in X$ *such that* $F(x) \subset \operatorname{int} C$*;*

(II) *There exists* $k \in \operatorname{int} C$ *such that* $-\psi_C^{-F}(x;k) \leq 0$ *for all* $x \in X$*.*

Proof. The proof is completed simply by replacing F by $-F$ in the proof of Corollary 3.1. □

Theorem 3.5. *Let* X *and* Y *be a nonempty set and a topological vector space,* C *a convex cone in* Y *with nonempty interior, and* $F : X \to 2^Y$ *a set-valued map, respectively. Then, exactly one of the following two systems is consistent:*

(I) *There exists* $x \in X$ *such that* $F(x) \cap (-\operatorname{int} C) \neq \emptyset$ *or* $F(x) \cap \operatorname{int} C \neq \emptyset$*;*

(II) *There exists* $k \in \operatorname{int} C$ *such that* $\varphi_C^F(x;k) \geq 0$ *and* $-\varphi_C^{-F}(x;k) \leq 0$ *for all* $x \in X$*.*

Proof. The proof is straightforward from the same way as the proofs of Theorems 3.1 and 3.3. □

Next, we present five types of alternative theorems for set-valued maps when we compare each image of set-valued map with the zero vector with respect to the closure of a convex cone.

Theorem 3.6. *Let* X *and* Y *be a nonempty set and a topological vector space,* C *a convex cone in* Y *with nonempty interior, and* $F : X \to 2^Y$ *a set-valued map, respectively. If* F *is compact-valued on* X*, then exactly one of the following two systems is consistent:*

(I) *There exists* $x \in X$ *such that* $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$*;*

(II) *There exists* $k \in \operatorname{int} C$ *such that* $\varphi_C^F(x;k) > 0$ *for all* $x \in X$*.*

Proof. First, we assume that system (I) is consistent. Then, there exists $x \in X$ such that $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$. By (i) of Proposition 2.3, $\varphi_C^F(x;k) \leq 0$ for all $k \in \operatorname{int} C$, which shows that system (II) is not consistent.

Next, we assume that system (II) is not consistent. Then, for all $k \in \operatorname{int} C$, there exists $x \in X$ such that $\varphi_C^F(x;k) \leq 0$. By (ii) of Proposition 2.3, system (I) is consistent. □

Theorem 3.7. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. Then, exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \subset -\operatorname{cl} C$;*

(II) *There exists $k \in \operatorname{int} C$ such that $\psi_C^F(x;k) > 0$ for all $x \in X$.*

Proof. First, we assume that system (I) is consistent. Then, there exists $x \in X$ such that $F(x) \subset -\operatorname{cl} C$. By (i) of Proposition 2.4, $\psi_C^F(x;k) \leq 0$ for all $k \in \operatorname{int} C$, which shows that system (II) is not consistent.

Next, we assume that system (II) is not consistent. Then, for all $k \in \operatorname{int} C$, there exists $x \in X$ such that $\psi_C^F(x;k) \leq 0$. By (ii) of Proposition 2.4, system (I) is consistent. □

Theorem 3.8. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. If F is compact-valued on X, then exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \cap \operatorname{cl} C \neq \emptyset$;*

(II) *There exists $k \in \operatorname{int} C$ such that $-\varphi_C^{-F}(x;k) < 0$ for all $x \in X$.*

Proof. The proof is completed simply by replacing F by $-F$ in the proof of Theorem 3.6. □

Theorem 3.9. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. Then, exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \subset \operatorname{cl} C$;*

(II) *There exists $k \in \operatorname{int} C$ such that $-\psi_C^{-F}(x;k) < 0$ for all $x \in X$.*

Proof. The proof is completed simply by replacing F by $-F$ in the proof of Theorem 3.7. □

Theorem 3.10. *Let X and Y be a nonempty set and a topological vector space, C a convex cone in Y with nonempty interior, and $F : X \to 2^Y$ a set-valued map, respectively. If F is compact-valued on X, then exactly one of the following two systems is consistent:*

(I) *There exists $x \in X$ such that $F(x) \cap (-\operatorname{cl} C) \neq \emptyset$ or $F(x) \cap \operatorname{cl} C \neq \emptyset$;*

