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SEPARATION IN $\mathbb{B}$-CONVEXITY

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To Terry Rockafellar on his 70th birthday.

Abstract: A subset B of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if for all $x_1, x_2 \in B$ and all $t \in [0,1]$ one has $tx_1 \vee x_2 \in B$. These sets were first investigated in [1] where it was shown that Carathéodory, Radon and Helly like Theorems hold. In this work we establish separation and Hahn-Banach like Theorems for $\mathbb{B}$-convex sets.

Key words: *$\mathbb{B}$-convexity, gauges, co-gauges, separation, $\mathbb{B}$-measurable maps*

Mathematics Subject Classification: *46A22, 52A30, 52A99,06F30*

1 Introduction

This paper continues the investigation of $\mathbb{B}$-convexity introduced in [1], more precisely, we establish geometric and functional Hahn-Banach like separation properties in $\mathbb{B}$-convexity. A subset B of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if for all $x_1, x_2 \in B$ and all $t \in [0,1]$ one has $tx_1 \vee x_2 \in B$; an easy induction shows that B is $\mathbb{B}$-convex if and only if, for all $x_1, \ldots, x_m \in B$ and all $t_1, \ldots, t_m \in [0,1]$ such that $\max\{t_1, \ldots, t_m\} = 1$ one has $\bigvee_{i=1}^m t_i x_i \in B$, where $\vee$ denotes the maximum with respect to partial order of $\mathbb{R}^n_+$ associated to the positive cone, that is, the coordinatewise supremum. For x and y in $\mathbb{R}^n_+$, $x \leq y$ means $y - x \in \mathbb{R}^n_+$.

The first section deals with the Stone-Kakutani separation property (the algebraic Hahn-Banach Theorem) which, as is well known, is a consequence of the Pash-Peano Property. So, we show that the Pash-Peano Property holds in $\mathbb{B}$-convexity. For the reader's convenience, we give the (standard) proof of the Stone-Kakutani Theorem as well as the generalized Stone-Kakutani and Pash-Peano Theorems which are due, at least in the framework of abstract convexities, to Van de Vel [4].

$\mathbb{B}$-convex sets are studied through the properties of their Minkowski gauge. Of particular interest are the $\mathbb{B}$-convex sets whose complements are also $\mathbb{B}$-convex; those sets are called half-spaces ($\mathbb{B}$-half-spaces would be more precise, but since these half-spaces will always, and only, appear in conjunction with $\mathbb{B}$-convex sets the possiblity of confusion is minimal). Analytic and geometric separation theorems are proved for disjoint $\mathbb{B}$-convex sets. It is shown that a closed $\mathbb{B}$-convex set is always the intersection of the closed half-spaces in which it is contained. A family of open half-spaces such that all convex sets are intersection of members of that family is given; furthermore, it is shown that the family in question is minimal with respect to that property.

We use the following notation: $I = \{1, \ldots, n\}$; $\mathbb{R}^n_+$ is the set of points with nonnegative coordinates and $\mathbb{R}^n_{++}$ is its interior, the set of points with strictly positive coordinates; for

$x \in \mathbb{R}^n_+$ the ith coordinate of x is denoted by x_i and $I(x) = \{i : x_i \neq 0\}$. We denote by $[\![A]\!]$ the $\mathbb{B}$-convex hull of A, that is, the intersection of all the $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ containing A. The $\mathbb{B}$-convex hull of a finite set is a $\mathbb{B}$-polytope; we recall that a set C is $\mathbb{B}$-convex if and only if it contains all the $\mathbb{B}$-polytopes spanned by its finite subsets and that the union of an up-directed family of $\mathbb{B}$-convex sets is $\mathbb{B}$-convex.*

2 Algebraic Separation

The following lemma says that $\mathbb{B}$-convexity is join-hull commutative; this property will be used in the proof of the Pasch-Peano Property from which algebraic separation of disjoint $\mathbb{B}$-convex sets is a consequence.

Lemma 2.1. *For all subsets S of $\mathbb{R}^n_+$ and for all point p we have*

$$[\![S \cup \{p\}]\!] = \bigcup_{x \in [\![S]\!]} [\![x, p]\!].$$

Proof. One inclusion is clear; we prove the other one, and we assume that S is not empty. If $y \in [\![S \cup \{p\}]\!]$, we can assume that $y \neq p$ otherwise there is nothing to prove; then, there exist $x_1, \dots, x_k \in S$ and $\eta_1, \dots, \eta_{k+1} \in [0,1]$ such that $\max_{1 \leq i \leq k+1} \eta_i = 1$ and $y = \eta_1 x_1 \vee \dots \vee \eta_k x_k \vee \eta_{k+1} p$. Since $y \neq p$ we have $\max_{1 \leq i \leq k} \eta_i \neq 0$, let $\mu = \max_{1 \leq i \leq k} \eta_i$ and $\mu_i = \eta_i / \mu$ for $i = 1, \dots, k$. Then $x = \mu_1 x_1 \vee \dots \vee \mu_k x_k \in [\![S]\!]$ and $y = \mu x \vee \eta_{k+1} p \in [\![x, p]\!]$ since $\max\{\mu, \eta_{k+1}\} = 1$. □

Proposition 2.1 (Pash-Peano). *For all quintuple (a, b_1, b_2, c_1, c_2) of points of $\mathbb{R}^n_+$ such that $c_i \in [\![a, b_i]\!]$ we have $[\![b_1, c_2]\!] \cap [\![b_2, c_1]\!] \neq \emptyset$.*

Proof. We have to show that the following system of equations has a solution

$$(\star) \quad \begin{cases} \eta_1 c_1 \vee \mu_2 b_2 & = & \eta_2 c_2 \vee \mu_1 b_1 \\ \max\{\eta_1, \mu_2\} & = & 1 \\ \max\{\eta_2, \mu_1\} & = & 1 \end{cases}$$

Since $c_i \in [\![a, b_i]\!]$ we can write $c_i = \rho_i a \vee \alpha_i b_i$ with $\max\{\rho_i, \alpha_i\} = 1$; a substitution in the first line of $(\star)$ yields

$$(\star_2) \quad (\eta_1 \rho_1) a \vee (\eta_1 \alpha_1) b_1 \vee \mu_2 b_2 = (\eta_2 \rho_2) a \vee (\eta_2 \alpha_2) b_2 \vee \mu_1 b_1$$

Given ρ_i and α_i, $i = 1, 2$, one can easily solve $(\star_2)$ for η_i and μ_i, $i = 1, 2$ in the following way

$$\begin{cases} (\sharp)_1 & \eta_1 = (\rho_2/\rho_1) \text{ and } \eta_2 = 1 & \text{if} & \rho_1 > \rho_2 \\ (\sharp)_2 & \eta_1 = 1 \text{ and } \eta_2 = (\rho_1/\rho_2) & \text{if} & \rho_2 > \rho_1 \\ (\sharp)_3 & \eta_1 = \eta_2 = 1 & \text{if} & \rho_1 = \rho_2 \\ (\sharp)_4 & \mu_i = \eta_i \alpha_i & & \end{cases}$$

This solution of $(\star_2)$ is also a solution of $(\star)$ if $\max\{\eta_1, \mu_2\} = \max\{\eta_2, \mu_1\} = 1$; let us see that this is indeed the case. There are, formally, three cases to consider, namely $(\sharp)_1$, $(\sharp)_2$ and $(\sharp)_3$. In the first case, $(\sharp)_1$, we have $\max\{\eta_2, \mu_1\} = \eta_2 = 1$ and $\max\{\rho_2, \alpha_2\} = 1$ with $1 \geq \rho_1 > \rho_2 \geq 0$ from which we obtain $\alpha_2 = 1$ and $\mu_2 = \eta_2 = 1$; we have shown that $\max\{\eta_1, \mu_2\} = 1$. The second case, that is $(\sharp)_2$, is treated similarly; as for $(\sharp)_3$ there is nothing to prove since $\eta_1 = \eta_2 = 1$. □

*A family of sets is up-directed if the union of any two members of the family is contained in a member of the family.

Theorem 2.1 (The Stone-Kakutani Property). *If C_1 and C_2 are disjoint $\mathbb{B}$-convex sets, then there exists a $\mathbb{B}$-convex set $D \subset \mathbb{R}^n_+$ such that $\mathbb{R}^n_+ \setminus D$ is also $\mathbb{B}$-convex, $C_1 \subset D$ and $C_2 \subset \mathbb{R}^n_+ \setminus D$.*

Proof. Let $\mathcal{Z}$ be the family of pairs of disjoint convex sets (D_1, D_2) such that $C_i \subset D_i$ partially ordered by $(D_1, D_2) \subset (D'_1, D'_2)$ if $D_i \subset D'_i$. The pair (C_1, C_2) belongs to $\mathcal{Z}$ and if $\mathcal{C} = \{(D_{1,\lambda}, D_{2,\lambda}) : \lambda \in \Lambda\}$ is a chain in $\mathcal{Z}$ then $(\bigcup_{\lambda\in\Lambda} D_{1,\lambda}, \bigcup_{\lambda\in\Lambda} D_{2,\lambda}) \in \mathcal{Z}$ since an up-directed union of $\mathbb{B}$-convex sets is $\mathbb{B}$-convex and, as can easily be seen, $(\bigcup_{\lambda\in\Lambda} D_{1,\lambda}) \bigcap (\bigcup_{\lambda\in\Lambda} D_{2,\lambda}) = \emptyset$; by Zorn's lemma there is a maximal element (H_1, H_2) in $\mathcal{Z}$. Assume that there is a point a in $\mathbb{R}^n_+ \setminus (H_1 \bigcup H_2)$; from the maximality of the pair (H_1, H_2) we have $[\![\, H_1 \bigcup \{a\} \,]\!] \cap H_2 \neq \emptyset$ and $[\![\, H_2 \bigcup \{a\} \,]\!] \cap H_1 \neq \emptyset$. Take a point c_1 in the first set and a point c_2 in the second set. By Lemma 2.1 there exists $b_i \in H_i$ such that $c_i \in [\![\, a, b_i \,]\!]$. By the Pash-Peano Property there exists a point u in $[\![\, b_1, c_2 \,]\!] \bigcap [\![\, b_2, c_1 \,]\!]$. From $b_1, c_2 \in H_1$ and $b_2, c_1 \in H_2$ we obtain $u \in H_1 \cap H_2$, which is impossible since the pair (H_1, H_2) is in $\mathcal{Z}$. □

Corollary 2.1 (Generalized Pash-Peano Property). *If $a, b_1, \dots, b_m, c_1, \dots, c_m$ are points in $\mathbb{R}^n_+$ such that $c_i \in [\![\, a, b_i \,]\!]$ for all $i = 1, \dots m$ then $\bigcap_{i=1}^m [\![\, B_i \cup \{c_i\} \,]\!] \neq \emptyset$ where $B_i = \{b_j : j \neq i\}$.*

Proof. For $m = 1$ there is nothing to prove, for $m = 2$ we have the Pash-Peano Property; let us assume that $m \geq 3$ and that the conclusion has been established for values less or equal to $m - 1$. Let $B_i^{'} = B_i \setminus \{b_m\}$; by the induction hypothesis there is a point p in $\bigcap_{i=1}^{m-1} [\![\, B'_i \cup \{c_i\} \,]\!] \subset \bigcap_{i=1}^{m-1} [\![\, B_i \cup \{c_i\}]\!]$; since $b_m \in B_i$ for $i \leq m - 1$ we have $[\![\, p, b_m \,]\!] \subset \bigcap_{i=1}^{m-1} [\![\, B_i \cup \{c_i\} \,]\!]$. If $[\![\, p, b_m \,]\!] \cap [\![\, B_m \cup \{c_m\} \,]\!] \neq \emptyset$ the proof is over, otherwise, by the Stone-Kakutani Property, there exists a $\mathbb{B}$-convex set D whose complement in $\mathbb{R}^n_+$ is also $\mathbb{B}$-convex and such that $[\![\, p, b_m \,]\!] \subset D$ and $[\![\, B_m \cup \{c_m\} \,]\!] \cap D = \emptyset$. From $c_m \in [\![\, a, b_m \,]\!]$, $b_m \in D$ and $c_m \in \mathbb{R}^n_+ \setminus D$ we see that $a \in \mathbb{R}^n_+ \setminus D$ and consequently $\bigcup_{i\neq m} [\![\, a, b_i \,]\!] \subset \mathbb{R}^n_+ \setminus D$ which implies $c_i \in \mathbb{R}^n_+ \setminus D$ for $i \neq m$. For $i \neq m$ we have $B_i^{'} \subset B_m$ and therefore $[\![\, B'_i \cup \{c_i\} \,]\!] \subset \mathbb{R}^n_+ \setminus D$ which yields the obvious contradiction $p \notin D$. □

Let us call **half-spaces**† those $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ whose complement is also $\mathbb{B}$-convex.

Proposition 2.2 (Generalized Stone-Kakutani). *If $C_1, \dots, C_m$ are $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ such that $\bigcap_{i=1}^m C_i = \emptyset$, then there exists half-spaces $D_1, \dots, D_m$ such that $\bigcap D_i = \emptyset$, $\bigcup D_i = \mathbb{R}^n_+$ and, for all $i = 1, \dots, m$, $C_i \subset D_i$.*

Proof. Let $\mathcal{Z}$ be the set of m-tuples $(D_1, \dots, D_m)$ of $\mathbb{B}$-convex subsets of $\mathbb{R}^n_+$ such that $\bigcap_{i=1}^m D_i = \emptyset$ and, for all $i = 1, \dots, m$, $C_i \subset D_i$, partially ordered in the obvious way, that is, $(D_1, \dots, D_m)$ is greater than $(D_1^{'}, \dots, D_m^{'})$ if, for all i, $D_i \supseteq D_i^{'}$. Using Zorn's Lemma we find a maximal element $(H_1, \dots, H_m)$. For each $j = 1, \dots, m$ there is, by the Stone-Kakutani Property, a half-space D_j containing H_j which does not intersect $\bigcap_{i\neq j} H_i$; the n-tuple obtained by replacing H_j by D_j in $(H_1, \dots, H_m)$ is greater than $(H_1, \dots, H_m)$; by maximality, $H_j = D_j$. We have to show that $\bigcup_{i=1}^m H_i = \mathbb{R}^n_+$; for a contradiction, assume that this is not the case and let $a \in \mathbb{R}^n_+ \setminus \bigcup_{i=1}^m H_i$. By maximality of $(H_1, \dots, H_m)$, there exists, for all j, a point c_j in $[\![\, H_j \cup \{a\} \,]\!] \bigcap \left(\bigcap_{i\neq j} H_i\right)$ and therefore a point $b_j \in H_j$ such that $c_j \in [\![\, b_j \cup \{a\} \,]\!] \bigcap \left(\bigcap_{i\neq j} H_i\right)$. From the generalized Pash-Peano Property we have

† Also called hemi-spaces or maximal $\mathbb{B}$-convex sets.

$\bigcap_{j=1}^m [\![\{c_i : i \neq j\} \cup \{b_j\}]\!] \neq \emptyset$; we have reached a contradiction since $[\![\{c_i : i \neq j\} \cup \{b_j\}]\!] \subset H_j$. □

3 Gauges and Co-Gauges

We need the following definitions and results (see [3] for details). A set $U \subset \mathbb{R}^n_+$ is called radiant if $(x \in U, t \in]0,1])$ implies $tx \in U$. A radiant set containing 0 is starshaped at 0. A set $V \subset \mathbb{R}^n_+$ is called co-radiant if $0 \notin V$ and $(x \in V, t \geq 1)$ implies $tx \in V$. The Minkowski gauge μ_U of the radiant set U is defined by

$$\mu_U(x) = \inf\{\lambda \in]0,+\infty[: x \in \lambda U\}, \qquad x \in \mathbb{R}^n_+.$$

The Minkowski co-gauge ν_V of the co-radiant set V is defined by

$$\nu_V(x) = \sup\{\lambda \in]0,+\infty[: x \in \lambda V\}, \qquad x \in \mathbb{R}^n_+.$$

If U is a radiant set then $V = \mathbb{R}^n_+ \setminus U$ is co-radiant and $\nu_V = \mu_U$. If V is co-radiant then $U = \mathbb{R}^n_+ \setminus V$ is radiant and $\mu_U = \nu_V$.

For each $x \neq 0$ consider the ray $R_x = \{\lambda x : \lambda > 0\}$. Let U be a radiant set. It follows from the definition that $\mu_U(0) = 0$. If $x \neq 0$ then

$$\mu_U(x) = 0 \iff R_x \subset U, \qquad \mu_U(x) = +\infty \iff R_x \cap U = \emptyset. \tag{3.1}$$

Let V be a co-radiant set. Then $\nu_V(0) = 0$. If $x \neq 0$ then

$$\nu_V(x) = +\infty \iff R_x \subset V, \qquad \nu_V(x) = 0 \iff R_x \cap V = \emptyset. \tag{3.2}$$

Note that both μ_U and ν_V are positively homogeneous functions.[‡]

A set $U \subset \mathbb{R}^n$ is called radially closed, or closed-along-rays in the terminology of [3], if

$$(x \in \mathbb{R}^n_+, \lambda_k > 0, \lambda_k x \in U, k = 1, 2, \dots, \lambda_k \to \lambda) \implies \lambda x \in U. \tag{3.3}$$

A radiant set U is radially closed if and only if $U = \{x \in \mathbb{R}^n_+ : \mu_U(x) \leq 1\}$. A co-radiant set V is radially closed if and only if $V = \{x \in \mathbb{R}^n_+ : \nu_V(x) \geq 1\}$.

A set U is called radially open (or open-along-rays) if its complement $\mathbb{R}^n_+ \setminus U$ is radially closed. It is easy to check that a radiant set U (co-radiant set V, respectively) is radially open if and only if $U = \{x \in \mathbb{R}^n_+ : \mu_U(x) < 1\}$ ($V = \{x \in \mathbb{R}^n_+ : \nu_V(x) > 1\}$, respectively).

A set $U \subset \mathbb{R}^n_+$ is called an upper semilattice if $(x, y \in U \implies x \vee y \in U)$. Note that each $\mathbb{B}$-convex set is an upper semilattice.

Proposition 3.1. *A subset of $\mathbb{R}^n_+$ containing 0 is $\mathbb{B}$-convex if and only if it is an upper semilattice starshaped at 0. A set which is radiant is $\mathbb{B}$-convex if and only if it is an upper semilattice.*

Proof. If B is $\mathbb{B}$-convex then $x \vee y \in B$ for all x and y in B; if $0 \in B$ then $tx = 0 \vee tx \in B$ for all $x \in B$ and all $t \in [0,1]$. Reciprocally, if B is starshaped at 0 (resp. radiant) then $tx \in B$ if $x \in B$ and $t \in [0,1]$ (resp. $t \in]0,1]$) and, if B is also an upper semilattice of $\mathbb{R}^n_+$ then $tx \vee y \in B$ for all x and y in B and $t \in [0,1]$. □

Lemma 3.1. (1) *Let $U \subset \mathbb{R}^n_+$ be a radiant set.*

[‡]A function $f : \mathbb{R}^n_+ \to [0,+\infty]$ is called positively homogeneous if $f(\lambda x) = \lambda f(x)$ for $\lambda \in]0,+\infty[$.

(*a*) *If U is $\mathbb{B}$-convex then*

$$\mu_U(tx \vee y) \le \max\{\mu_U(x), \mu_U(y)\} \quad \text{for all} \quad x, y \in \mathbb{R}^n_+,\ t \in [0,1]. \tag{3.4}$$

(*b*) *If U is radially closed or radially open and (3.4) holds then U is $\mathbb{B}$-convex.*

(2) *Let $V \subset \mathbb{R}^n_+$ be a co-radiant set.*

(*a*) *If V is $\mathbb{B}$-convex then*

$$\nu_V(tx \vee y) \ge \min\{\nu_V(x), \nu_V(y)\} \quad \text{for all} \quad x, y \in \mathbb{R}^n_+,\ t \in [0,1]. \tag{3.5}$$

(*b*) *If V is radially closed or radially open and (3.5) holds then V is $\mathbb{B}$-convex.*

Proof. (1) Let U be a radiant $\mathbb{B}$-convex set. Consider points $x, y \in \mathbb{R}^n_+$ and $t \in [0,1]$. Assume for the sake of definiteness that $\mu_U(tx) \le \mu_U(y) < +\infty$. If $r > \mu_U(y)$ then $tx \in rU, y \in rU$. From $r^{-1}(tx \vee y) = r^{-1}tx \vee r^{-1}y$ we have $r^{-1}(tx \vee y) \in U$; this shows that (3.4) holds. If either $\mu_U(tx) = +\infty$ or $\mu_U(y) = +\infty$ then (3.4) is obvious. Let U be radiant. If U is radially closed, then $U = \{x \in \mathbb{R}^n_+ : \mu_U(x) \le 1\}$, if U is radially open then $U = \{x \in \mathbb{R}^n_+ : \mu_U(x) < 1\}$. In both cases (3.4) implies that U is a $\mathbb{B}$-convex set.