(II) *There exists $k \in \operatorname{int} C$ such that $\varphi_C^F(x;k) > 0$ and $-\varphi_C^{-F}(x;k) < 0$ for all $x \in X$.*

Proof. The proof is straightforward from the same way as the proofs of Theorems 3.6 and 3.8. □

4 Optimality Conditions

Throughout this section, let X be a nonempty set, and let Y and Z be ordered topological vector spaces with convex cones C and D, respectively. We assume that $C \neq Y$ and $\operatorname{int} C \neq \emptyset$. Let $F : X \to 2^Y$ and $G : X \to 2^Z$ be set-valued maps. A constrained set-valued optimization problem is written as

$$\text{(MP)} \quad \min_K \; F(x) \quad \text{subject to} \quad G(x) \cap (-D) \neq \emptyset,$$

where K is a convex cone in Y. The feasible set of problem (MP) is defined by $V = \{x \in$

$X : G(x) \cap (-D) \neq \emptyset\}$. Problem (MP) is to find all solutions $x_0 \in V$ such that there exists $y_0 \in F(x_0)$ and for each $x \in V$, there exists no $y \in F(x)$ satisfying $y_0 \in y + K \setminus \{0_Y\}$. Such solution x_0 is called an efficient solution of problem (MP) with respect to K, and in case of $\operatorname{int} K$ instead of K, its solution is called a weakly efficient solution (traditionally in vector optimization). Since the constraint in problem (MP) is reduced to $G(x) \leq 0$ when G is a real-valued function and D is the cone of nonnegative reals, it is a generalization of the inequality constraints of a standard nonlinear programming problem. Thus, we consider the following optimization problems:

(MP1) $\quad \min_{\operatorname{int} C} F(x) \quad$ subject to $G(x) \cap (-D) \neq \emptyset$;
(MP2) $\quad \min_{C} F(x) \quad$ subject to $G(x) \cap (-D) \neq \emptyset$.

Definition 4.1. A point $x_0 \in V$ is said to be *a weakly efficient solution* of (MP1) if there exists $y_0 \in F(x_0)$ and for each $x \in V$, there exists no $y \in F(x)$ satisfying $y_0 \in y + \operatorname{int} C$, that is,

$$F(V) \cap (y_0 - \operatorname{int} C) = \emptyset; \tag{4.1}$$

A pair (x_0, y_0) is said to be *a weakly efficient element* for (MP1) if $x_0 \in V$ and $y_0 \in F(x_0)$ satisfies (4.1).

Definition 4.2. A point $x_0 \in V$ is said to be *an efficient solution* of (MP2) if there exists $y_0 \in F(x_0)$ and for each $x \in V$, there exists no $y \in F(x)$ satisfying $y_0 \in y + C \setminus \{0_Y\}$, that is,

$$F(V) \cap (y_0 - C \setminus \{0_Y\}) = \emptyset; \tag{4.2}$$

A pair (x_0, y_0) is said to be *an efficient element* for (MP2) if $x_0 \in V$ and $y_0 \in F(x_0)$ satisfies (4.2).

Definition 4.3. Let $k \in \operatorname{int} C$. Consider the following scalar minimization problem

$$\min_{x \in V} \varphi_C^F(x; k). \tag{4.3}$$

Let $x_0 \in V$ be given. Then, a pair (x_0, y_0) is said to be *an optimal element* for the problem if the following conditions hold:

(i) $\varphi_C^F(x; k) \geq \varphi_C^F(x_0; k)$ for all $x \in V$;

(ii) $\varphi_C^F(x_0; k) = h_C(y_0; k)$ and $y_0 \in F(x_0)$.

Remark 4.1. Under $k \in \operatorname{int} C$, we have the following: a pair (x_0, y_0) is an optimal element for (4.3) if and only if $x_0 \in V$ and $y_0 \in F(x_0)$ satisfies

$$h_C(y; k) \geq h_C(y_0; k) \text{ for all } y \in F(V).$$

Definition 4.4. Let $k \in \operatorname{int} C$. Consider problem (4.3). Let $x_0 \in V$ be given. Then, a pair (x_0, y_0) is said to be *a strict optimal element* if the following conditions hold:

(i) $\varphi_C^F(x; k) > \varphi_C^F(x_0; k)$ for all $x \in V \setminus \{x_0\}$;

(ii) $\varphi_C^F(x_0; k) = h_C(y_0; k)$ and $y_0 \in F(x_0)$;

(iii) $h_C(y; k) > h_C(y_0; k)$ for all $y \in F(x_0) \setminus \{y_0\}$.