(2) Let V be a co-radiant and $\mathbb{B}$-convex set. Consider points $x, y \in V$ and a number $t \in [0,1]$. Assume for the sake of definiteness that $+\infty \ge \nu_V(tx) \ge \nu_V(y) > 0$. Let $0 < r < \nu_V(y) \le \nu_V(ty)$. Since V is co-radiant it follows that $tx \in rV$, $y \in rV$. $\mathbb{B}$-convexity of V implies $\mathbb{B}$-convexity of rV, so $(tx \vee y) \in rV$. This means that $\nu_V(tx \vee y) \ge \nu_V(y)$, hence (3.5) holds. If either $\nu_V(x) = 0$ or $\nu_V(y) = 0$ then (3.5) is obvious. If V is either radially closed or open and (3.5) holds then clearly V is a $\mathbb{B}$-convex set. □

Corollary 3.1. *If a radiant set $U \subset \mathbb{R}^n_+$ is $\mathbb{B}$-convex then*

$$\mu_U(x \vee y) \le \max\{\mu_U(x), \mu_U(y)\} \quad \text{for all} \quad x, y \in \mathbb{R}^n_+. \tag{3.6}$$

If U is radiant, either radially closed or radially open, and (3.6) holds then U is $\mathbb{B}$-convex.

Proposition 3.2. (1) *Let B be a half-space containing 0. Then, for all $x, y \in \mathbb{R}^n_+$ and $t \in [0,1]$,*

$$\min\{\mu_B(x), \mu_B(y)\} \le \mu_B(tx \vee y) \le \max\{\mu_B(x), \mu_B(y)\}. \tag{3.7}$$

If B is $\mathbb{B}$-convex, either radially closed or open and the left-hand inequality in (3.7) holds then B is a half-space.

(2) *Let B be a half-space that does not contain 0. Then, for all $x, y \in \mathbb{R}^n_+$ and $t \in [0,1]$,*

$$\min\{\nu_B(x), \nu_B(y)\} \le \nu_B(tx \vee y) \le \max\{\nu_B(x), \nu_B(y)\}. \tag{3.8}$$

If B is $\mathbb{B}$-convex, either radially closed or open and the right-hand inequality in (3.8) holds then B is a half-space.

Proof. (1) Let B be a half-space and $0 \in B$. Let $x, y \in \mathbb{R}^n_+$ and $t \in [0,1]$. Since B is $\mathbb{B}$-convex and radiant it follows from (3.4) that the right-hand inequality in (3.7) holds. Since $C = \mathbb{R}^n_+ \setminus B$ is $\mathbb{B}$-convex and co-radiant it follows from (3.5) that

$$\nu_C(tx \vee y) \ge \min\{\nu_C(x), \nu_C(y)\}.$$

From $\mu_B = \nu_C$ we obtain the left-hand inequality in (3.7).

Let B be $\mathbb{B}$-convex and radially closed and $C = \mathbb{R}^n_+ \setminus B$. Then C is radially open. The left-hand inequality in (3.7) can be presented as $\min\{\nu_C(x), \nu_C(y)\} \leq \nu_C(tx \vee y)$. Applying Lemma 3.1 we conclude that C is $\mathbb{B}$-convex, hence B is a half-space. A similar argument can be used if B is radially open.

(2) Let B be a half-space and $0 \notin B$. Let $x, y \in \mathbb{R}^n_+$ and $t \in [0,1]$. Since B is $\mathbb{B}$-convex and co-radiant it follows from (3.5) that the left-hand inequality in (3.8) holds. Since $C = \mathbb{R}^n_+ \setminus B$ is $\mathbb{B}$-convex and radiant it follows from (3.4) that

$$\mu_C(tx \vee y) \leq \max\{\mu_C(tx), \mu_C(y)\}.$$

This implies the right-hand inequality in (3.8).

Let B be $\mathbb{B}$-convex and radially closed and $C = \mathbb{R}^n_+ \setminus B$. Then C is radially open. The right-hand inequality in (3.8) can be presented as $\mu_C(tx \vee y) \leq \max\{\mu_C(x), \mu_C(y)\}$. Applying Lemma 3.1 we conclude that C is $\mathbb{B}$-convex, hence B is a half-space. A similar argument can be used if B is radially open. □

Corollary 3.2. *Let $B \subset \mathbb{R}^n_+$ be a conic set, that is $(x \in B, \lambda \geq 0) \implies \lambda x \in B$. Then*

$$\mu_B(x) = \nu_C(x) = \begin{cases} 0 & \text{if } x \in B \\ +\infty & \text{if } x \in C \end{cases}, \tag{3.9}$$

where $C = \mathbb{R}^n_+ \setminus B$. If B is an upper semilattice then B is a half-space.

Indeed, since B and C are both radiant and co-radiant sets we can consider μ_B and ν_C and (3.9) follows from (3.1) and (3.2).

Assume that B is an upper semilattice. Then for each $x, y \in B$ and $t \in [0,1]$ we have $\mu_B(x) = \mu_B(y) = \mu_B(tx \vee y) = 0$. It follows from Proposition 3.2 that B is a half space.

Gauges and co-gauges are defined for arbitrary radiant and co-radiant sets; if those sets are also $\mathbb{B}$-convex then the gauge and the co-gauge have additional algebraic properties but the $\mathbb{B}$-convexity structure has in itself little bearing on the continuity properties. For the sake of completeness we state the following result which follows from Propositions 5.2 and 5.10 of [3].

Proposition 3.3. *Let U be a radially closed $\mathbb{B}$-convex containing 0, then*
(1) μ_U is lower semicontinuous if and only U is closed;
(2) μ_U is continuous if and only if U is closed, 0 is in the relative interior of U (with respect to $\mathbb{R}^n_+$) and, for all $x \in \mathbb{R}^n_+$, R_x does not intersect the boundary of U more than once.

4 $\mathbb{B}$-Measurable Maps and Half-Spaces

A map $f : \mathbb{R}^n_+ \to [0, \infty]$ is called $\mathbb{B}$-measurable if, for all $x, y \in \mathbb{R}^n_+$ and $t \in [0,1]$,

$$\min\{f(x), f(y)\} \leq f(tx \vee y) \leq \max\{f(x), f(y)\}. \tag{4.10}$$

A $\mathbb{B}$-measurable map is characterized by the fact that inverse images of intervals are $\mathbb{B}$-convex; it follows that level sets $\{x \in \mathbb{R}^n_+ : f(x) = \lambda\}$ of $\mathbb{B}$-measurable maps are $\mathbb{B}$-convex sets, [1]. The gauge of a half-space containing 0 and the co-gauge of a half-space that does not contain 0 are positively homogeneous $\mathbb{B}$-measurable maps. One can easily check that for an homogeneous $\mathbb{B}$-measurable map $f : \mathbb{R}^n_+ \to [0, \infty]$ and for all $\lambda > 0$, $\{x \in \mathbb{R}^n_+ : f(x) < \lambda\}$

is a radially open half-space containing 0, $\{x \in \mathbb{R}^n_+ : f(x) \leq \lambda\}$ is a radially closed half-space and $\{x \in \mathbb{R}^n_+ : f(x) = 0\}$ is a conic set and an upper semilattice of $\mathbb{R}^n_+$.

A subset $\{u_1, \dots, u_m\}$ of $\mathbb{R}^n_+$ is a spanning set if, for all $x \in \mathbb{R}^n_+$ there exist positive real numbers $r_1, \dots, r_m$ such that $x = r_1 u_1 \vee \dots \vee r_m u_m$. It is easy to see that for a minimal spanning set $\{u_1, \dots, u_m\}$ we have $m = n$ and, up to relabeling, $u_k = t_k e_k$ where $e_1, \dots, e_n$ are the vectors of the canonical bases of $\mathbb{R}^n$ and $t_k > 0$. We consider only spanning sets of this form later on.

Lemma 4.1. *Let* $f : \mathbb{R}^n_+ \to [0, \infty]$ *be a positively homogeneous* $\mathbb{B}$*-measurable map, then*
(1) f *takes only finite values if and only if there exists a spanning set* $\{u_1, \dots, u_m\}$ *such that* $\max\{f(u_1), \dots, f(u_m)\} < +\infty$*;*
(2) $\operatorname{dom} f := \{x \in \mathbb{R}^n_+ : f(x) < +\infty\}$, $\{x \in \mathbb{R}^n_+ : f(x) = +\infty\} \cup \{0\}$ *and* $\{x \in \mathbb{R}^n_+ : f(x) = 0\}$ *are conic upper semilattices;*
(3) *if* $\operatorname{dom}(f) = \mathbb{R}^n_+$ *then*

- (a) *if* $\min\{f(x), f(y)\} \neq 0$ *then* $f(x \vee y) = \max\{f(x), f(y)\}$*;*
- (b) *if* $\{u \in \mathbb{R}^n_+ : f(u) = 0\} = \{0\}$ *then there exists a unique* $a \in \mathbb{R}^n_{++}$ *such that, for all* $x \in \mathbb{R}^n_+$, $f(x) = \max_{1 \leq i \leq n}\{x_i a_i\}$*;*
- (c) *if* $\{u \in \mathbb{R}^n_+ : f(u) = 0\} \neq \{0\}$ *then there exists a partition of* $I = \{1, \dots, n\}$ *into two subsets* I^0 *and* I^+ *and there exists a unique* $a \in \mathbb{R}^n_+$ *such that*
  - (α) $f(e_i) = 0$ *if and only if* $i \in I^0$ *and* $a_i = 0$ *if* $i \in I^0$*;*
  - (β) *if* $f(x) \neq 0$ *then* $f(x) = \max_{i \in I}\{a_i x_i\} = \max_{i \in I^+}\{a_i x_i\}$.

Proof. Part (1) follows from $f(r_1 u_1 \vee \dots \vee r_m u_m) \leq \max_{1 \leq i \leq m}\{r_i f(u_i)\}$ and part (2) from (4.10) and the fact that $f(x) = t^{-1} f(tx)$ if $f(x) \neq +\infty$ and $t > 0$.

Assume that f does not take the value $+\infty$ and let x and y in $\mathbb{R}^n_+$ such that $0 < f(x) \leq f(y)$; then $f(x)^{-1}x$ and $f(y)^{-1}y$ belong to $L(f;1) = \{u \in \mathbb{R}^n_+ : f(u) = 1\}$ which is $\mathbb{B}$-convex since f is $\mathbb{B}$-measurable, and therefore

$$(f(x)f(y)^{-1})f(x)^{-1}x \vee f(y)^{-1}y \in L(f;1),$$

that is $f(f(y)^{-1}(x \vee y)) = 1$ or $f(x \vee y) = f(y) = \max\{f(x), f(y)\}$. By induction we can show that if $\min_{1 \leq j \leq m} f(x_j) > 0$ then

$$f(x_1 \vee \dots \vee x_m) = \max\{f(x_1), \dots, f(x_m)\}. \tag{4.11}$$

If $f(e_i) > 0$ for all $i \in I$ then due to (4.11) and positive homogeneity of f we have for all $x \in \mathbb{R}^n_+$:

$$f(x) = \max_{\{i : x_i \neq 0\}}\{x_i f(e_i)\} = \max_{1 \leq i \leq n}\{x_i f(e_i)\};$$

consequently, if $L(f;0) = \{u \in \mathbb{R}^n_+ : f(u) = 0\}$ is not $\{0\}$ then there is at least one index i for which $f(e_i) = 0$. Let $I^0 = \{i : f(e_i) = 0\}$ and $I^+ = \{i : f(e_i) \neq 0\}$ and for $x \in \mathbb{R}^n_+$ let $x_+ = \bigvee_{i \in I^+} x_i e_i$ and $x_0 = \bigvee_{i \in I^0} x_i e_i$; from the previous computations we have $f(x_+) = \max_{i \in I^+}\{x_i f(e_i)\} = \max_{i \in I}\{x_i f(e_i)\}$. Since f is $\mathbb{B}$-measurable and positively homogenous we have

$$0 \leq f(x_0) \leq \max_{i \in I^0}\{f(x_i e_i)\} = \max_{i \in I^0}\{x_i f(e_i)\} = 0,$$

and therefore $f(x_0) = 0$; from $x = x_+ \vee x_0$ we obtain $f(x) \leq \max\{f(x_+), f(x_0)\} = f(x_+)$. If $f(x) \neq 0$ we obtain, taking into account that $x = x \vee x_+$,

$$f(x) = \max\{f(x), f(x_+)\} = f(x_+) = \max_{i \in I}\{x_i f(e_i)\}.$$

If $f(x) = \max_{i \in I}\{x_i f(e_i)\}$ for all x such that $f(x) \neq 0$ then $f(e_i) = a_i$ for all indices i such that $f(e_i) \neq 0$. □

Corollary 4.1. *If $U \subset \mathbb{R}^n_+$ is a half-space containing 0 in its relative interior and no half-rays then there exists a unique $(a_1, \dots, a_n) \in \mathbb{R}^n_{++}$ such that $\mu_U(x) = \max_{1 \leq i \leq n}\{a_i x_i\}$. If U is closed and bounded then $\overline{x} = (a_1^{-1}, \dots, a_n^{-1})$ is the maximal element of U and $U = \{x \in \mathbb{R}^n_+ : x \leq \overline{x}\}$; $\overline{x}$ is also the unique point of U where the map $(x_1, \dots, x_n) \mapsto x_1 + \dots + x_n$ attains its maximum value.*

Proof. Since 0 is in the relative interior of U, the domain of μ_U is $\mathbb{R}^n_+$ and if $\mu_U(x) = 0$ then $x = 0$ since there are no half-rays in U. This proves the first part. To prove the second part notice first that $U = \{x \in \mathbb{R}^n_+ : \mu_U(x) \leq 1\}$, since U is closed and radial, and therefore, from $\max_{1 \leq i \leq n}\{\overline{x}_i a_i\} = 1$, we have $\overline{x} \in U$. The inequality $y \leq \overline{x}$, for $y \in \mathbb{R}^n_+$, is clearly equivalent to $y_i a_i \leq 1$ for all i, this proves the second part. If $y \leq \overline{x}$ then $y_1 + \dots + y_n \leq \overline{x}_1 + \dots + \overline{x}_n$. □

Corollary 4.2. *A compact nonempty $\mathbb{B}$-convex set B has a unique maximal element $\overline{x}$, it is the unique point of B where the map $(x_1, \dots, x_n) \mapsto x_1 + \dots + x_n$ attains its maximum value. If $f : B \to \mathbb{R}_+$ is $\mathbb{B}$-measurable and positevely homogeneous then either $f(\overline{x}) = 0$ or $f(\overline{x}) = \max\{f(x) : x \in B\}$.*

Proof. If $y \in B$ then $y \vee \overline{x} \in B$ and therefore $\sum_{i=1}^n \max\{y_i, \overline{x}_i\} \leq \sum_{i=1}^n \overline{x}_i$, this shows that $y_i \leq \overline{x}_i$. If $f : B \to \mathbb{R}_+$ is $\mathbb{B}$-measurable and $f(\overline{x}) \neq 0$ then, for all $y \in B$, $\min\{f(\overline{x}), f(y)\} \neq 0$ and therefore $f(\overline{x} \vee y) = \max\{f(\overline{x}), f(y)\}$. But $\overline{x} \vee y = \overline{x}$, therefore $f(\overline{x}) \geq f(y)$. □

5 Separation of $\mathbb{B}$-Convex Sets by a Map

Let us say that two sets A and B are

(a) **weakly separated** by a map f if $\sup_{x \in A} f(x) \leq \inf_{x \in B} f(x)$ or $\sup_{x \in B} f(x) \leq \inf_{x \in A} f(x)$;

(b) **separated** by a map f if there exists a real number r such that either $\forall (x, y) \in A \times B$ $f(x) < r < f(y)$ or $\forall (x, y) \in A \times B$ $f(y) < r < f(x)$;

(c) **strictly separated** by a map f if $\sup_{x \in A} f(x) < \inf_{x \in B} f(x)$ or $\sup_{x \in B} f(x) < \inf_{x \in A} f(x)$.

The following lemma will be used in the examination of separation properties of $\mathbb{B}$-convex sets.

Lemma 5.1. *If C_1 and C_2 are $\mathbb{B}$-convex sets then $[\![C_1 \cup C_2]\!] = \{sx \vee ty : (x, y) \in C_1 \times C_2$ and $s \geq 0, t \geq 0, \max\{s, t\} = 1\}$.*

Proof. One inclusion being obvious,we have to show that

$$C = \{sx \vee ty : (x, y) \in C_1 \times C_2 \text{ and } s \geq 0, t \geq 0, \max\{s, t\} = 1\}$$

is $\mathbb{B}$-convex. Let $u_i = s_i x_i \vee t_i y_i$, $i = 1, 2$, with $x_i \in C_1$, $y_i \in C_2$, $0 \le \min\{s_i, t_i\}$, $\max\{s_i, t_i\} = 1$ and let $t \in [0, 1]$. Then

$$\max\{\max\{ts_1, s_2\}, \max\{tt_1, t_2\}\} = \max\{ts_1, tt_1, s_2, t_2\} = 1.$$

Let $\alpha = \max\{ts_1, s_2\}$ and $\beta = \max\{tt_1, t_2\}$; if $\alpha = 0$ then either $s_1 = s_2 = 0$, in which case $t_1 = t_2 = 1$ and $tu_1 \vee u_2 = ty_1 \vee y_2 \in C_2 \subset C$ or $t = s_2 = 0$, in which case $tu_1 \vee u_2 = u_2 = y_2 \in C_2 \subset C$; similarly, if $\beta = 0$ we have $tu_1 \vee u_2 \in C_1 \subset C$. We can now assume that $\alpha \neq 0$ and $\beta \neq 0$; Then $x = (\alpha^{-1} ts_1)x_1 \vee (\alpha^{-1} s_2)x_2 \in C_1$ and $y = (\beta^{-1} tt_1)y_1 \vee (\beta^{-1} t_2)y_2 \in C_2$, therefore $tu_1 \vee u_2 = \alpha x \vee \beta y \in C$. □

Theorem 5.1. (1) *Two disjoint $\mathbb{B}$-convex sets C_1 and C_2 can be weakly separated by a $\mathbb{B}$-measurable positively homogenous map.*

(2) *Furthermore, if there exists a vector $u \in \mathbb{R}^n_{++}$ such that $C_1 \cup C_2 \subset \mathbb{R}^n_+ + u$, then C_1 and C_2 can be weakly separated by a finite $\mathbb{B}$-measurable positively homogenous map.*

(3) *If, on top of the conditions from* (1) *and* (2) *above, we also have* $\inf_{(x,y)\in C_1\times C_2} ||x - y||_\infty > 0$, *then C_1 and C_2 can be separated by a finite $\mathbb{B}$-measurable positively homogenous map.*

(4) *Under all the previous assumptions, if either C_1 or C_2 is bounded then they can be strictly separated by a finite $\mathbb{B}$-measurable positively homogenous map.*

Proof. (1). Let C_i, $i = 1, 2$ be disjoint $\mathbb{B}$-convex sets of $\mathbb{R}^n_+$; by Theorem 2.1 there exists a half-space B such that $C_1 \subset B$ and $C_2 \subset \mathbb{R}^n_+ \setminus B$, where, without loss of generality, we can assume that $0 \in B$. Then , for all $x \in C_1$, $\mu_B(x) \le 1$ and for all $x \in C_2$, $1 \le \mu_B(x)$.

(2) First, if $C_1 \cup C_2 \subset \mathbb{R}^n_+ + u$ with $u \in \mathbb{R}^n_{++}$ then $C_1 \cup C_2 \subset \mathbb{R}^n_{++} + 2^{-1}u$; we can therefore assume that $C_1 \cup C_2 \subset \mathbb{R}^n_{++} + u$. Let $B_\delta = \{x \in \mathbb{R}^n_+ : ||x||_\infty \le \delta\}$ and assume that

$$[\![B_\delta \cup C_1]\!] \cap C_2 \neq \emptyset \quad \text{for all} \quad \delta > 0.$$

By Lemma 5.1 there are two possibilities;

(a) either there exists a sequence of elements of the form $x_k \vee y_k \in C_2$ with $||x_k||_\infty \le \delta_k$, $y_k \in C_1$ and δ_k decreases to 0, or

(b) there exists a sequence of elements of the form $x_k \vee t_k y_k \in C_2$ with x_k and y_k as in (a) and $t_k \in [0, 1]$.