Remark 4.2. Under $k \in \operatorname{int} C$, we have the following: a pair (x_0, y_0) is a strict optimal element for (4.3) if and only if $x_0 \in V$ and $y_0 \in F(x_0)$ satisfies

$$h_C(y; k) > h_C(y_0; k), \text{ for all } y \in F(V) \setminus \{y_0\}.$$

Theorem 4.1. (Sufficient condition for (MP1).) *Let $\bar{x} \in V$ and $\bar{y} \in F(\bar{x})$. If there exists $k \in \operatorname{int} C$ such that $(\bar{x}, \bar{y})$ is an optimal element for* (4.3)*, then $(\bar{x}, \bar{y})$ is a weakly efficient element for* (MP1).

Proof. Assume that $(\bar{x}, \bar{y})$ is not a weakly efficient element for (MP1). Then, there exist $x \in V$ and $y \in F(x)$ such that $\bar{y} \in y + \operatorname{int} C$. Since $k \in \operatorname{int} C$, it follows from (i) of Lemma 2.4 that $h_C(\bar{y}; k) > h_C(y; k)$. By Remark 4.1, it contradicts the assumption that $(\bar{x}, \bar{y})$ is an optimal element for (4.3). □

Theorem 4.2. (Necessary and sufficient condition for (MP1).) *Let $\bar{x} \in V$ and $\bar{y} \in F(\bar{x})$. Then $(\bar{x}, \bar{y})$ is a weakly efficient element for* (MP1) *if and only if there exists $k \in \operatorname{int} C$ such that $h_C(y - \bar{y}; k) \geq 0$ for all $y \in F(V)$.*

Proof. Suppose first that $(\bar{x}, \bar{y})$ is a weakly efficient element for (MP1). By definition, we have $(F(V) - \bar{y}) \cap (-\operatorname{int} C) = \emptyset$. By applying Theorem 3.1 to $F(V) - \bar{y}$ instead of $F(x)$, there exists $k \in \operatorname{int} C$ such that $h_C(y - \bar{y}; k) \geq 0$ for all $y \in F(V)$.

Conversely, suppose that there exists $k \in \operatorname{int} C$ such that $h_C(y - \bar{y}; k) \geq 0$ for all $y \in F(V)$. Assume that $(\bar{x}, \bar{y})$ is not a weakly efficient element for (MP1). Then, there exist $x \in V$ and $y \in F(x)$ such that $y - \bar{y} \in -\operatorname{int} C$. Since $k \in \operatorname{int} C$, it follows from (i) of Lemma 2.1 that $h_C(y - \bar{y}; k) < 0$, which contradicts the assumption. □

Theorem 4.3. (Sufficient condition for (MP2).) *Let $\bar{x} \in V$ and $\bar{y} \in F(\bar{x})$. If there exists $k \in \operatorname{int} C$ such that $(\bar{x}, \bar{y})$ is a strict optimal element for* (4.3)*, then $(\bar{x}, \bar{y})$ is an efficient element for* (MP2).

Proof. By applying the same argument as the proof of Theorem 4.1 to problem (MP2), the proof is straightforward from (ii) of Lemma 2.4 and Remark 4.2. □

Theorem 4.4. (Necessary and sufficient condition for (MP2).) *Let $\bar{x} \in V$ and $\bar{y} \in F(\bar{x})$. If F is compact-valued on V and C is closed, then $(\bar{x}, \bar{y})$ is an efficient element for* (MP2) *if and only if there exists $k \in \operatorname{int} C$ such that $h_C(y - \bar{y}; k) > 0$ for all $y \in F(V) \setminus \{\bar{y}\}$.*

Proof. For problem (MP2), by using the same argument as the proof of Theorem 4.2, it follows from Theorem 3.6 that the necessity is shown. By (i) of Lemma 2.2, we can also show the sufficiency. □

5 Conclusions

Based on a nonlinear scalarization technique for sets, we establish five types of alternative theorems for set-valued maps without any convexity assumption. Moreover, we obtain optimality conditions for set-valued optimization problems.

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