In the first case, since $u \in \mathbb{R}^n_{++}$ and $u \le y_k$, we have $x_k \vee y_k = y_k$ if k is large enough; we would then have $y_k \in C_1 \cap C_2$, which is impossible; (b) is therefore the case. Let $u_k = x_k \vee t_k y_k$; since $u_k \in C_2$ we have $u_k = u \vee u_k$ or, $u_k = (u \vee x_k) \vee t_k y_k$. From $u \in \mathbb{R}^n_{++}$ we have $u \vee x_k = u$ for k large enough, therefore, $u_k = u \vee t_k y_k$, but $u_k \in \mathbb{R}^n_{++} + u$ and consequently, $u_k = t_k y_k$ for k large enough. In conclusion, we have shown that, if, for all $\delta > 0$, $[\![B_\delta \cup C_1]\!] \cap C_2 \neq \emptyset$ then there exists $y \in C_1$ and $t \in]0, 1[$ such that $ty \in C_2$. Let us see that we cannot have for all $\delta > 0$, $[\![B_\delta \cup C_1]\!] \cap C_2 \neq \emptyset$ and $[\![B_\delta \cup C_2]\!] \cap C_1 \neq \emptyset$. If this were the case there would exist $y \in C_1$, $z \in C_2$ and $t, s \in]0, 1[$ such that $ty \in C_2$ and $sz \in C_1$; we have $ty \in [\![0, y]\!]$ and $sz \in [\![0, z]\!]$, by Proposition 2.1 there exists a point w in $[\![y, sz]\!] \cap [\![z, ty]\!]$. From $[\![y, sz]\!] \subset C_1$ and $[\![z, ty]\!] \subset C_2$ we obtain $w \in C_1 \cap C_2$. We can assume that there exists $\delta > 0$ such that $[\![B_\delta \cup C_1]\!] \cap C_2 \neq \emptyset$; let $C = [\![B_\delta \cup C_1]\!]$ and find a half-space B such that $C \subset B$ and $C_2 \cap B = \emptyset$. From $0 \in C$ we have $0 \in B$ and from $\delta e_i \in C$, $i = 1, \ldots, n$, we have $\mu_B(e_i) \le \delta^{-1}$ and, by part (1) of Lemma 4.1, μ_B takes only finite values.

(3) Let $\eta = \inf_{(x,y)\in C_1\times C_2} ||x-y||_\infty$ and, for all subset $S \subset \mathbb{R}^n_+$ let $B_\delta(S) = \{x \in \mathbb{R}^n_+ : \exists y \in S \text{ s.t. } ||x-y||_\infty \leq \delta\}$; if C is $\mathbb{B}$-convex then, for all $\delta \geq 0$, $B_\delta(C)$ is also $\mathbb{B}$-convex by Proposition 2.4.2 . of [1], which also implies that the closure of a $\mathbb{B}$-convex set is $\mathbb{B}$-convex. Since $\eta > 0$ we can choose $u \in \mathbb{R}^n_{++}$, $\mu > 0$ such $\mu < ||u||_\infty$ and $B_\mu(C_1)\cap B_\mu(C_2) = \emptyset$ and also $B_\mu(C_1)\cup B_\mu(C_2) \subset \mathbb{R}^n_{++}+u$, since, by hypothesis, $C_1\cup C_2 \subset \mathbb{R}^n_+ + v$ for some $v \in \mathbb{R}^n_{++}$. From part (2) we find a half-space B such that $0 \in B$, $B_\mu(C_1) \subset B$, $B_\mu(C_2) \subset \mathbb{R}^n_+ \setminus B$, μ_B is finite valued and $\sup_{x\in B_\mu(C_1)} \mu_B(x) \leq 1 \leq \inf_{x\in B_\mu(C_2)} \mu_B(x)$. If $x \in C_1$ and $t = (1+\mu/(2||x||_\infty))$ then then $tx \in B_\mu(C_1) \subset B$, and therefore,

$$\mu_B(x) \leq \left(1 + \frac{\mu}{2||x||_\infty}\right)^{-1} < 1.$$

If $y \in C_2$ then $y \in \mathbb{R}^n_+ + u$ and therefore $||y||_\infty > \mu$ and therefore $s = (1-\mu/(2||y||_\infty))$ is strictly positive. Since $||sy-y||_\infty = \mu/2$ we have $sy \in B_\mu(C_2) \subset \mathbb{R}^n_+ \setminus B$, and therefore $sy \notin B$ which implies that

$$1 < \left(1 - \frac{\mu}{2||y||_\infty}\right)^{-1} \leq \mu_B(y).$$

(4) Now, assume that C_1 is bounded. There is a $v \in \mathbb{R}^n_+$ such that, for all $x \in C_1$, $||x||_\infty \leq ||v||_\infty$ and therefore

$$\sup_{x\in C_1} \mu_B(x) \leq \left(1 + \frac{\mu}{||v||_\infty}\right)^{-1} < 1 \leq \inf_{y\in C_2} \mu_B(y).$$

If C_2 is bounded there is a $v \in \mathbb{R}^n_+$ such that, for all $y \in C_2$, $||y||_\infty \leq ||v||_\infty$, and $\mu < ||v||_\infty$; we then have

$$\sup_{x\in C_1} \mu_B(x) \leq 1 < \left(1 - \frac{\mu}{||v||_\infty}\right)^{-1} \leq \inf_{y\in C_2} \mu_B(y).$$

□

Theorem 5.2. *A point can be strictly separated from any closed $\mathbb{B}$-convex set to which it does not belong by a finite $\mathbb{B}$-measurable positively homogenous map.*

Proof. Let $C \subset \mathbb{R}^n_+$ be a closed $\mathbb{B}$-convex set and $x \in \mathbb{R}^n_+$ a point which is not in C. If $x = 0$ we take $f(x) = ||x||_\infty$; from $0 \notin C$, and C closed, we have $f(x) = 0 < \inf_{y\in C} f(y)$; we can assume now that $x \neq 0$.

Let us see that we cannot have, for all $\delta > 0$, $[\![B_\delta \cup \{x\}]\!] \bigcap C \neq \emptyset$ and $x \in [\![B_\delta \cup C]\!]$. First, notice that $[\![B_\delta \cup \{x\}]\!]$ is closed and bounded; if $[\![B_\delta \cup \{x\}]\!] \bigcap C \neq \emptyset$ for all $\delta > 0$ then, since $[\![B_\delta \cup \{x\}]\!] \bigcap C$ is compact and

$$[\![B_\eta \cup \{x\}]\!] \subset [\![B_\delta \cup \{x\}]\!] \quad \text{if} \quad \eta \leq \delta,$$

we conclude that $\bigcap_{\delta>0}([\![B_\delta \cup \{x\}]\!] \bigcap C)$ is not empty.

An element y of $\bigcap_{\delta>0}[\![B_\delta \cup \{x\}]\!]$ is of the form tx with $t \in [0,1]$; indeed, we have $y = x_k \vee t_k x$ where we can assume that t_k converges to some $t^\star \in [0,1]$ and x_k converges to 0 in $\mathbb{R}^n_+$, therefore $x_k \vee t_k x$ converges to $t^\star x$.

If $x \in \bigcap_{\delta>0}[\![B_\delta \cup C]\!]$ then we can find a sequence δ_k which decreases to 0, a sequence $t_k \in [0,1]$, a sequence $u_k \in \mathbb{R}^n_+$ with $||u_k||_\infty \leq \delta_k$ and a sequence $y_k \in C$ such that, for all k, $x = u_k \vee t_k y_y$. Since $x \neq 0$ there is at least one index i for which $x_i > 0$; from

$x_i = \max\{u_{k,i}, t_k y_{k,i}\}$ for all k there exists $k(i)$ such that, for all $k \geq k(i)$, $x_i = t_k y_{k,i}$. If $x_j = 0$ then $0 = \max\{u_{k,j}, t_k y_{k,j}\}$ and in particular $0 = t_k y_{k,j}$, that is, $x_j = t_k y_{k,j}$; let $K = \max\{k(i) : x_i > 0\}$ then, for $k \geq K$, we have $x = t_k y_k$. In conclusion, if for all $\delta > 0$, $[\![B_\delta \cup \{x\}]\!] \bigcap C \neq \emptyset$ and $x \in [\![B_\delta \cup C]\!]$ then there exits $t \in [0,1]$ such that $tx \in C$ and there exist $s \in [0,1]$ and $y \in C$ such that $x = sy$; but $(tx) \vee sy \in C$, since tx and y are in C, and $(tx) \vee sy = (tx) \vee x = x$; we have obtained $x \in C$, which is not the case by hypothesis. So, either

(a) there exists $\delta > 0$ such that $[\![B_\delta \cup \{x\}]\!] \bigcap C = \emptyset$ or

(b) there exists $\delta > 0$ such that $x \notin [\![B_\delta \cup C]\!]$.

If (a) is the case, we find $\mu > 0$ such that $B_\mu([\![B_\delta \cup \{x\}]\!]) \bigcap C = \emptyset$, which is possible since $[\![B_\delta \cup \{x\}]\!]$ is compact and C is closed. There is a half-space B such that $B_\mu([\![B_\delta \cup \{x\}]\!]) \subset B$ and $C \cap B = \emptyset$; μ_B is finite and, proceeding as in the proof of Theorem 5.1 we obtain

$$\mu_B(x) \leq \left(1 + \frac{\mu}{||x||_\infty}\right)^{-1} < 1 \leq \inf_{y \in C} \mu_B(y).$$

If (b) is the case, we find $\mu > 0$ such that $B_\mu(x) \cap [\![B_\delta \cup C]\!] = \emptyset$ and a half-space B such that $[\![B_\delta \cup C]\!] \subset B$ and $B_\mu(x) \cap B = \emptyset$; we can always assume that $\mu < ||x||_\infty$. Proceeding again as in the proof of Theorem 5.1 we obtain

$$\sup_{y \in C} \mu_B(y) \leq 1 < \left(1 - \frac{\mu}{||x||_\infty}\right)^{-1} \leq \mu_B(x).$$

□

6 Separation of a Point from a 𝔹-Convex Set

This section contains an explicit construction of a family $\mathcal{M}$ of open half-spaces such that all 𝔹-convex sets are intersections of members of $\mathcal{M}$ and $\mathcal{M}$ is minimal with respect to that property. For a point $z \in \mathbb{R}^n_{++}$ and a closed 𝔹-convex set to which it does not belong a separating map is explicitely given.

We also show that the 𝔹-convexity structure on $\mathbb{R}^n_{++}$ can be reconstructed from $n+1$ of partial orders. For $z \in \mathbb{R}^n_+$ let

$$N_0(z) = \{x : 0 \leq x \leq z\}$$

and

$$N_j(z) = \{x \in \mathbb{R}^n_+ : x_j \geq z_j \text{ and } \forall i \; z_i x_j \geq z_j x_i\}.$$

The set $N_0(z)$ is closed convex and radiant and the sets $N_j(z)$, for $j \geq 1$ are closed convex and co-radiant. If $z_j = 0$ then $N_j(z) = \mathbb{R}^n_+$; if $z_j \neq 0$ and $x \in N_j(z)$ then $x_j \neq 0$ and $x_i \neq 0$ implies $z_i \neq 0$, in other words $I(x) \subset I(z)$. For $z_j \neq 0$ let $F_j(z)$ be the j-th face of the polytope $N_0(z)$, that is

$$F_j(z) = \{x \in \mathbb{R}^n_+ : x_j = z_j \text{ and } \forall i \, x_i \leq z_i\}$$

then $N_j(z)$ is the conic hull of $F_j(z)$, that is

$$N_j(z) = \{tx : t \geq 1 \text{ and } x \in F_j(z)\}.$$

For $j \in \{0, 1 \dots, n\}$ let

$$M_j(z) = \mathbb{R}^n_+ \setminus N_j(z) \text{ and } U_j(z) = \{x \in \mathbb{R}^n_+ : z \in N_j(x)\}.$$

Notice that for, $j \neq 0$, $x \in M_j(z)$ if and only if $z_j > 0$ and either $x_j < z_j$ or $x_j \geq z_j$ and there exists an index l such that $x_j z_l < x_l z_j$ which is equivalent to

$$\begin{cases} (a) \text{ either } z_l = 0 \text{ and } x_l > 0 \quad \text{or} \\ (b) \frac{x_j}{z_j} < \max\left\{1, \max_{i \in I(z), i \neq j} \frac{x_i}{z_i}\right\} \text{ where } I(z) = \{i : z_i > 0\}. \end{cases}$$

Lemma 6.1. *The sets $N_j(z)$ are closed and $\mathbb{B}$-convex, the sets $M_j(z)$ are open and $\mathbb{B}$-convex; they are therefore half-spaces.*

Proof. That $N_0(z)$ is closed is obvious; it is a radiant upper semilattice and therefore $\mathbb{B}$-convex. If x and y belong to $M_0(z)$ then there are indices i and j such that $x_i > z_i$ and $y_l > z_l$; if $t \in [0, 1]$ then $\max\{x_i, ty_i\} \geq x_i > z_i$ and $\max\{tx_l, y_l\} \geq y_l > z_l$. This proves that $M_0(z)$ is $\mathbb{B}$-convex.

Let $j \in \{1, \dots, n\}$; if $z_j = 0$ then $N_j(z) = \mathbb{R}^n_+$ and $M_j(z) = \emptyset$ are $\mathbb{B}$-convex. Assume that $z_j > 0$ and let $x, y \in N_j(z)$ and $t \in [0, 1]$. From $x_j \geq z_j$ and $y_j \geq z_j$ we have $\max\{tx_j, y_j\} \geq z_j$, and from $x_j z_i \geq x_i z_j$ we get $tx_j z_i \geq tx_i z_j$, now, from $y_j z_i \geq y_i z_j$ and the positivity of the coefficients we have $\max\{tx_j, y_j\}z_i = \max\{tx_j z_i, y_j z_i\} \geq \max\{tx_i z_j, y_i z_j\} \geq \max\{tx_i, y_i\}z_j$; we have shown that $tx \vee y \in N_j(z)$.

Now let $x, y \in M_j(z)$ and $t \in]0, 1]$, we show that $tx \vee y \in M_j(z)$. There are two cases to consider:
(a) there exists an index l such that either $x_l > 0$ and $z_l = 0$ or $y_l > 0$ and $z_l = 0$; then, in both cases $\max\{tx_l, y_l\} > 0$ and $z_l = 0$;
(b) for $i \in I(z)$ let $a_i = x_i z_i^{-1}$ and $b_i = y_i z_i^{-1}$, then $a_j < \max\left\{1, \max_{i \in I(z), i \neq j} a_i\right\}$ and $b_j < \max\left\{1, \max_{i \in I(z), i \neq j} b_i\right\}$.

From $t \in]0, 1]$ we have $\max\{ta_j, b_j\} < \max\{t, 1, \max_{i \in I(z), i \neq j} ta_i, \max_{i \in I(z), i \neq j} b_i\} = \max\{1, \max_{i \in I(z), i \neq j} ta_i, \max_{i \in I(z), i \neq j} b_i\} = \max\{1, \max_{i \in I(z), i \neq j}\{ta_i, b_i\}\}$. □

Lemma 6.2. *For all $j \in \{0, \dots, n\}$ the binary relation U_j is a partial order on $\mathbb{R}^n_{++}$.*

Proof. For simplicity let us write $z \preceq_j x$ for $z \in U_j(x)$. For $j = 0$ $\preceq_j$ is simply the partial order associated to the positive cone $\mathbb{R}^n_+$; we assume that $j \geq 1$. From the definition of N_j we have $z \in N_j(z)$ for all $z \in \mathbb{R}^n_+$. Assume that $z \preceq_j x$ and $x \preceq_j z$ then $z \in N_j(x)$ and $x \in N_j(z)$; from $x_j \geq z_j$ and $z_j \geq x_j$ we have $x_j = z_j$ and, from $z_i x_j \geq z_j x_i$ and $z_j x_i \geq z_i x_j$ for all i we have, taking into account that x and z are in $\mathbb{R}^n_{++}$, $x_i = z_i$ for all i. If $z \preceq_j y$ and $y \preceq_j x$ then $y_j \leq z_j$ and $x_j \leq y_j$, also $y_i z_j \geq y_j z_i$ and $x_i y_j \geq x_j y_i$ for all i; multiplying the second inequality by $z_j z_i$ we obtain $(x_i z_j)(y_j z_i) \geq (x_j z_i)(y_i z_j)$, and from the first inequality, $(x_i z_j)(y_j z_i) \geq (x_j z_i)(y_j z_i)$ which yields $(x_i z_j) \geq (x_j z_i)$. □

Theorem 6.1. (1) *Let A be a nonempty set and let B be the $\mathbb{B}$-convex radiant set spanned by A (that is the intersection of all the $\mathbb{B}$-convex radiant sets containing A). Then $z \in B$ if and only if, for all $j \in \{1, \dots n\}$ $N_j(z) \cap A \neq \emptyset$.*
(2) $[\![A]\!] = \{z \in \mathbb{R}^n_+ : \forall j \in \{0, \dots, n\}\ N_j(z) \cap A \neq \emptyset\}$.

Proof. First notice that, for an arbitrary nonempty set A, $N_j(z) \cap A \neq \emptyset$ if $j \geq 1$, $z_j = 0$, and $N_0(0) \cap A \neq \emptyset$ if and only if $0 \in A$; in other words, we can assume that $z \neq 0$ and $j \in I(z)$.

$(1-a)$ We assume that $A=B$. Let $b_j \in N_j(z)\cap B$; for all $j\in I(z)$ we have $b_{j,j}\geq z_j$, let

$$t=\frac{b_{k,k}}{z_k}=\min\left\{\frac{b_{i,i}}{z_i}:i\in I(z)\right\} \text{ and, for } l\in I(z), \mu_l=t\frac{z_l}{b_{l,l}}.$$

We have $t\geq 1$, $0<\mu_l\leq 1$ and $\mu_k=1$. Since B is $\mathbb{B}$-convex and radiant we have $t^{-1}\bigvee_{l\in I(z)}\mu_l b_l \in B$, that is $v=\bigvee_{l\in I(z)}(z_l/b_{l,l})b_l\in B$. Let us see that $v=z$. If $j\notin I(z)$ then $v_j=0=z_j$. If $j\in I(z)$ then $b_{j,j}/z_j=\max_{i\in I(z)} b_{j,i}/z_i$ and $v_j=\big(\max_{l\in I(z)}(z_l/b_{l,l})(b_{l,j}/z_j)\big)\, z_j=z_j$ since $(b_{l,l}/z_l)\geq(b_{l,j}/z_j)$ for all $l\in I(z)$.
$(1-b)$ If $A\bigcap N_j(z)=\emptyset$ then $A\subset M_j(z)$ and, since $M_j(z)$ is radiant, we have $B\subset M_j(z)$, in other words, $B\bigcap N_j(z)=\emptyset$. This shows that, for $j\in\{1,\dots,n\}$, $B\bigcap N_j(z)\neq\emptyset$ if and only if $A\bigcap N_j(z)\neq\emptyset$.

(2) Let $B=[\![A]\!]$; since the sets $N_j(z)$ are half-spaces we have, as in $(1-b)$, $B\bigcap N_j(z)\neq\emptyset$ if and only if $A\bigcap N_j(z)\neq\emptyset$, for $j\in\{0,\dots,n\}$. The set $\bigcup_{0<t\leq 1} tB$ is radiant, and $\mathbb{B}$-convex. From $N_j(z)\cap B\neq\emptyset$ for all $j\in I(z)$ we have $z=tu$ with $0<t\leq 1$ and $u\in B$; if $N_0(z)\cap B\neq\emptyset$ then there exists $v\in B$ such that $v\leq z$. From $z=v\vee z=v\vee tu$ we have $z\in B$. □

Given a binary relation R on a set X and a subset A of X we write $R(A)$ for $\bigcup_{a\in A}R(a)$. Corollary 6.1 below makes clear the content of Theorem 6.1; it says that the $\mathbb{B}$-convex hull operator $A\mapsto[\![A]\!]$ is determined by the $n+1$ binary relations U_j; in particular, on $\mathbb{R}^n_{++}$, $[\![A]\!]$ is obtained by first taking the upper-set generated by A for each of the partial orders U_j, that is $\bigcup_{a\in A}\{x\in\mathbb{R}^n_{++}: a\preceq_j x\}$, and then by taking the intersection of all these sets.

Corollary 6.1. *For all subset A of $\mathbb{R}^n_+$ we have*

$$[\![A]\!]=\bigcap_{j=0}^{n}U_j(A). \tag{6.12}$$

As a consequence, the $\mathbb{B}$-convex hull of a finite set is always the union of a finite number of linear polytopes.

Proof. Notice that $A\cap N_j(z)\neq\emptyset$ if and only if $z\in U_j(A)$, this proves the first part. If A is finite then $[\![A]\!]$ is compact, choose r and s such that $[\![A]\!]\subset[r,s]^n$, then $[\![A]\!]=\bigcap_{j=0}^n[U_j(A)\cap[r,s]^n]=\bigcap_{j=0}^n\bigcup_{a\in A}U_j(a)\cap[r,s]^n$. The set $U_j(a)\cap[r,s]^n=\{x\in\mathbb{R}^n_+:\forall i\ a_jx_i\geq a_ix_j \text{ and } r\leq x_i\leq t\}$ is compact and defined by a finite number of linear inequalities, it is therefore a linear polytope. The intersection of a finite number of linear polytopes is again a linear polytope. This completes the proof. □

Corollary 6.1 gives a effective procedure to construct $[\![A]\!]$ if A is a finite set. First, $x\in U_j(a)$ is the solution set of the following system of linear inequalities

$$\begin{cases} 0\leq x_j\leq a_j \\ 0\leq a_jx_i-a_ix_j \text{ for all } i.\end{cases}$$

which can be solved without the simplex algorithm. The description of $U_0(a)$ is even simpler: $U_0(a)=\{x\in\mathbb{R}^n_+: a\leq x\}$. Using distributivity of the intersection over the reunion in (6.12) we have

$$[\![A]\!]=\bigcup_{(a^0,\dots,a^n)\in A^{n+1}}\bigcap_{j=0}^{n}U_j(a^j). \tag{6.13}$$

Each of the set $\bigcap_{j=0}^{n} U_j(a^j)$ is the solution set of a system of linear inequalities which corresponds to one of the convex polytopes whose reunion makes the $\mathbb{B}$-convex hull of A.

The set $U_0(a)$ which is closed and co-radiant can be described by its co-gauge, and the sets $U_j(a)$, which are closed and radiant can be described by their gauges, $U_0(a) = \{x \in \mathbb{R}^n_+ : \nu_{U_0(a)}(x) \geq 1\}$ and $U_j(a) = \{x \in \mathbb{R}^n_+ : \mu_{U_j(a)}(x) \leq 1\}$. From Corollary 6.1, we have, for an arbitrary subset A of $\mathbb{R}^n_+$, that $x \in [\![A]\!]$ if and only if there exists $a' \in A$ and, for all $j \in \{1, \ldots, n\}$, there exists $a \in A$ such that $\mu_{U_j(a)}(x) \leq 1 \leq \nu_{U_0(a')}(x)$; we obtain the following characterization of the $\mathbb{B}$-convex hull of a finite set A:

$$[\![A]\!] = \{x \in \mathbb{R}^n_+ : \max_{j\in I} \min_{a\in A} \mu_{U_j(a)}(x) \leq 1 \leq \max_{a\in A} \nu_{U_0(a)}(x) \} \tag{6.14}$$

One can check that the co-gauge $U_0(a)$ and the gauges of the sets $U_j(a)$ are given by the following formulas:

$$\nu_{U_0(a)}(x) = \begin{cases} +\infty & \text{if } a = 0 \\ 0 & \text{if } \exists i \text{ s.t. } a_i > 0 \text{ and } x_i = 0 \\ \min_{i\in I(a)} x_i/a_i & \text{otherwise.} \end{cases}$$

and

$$\mu_{U_j(a)}(x) = \begin{cases} +\infty & \text{if } a_j = 0 \text{ and } x_j > 0 \\ +\infty & \text{if } \exists i \text{ s.t. } x_i a_j < x_j a_i \\ 0 & \text{if } a_j = x_j = 0 \\ x_j/a_j & \text{otherwise.} \end{cases}$$

If $a \neq 0$ then for all $x \in \mathbb{R}^n_{++}$ $\nu_{U_0(a)}(x) = \min_{i\in I(a)}(x_i/a_i)$, $\nu_{U_0(a)}$ is therefore continuous on $\mathbb{R}^n_{++}$. For a finite set $A \subset \mathbb{R}^n_+$ (6.14) can be interpreted as a separation formula.

Proposition 6.1. *For a finite set A let $\nu_A = \max_{a\in A} \nu_{U_0(a)}$ and $\mu_{j,A} = \min_{a\in A} \mu_{U_j(a)}$. If $x \notin [\![A]\!]$, then either $\nu_A(x) < 1 \leq \inf_{z\in[\![A]\!]} \nu_A(z)$ or there exists $j \geq 1$ such that $\max_{z\in[\![A]\!]} \mu_{j,A}(z) \leq 1 < \mu_{j,A}(x)$. Furthermore, $\mu_{j,A}$ is positively homogeneous and lower semicontinuous, and $\nu_{j,A}(z)$ is positively homogeneous and upper semicontinuous.*

Proposition 5.2 asserts that a closed $\mathbb{B}$-convex set and a point that does not belong to this set can be the strongly separated by a finite $\mathbb{B}$-measurable positively homogeneous map. If this point is strictly positive then an explicit expression for separating maps can be given. To $z \in \mathbb{R}^n_{++}$ we associate $n+1$ maps on $\mathbb{R}^n_+$ defined as follows:

$$\theta_z(x) = \max_{i\in I} \frac{x_i}{z_i}$$

and, for $j \geq 1$

$$\theta^j_z(x) = \begin{cases} 0 & \text{if } x_j = 0 \text{ or } (x_j/z_j) < \theta_z(x) \\ (x_j/z_j) & \text{otherwise.} \end{cases}$$

Notice that, for all $z \in \mathbb{R}^n_{++}$ and for all $j \geq 1$, $\theta_z(z) = 1$ and $\theta^j_z(z) = 1$.

Proposition 6.2. *Let B be a closed $\mathbb{B}$-convex set and $z \in \mathbb{R}^n_{++}$ a point that does not belong to B. Then the following alternative holds:*
(1) *either, for all $x \in B, 1 < \theta_z(x)$ or*
(2) *there exists $j \geq 1$ such that, for all $x \in B$, $\theta^j_z(x) < 1$.*

Proof. If $z \notin B$ there are two possibilites: either (a) $B \cap N_0(z) = \emptyset$ or (b) there exists $j \geq 1$ such that $B \cap N_j(z) = \emptyset$, as follows from Theorem 6.1.

Assume that (a) is the case. Since $N_0(z)$ is closed and radiant we have $N_0(z) = \{x \in \mathbb{R}^n_+ : \mu_{N_0(z)}(x) \le 1\}$, and therefore, for all $x \in B$, $1 < \mu_{N_0(z)}(x)$. Since $z \in \mathbb{R}^n_{++}$ an easy computation yields $\mu_{N_0(z)}(x) = \theta_z(x)$ for all $x \in B$.

Now, assume that (b) is the case. The set $N_j(z)$ is closed and co-radiant, it is therefore equal to $\{x \in \mathbb{R}^n_+ : 1 \le \nu_{N_j(z)}(x)\}$, and consequently, for all $x \in B$, $\nu_{N_j(z)}(x) < 1$. Again, taking into account that $z \in \mathbb{R}^n_{++}$ we find

$$\nu_{N_j(z)}(x) = \begin{cases} 0 & \text{if } x_j = 0 \text{ or } \exists i \text{ s.t. } z_i x_j < z_j x_i \\ (x_j/z_j) & \text{otherwise.} \end{cases}$$

If $\nu_{N_j(z)}(x) < 1$ then, either $x_j = 0$ or $(x_j/z_j) < \max_{i\in I}(x_i/z_i)$ or $\max_{i\in I}(x_i/z_i) = (x_j/z_j) < 1$, and therefore $\nu_{N_j(z)}(x) = \theta^j_z(x)$ for all $x \in B$. □

We have seen that U_j, as a binary relation on $\mathbb{R}^n_{++}$, is a partial order; it has another noticeable property. First, let us say that a binary relation $R \subset B \times B$ on a $\mathbb{B}$-convex set B is a Ky Fan relation if it has the following properties:
(1) for all $x \in \mathbb{R}^n_+$ $\quad x \in R(x)$;
(2) for all $x \in \mathbb{R}^n_+$ the set $\mathbb{R}^n_+ \setminus R^{-1}(x)$ is $\mathbb{B}$-convex.

Ky Fan maps appeared in Ky Fan's proof of his famous inequality (without being named of course); in the framework of classical convexity a Ky Fan map with closed values, one of which is compact, verifies $\bigcap_{x\in B} R(x) \neq \emptyset$. This statement, as is well known, is equivalent to Brouwer's fixed point Theorem, and to Ky Fan's inequality; it can be proved by Dugundji-Granas generalization of the Knaster-Kuratowski-Mazurkiewicz Theorem, namely: if R is a binary relation on a (classical) convex set X such that $R(x)$ is closed for all $x \in X$ and, for all non empty finite subset A of X the convex hull of A is contained in $\bigcup_{a\in A} R(a)$ (this is the KKM Property), then, for all non empty finite subset A of X, $\bigcap_{a\in A} R(x) \neq \emptyset$, [2] for more details. This result, which is also equivalent to Brouwer's fixed point Theorem, is of great importance in mathematical economics.

Let us see that if R is a Ky Fan map on a $\mathbb{B}$-convex set B, then for all non empty finite subset A of B one has $[\![A]\!] \subset R(A)$ (this is of course the KKM property). Indeed, if $x \notin R(A)$ then $A \subset \mathbb{R}^n_+ \setminus R^{-1}(x)$ which, by (2), implies $[\![A]\!] \subset \mathbb{R}^n_+ \setminus R^{-1}(x)$; from (1) we have $x \notin \mathbb{R}^n_+ \setminus R^{-1}(x)$, and therefore $x \notin [\![A]\!]$.

Now let us come to the reason for this diversion on Ky Fan maps: for all $j \in \{0, \dots, n\}$, U_j is a Ky Fan map on $\mathbb{R}^n_+$. This is easily seen, U_j is reflexive, and $U_j^{-1}(x) = N_j(x)$, which, as we have seen, is a half space, therefore $\mathbb{R}^n_+ \setminus U_j^{-1}(x) = \mathbb{R}^n_+ \setminus N_j(x) = M_j(x)$ is $\mathbb{B}$-convex. The interesting part of the formula $[\![A]\!] = \bigcap_{j=0}^n U_j(A)$ is not so much that the $\mathbb{B}$-convex hull of A is obtained as an intersection of images of A under Ky Fan maps but that those maps are finite in number and also partial orders.

Propositions 6.1 implies that a $\mathbb{B}$-polytope is an intersection of closed radiant and co-radiant sets; our next result will show than an arbitrary $\mathbb{B}$-convex set is an intersection of open half-spaces. More precisely, let

$$\mathcal{M} = \{M_j(z) : z \in \mathbb{R}^n_+,\, j \in I(z)\} \cup \{M_0(0)\},$$

members of $\mathcal{M}$ are open half-spaces, we show that arbitrary $\mathbb{B}$-convex sets are intersections of members of $\mathcal{M}$ and that $\mathcal{M}$ is a minimal set of open half-spaces with that property.

Lemma 6.3. (1) *Let $z \in \mathbb{R}^n_+$ and $u \in M_0(z)$. Then $u + \lambda e_i \in M_0(z)$ for all $i \in I$ and all $\lambda > 0$;*

(2) *Let* $z \in \mathbb{R}^n_+$ *and* $u \in M_j(z)$ *for* $j \in I(z)$. *Then there exists* $\lambda_j > 0$ *such that* $u + \lambda_j e_j \notin M_j(z)$ *and* $u + \lambda e_i \in M_j(z)$ *for all* $i \in I$, $i \neq j$.

Proof. It follows directly from the definition of sets $M_j(z)$. □

Lemma 6.4. *Let* $u \in \mathbb{R}^n_+$, $u \neq 0$ *and* $z \in N_j(u)$, $j \in \{0\} \cup I(u)$. *Then* $I(z) \subset I(u)$ *and* $N_j(z) \subset N_j(u)$.

Proof. The result is obvious if $j = 0$ so we consider only case $j \in I(u)$. Since $z \in N_j(u)$ it follows then there exists $t \geq 1$ such that $z_j = tu_j$ and $z_i \leq tu_i$ for $i \in I(u)$. It follows from this that $I(z) \subset I(u)$. Let $x \in N_j(z)$. Then there exists $\tau \geq 1$ such that $x_j = \tau z_j$ and $x_i \leq \tau z_i$ for $i \in I(z)$. We have $x_j = (\tau t)u_j$ and $x_i \leq (\tau t)u_i$ for $i \in I(z)$. Let $i \in I(u) \setminus I(z)$. Then $z_i = 0$, hence $x_i = 0$. Since $u_i > 0$ it follows that $x_i \leq (\tau t)u_i$ for such i as well. Thus the result follows. □

Lemma 6.5. *Let* $M_j(z) \subset M_j(u)$, $j \in \{0\} \cup I(u)$. *Then* $z \in M_j(u)$.

Proof. The result easily follows from Lemma 6.4 if $j = 0$, so we consider only case $j \in I(u)$. We have $N_j(z) \supset N_j(u)$. Assume that $z \notin M_j(u)$. Then $z \in N_j(u)$. In view of Lemma 6.4 we have $N_j(z) \subset N_j(u)$, that is $N_j(z) = N_j(u)$. Since $M_j(z) \neq M_j(u)$, this is impossible. Hence $z \in M_j(u)$. □

Lemma 6.6. *Let* $M \in \mathcal{M}$ *and* $\mathcal{M}' = \mathcal{M} \setminus \{M\}$ *then* $M \neq \bigcap\{M' \in \mathcal{M}' : M \subset M'\}$.

Proof. If $M = \mathbb{R}^n_+ \setminus \{0\}$ the conclusion holds trivially since $0 \in M'$ if $M' \neq M$; we can assume that $M \neq \mathbb{R}^n_+ \setminus \{0\}$. Let $M = M_j(z)$ with $z_j > 0$. If $M_j(z) \subset M_k(u)$ and $k \neq 0$ it easily follows from Lemma 6.3 that $k = j$, hence $M_j(u) \supset M_j(z)$. Due to Lemma 6.5 we conclude that $z \in M_j(u) = M'$. We have shown $z \in \bigcap\{M' \in \mathcal{M}' : M \subset M'\}$, and since $z \notin M_j(z)$ the proof is complete. □

Theorem 6.2. *For all* $A \subset \mathbb{R}^n_+$ *we have*

$$[\![A]\!] = \bigcap \{M \in \mathcal{M} : A \subset M\}$$

and $\mathcal{M}$ *is a minimal family of open half-spaces with the property above.*

Proof. If $z \notin [\![A]\!]$ and $z \neq 0$ then, by Theorem 6.1, there exists $j \in I(z)$ such that $A \bigcap N_j(z) = \emptyset$, in other words, $A \subset M_j(z)$, and, obviously, $z \notin M_j(z)$. If $z = 0$ then $A \subset \mathbb{R}^n_+ \setminus \{0\} = M_0(0)$. The remainder of the proof is contained in the previous lemmas. □

7 The Geometric Hahn-Banach Theorem in $\mathbb{B}$-Convexity

None of the results given so far imply that a closed $\mathbb{B}$-convex set is the intersection of the closed half-spaces containing. It is the purpose of this last section to establish this fact.

For $\delta > 0$ let $B_\delta[S]$ be the δ-neighbourhood of the set with respect to the norm $|| \cdot ||_\infty$ and let us say that two sets S_1 and S_2 are non-proximate if $\inf_{(x,y) \in S_1 \times S_2} ||x - y||_\infty > 0$ or, equivalently, if there exists $\delta > 0$ such that $B_\delta[S_1] \cap B_\delta[S_2] = \emptyset$. We recall that if C is $\mathbb{B}$-convex then $B_\delta[C]$ is $\mathbb{B}$-convex, as well as the closure of C and its interior, which follows from [1] Lemma 2.2.12 and Section 2.4.

Theorem 7.1. *If C_1 and C_2 are non-proximate $\mathbb{B}$-convex sets of $\mathbb{R}^n_+$ then there exists a closed half-space D such that $C_1 \subset \text{int}D$ and $C_2 \subset \mathbb{R}^n_+ \setminus D$.*

Proof. Choose $\delta > 0$ such that $B_\delta[C_1] \cap B_\delta[C_2] = \emptyset$; by the Stone-Kakutani Property there exists a half space L such that $B_\delta[C_1] \subset L$ and $B_\delta[C_2] \subset \mathbb{R}^n_+ \setminus L$. Let D be the closure of L; C_1 is in the interior of D and C_2 is in $\mathbb{R}^n_+ \setminus D$. Furthermore, D, being the closure of $\mathbb{B}$-convex set, is $\mathbb{B}$-convex, and, from $\mathbb{R}^n_+ \setminus \overline{L} = \text{int}(\mathbb{R}^n_+ \setminus L)$ (with respect to the relative topology of $\mathbb{R}^n_+$), we have that $\mathbb{R}^n_+ \setminus D$ is also $\mathbb{B}$-convex. □

Corollary 7.1. *A closed $\mathbb{B}$-convex set is the intersection of the closed half-spaces in which it is contained.*

In the previous sections we have shown that disjoint $\mathbb{B}$-convex sets can be separated by an homogeneous $\mathbb{B}$-measurable map; unfortunately, those maps do not have to be continuous. To achieve separation by continuous maps we have to use the larger class of $\mathbb{B}$-measurable maps, in other words, if we drop the positive homogeneity property then continuous separation is possibe; this is a consequence of results from [1] and a Theorem of Van de Vel, Theorem 2.7 in [4], which is akin to the Tietze-Urysohn Theorem . For the reader's conveniance, and also because Van de Vel's Theorem is more general than needed here, we reproduce in a simplified form, and in the context of $\mathbb{B}$-convexity, the main argument of the proof.

Theorem 7.2. *If C_1 and C_2 are non-empty non-proximate $\mathbb{B}$-convex sets of $\mathbb{R}^n_+$ then there exists a continuous $\mathbb{B}$-measurable map $f : \mathbb{R}^n_+ \to [0,1]$ such that $f(C_1) = \{0\}$ and $f(C_2) = \{1\}$.*

Proof. Let $\mathbb{D}$ be the set of dyadic numbers strictly between 0 and 1. As in Theorem 7.1 let $\delta > 0$ such that $B_\delta[C_1] \cap B_\delta[C_2] = \emptyset$ and a half space L such that $B_\delta[C_1] \subset L$ and $B_\delta[C_2] \subset \mathbb{R}^n_+ \setminus L$. Put $L = H_{1/2}$; considering the $\delta/2$ neighbourhoods one can see that $(C_1, \mathbb{R}^n_+ \setminus H_{1/2})$ and $(H_{1/2}, C_2)$ are two pairs of non-proximate $\mathbb{B}$-convex sets. The initial procedure applied to the pair $(C_1, \mathbb{R}^n_+ \setminus H_{1/2})$ yields a half-space which we call $H_{1/4}$, that same procedure applied to the pair $(H_{1/2}, C_2)$ yields a half-space which we call $H_{3/4}$. By induction one obtains a family $\{H_d : d \in \mathbb{D}\}$ of half-spaces such that:
(1) $d \mapsto H_d$ is increasing.
(2) for all $d \in \mathbb{D}$, $(C_1, \mathbb{R}^n_+ \setminus H_d)$ and (H_d, C_2) are pairs of non-proximate $\mathbb{B}$-convex sets.
(3) if $d_1 < d_2$ then $(H_{d_1}, \mathbb{R}^n_+ \setminus H_{d_2})$ is a pair of non-proximate $\mathbb{B}$-convex sets. The map $f : \mathbb{R}^n_+ \to [0,1]$ defined as follows has the required properties:

$$f(x) = \begin{cases} 1 & \text{if } x \notin \bigcup\{H_d : d \in \mathbb{D}\} \\ \inf\{d \in \mathbb{D} : x \in H_d\} & \text{otherwise.} \end{cases}$$

□

Theorem 7.2 improves part (3) of Theorem 5.1 in as much as the separation is done through a continuous map, but, on the other hand, we have much less information on the map, since we cannot expect a map taking its values in $[0,1]$ to be positively homogeneous. As a consequence of Theorem 7.2 we have again that closed $\mathbb{B}$-convex sets are intersections of closed half-spaces and of open half-spaces.

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