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A NOVEL HYBRID APPROACH TO SOLVE NON-LINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS VIA NEW INTEGRAL TRANSFORM

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Abstract. In this paper, the homotopy analysis transform technique is a mathematical approach that has been developed to solve non-linear partial differential equations, particularly those that are fractional in nature. This approach combines homotopy analysis method(HAM) and transform method to derive analytical solutions that are accurate and efficient. The method involves constructing a homotopy, which is a continuous deformation of one problem into another, and then applying the transform method to obtain an analytical solution. One of the advantages of this approach is that it provides a reliable and accurate solution with a short processing time, making it a valuable tool in various scientific and engineering applications. To ensure the reliability and validity of the suggested approach, comparison tests are usually carried out to compare the results with other established techniques in the literature. The fact that the suggested methodology is in agreement with other approaches which are available in the literature. The use of tables and graphs to display the numerical results also enhances the visual representation of the data and makes it easier to interpret.

1. Introduction

In the field of fractional calculus, integrals and derivatives of arbitrary order are examined and utilized in a variety of contexts. The applications that fractional calculus finds in numerical analysis and other fields of physics and engineering, presumably including fractal phenomena, have sparked great interest in it in recent years. Fractional calculus is a topic that is both old and new. There are clear physical and geometric ramifications for integrals and derivatives of integer order. However, fractional-order integration and differentiation are not the case; they rapidly expand fields with theoretical and practical applications. It has been applied in the past several years to the study of viscoelastic materials and a wide range of scientific and technical fields, including probability, electrical networks, fluid flow, rheology, diffusive transport, and electromagnetic theory. The fundamental mathematical notions of fractional calculus were created long ago by mathematicians such as Leibniz, Liouville, Riemann, and others [7, 8, 18, 26].

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Fractional differential equations make use of fractional derivatives. Their capacity to simulate intricate phenomena has drawn a great deal of attention. Investigating fractional differential equations therefore becomes a crucial part of this subject. There are many analytical and numerical methods exist to solve this type of fractional differential equations such as Adomian decomposition method(ADM) [10,23–25], variational iteration method(VIM) [12,15], residual power series method(RPSM) [2], reduced differential transform method(RDTM) [31], new iterative method(NIM) [4, 5], homotopy perturbation method(HPM) [13, 14], homotopy analysis method (HAM) [17,19–22,32], modified homotopy analysis method(modified-HAM) [27,30], collocations methods [3,33], wavelets methods [6], finite element method(FEM) [29], finite volume method(FVM) [11] etc.

In this study, we investigate some non-linear fractional partial differential using the homotopy analysis transform technique(HATT). The HATT is a simple conjunction of the new integral transform [28] and the homotopy analysis method. Section 2 is all about the basic definitions from fractional calculus and integral transform. In Section 3, the procedure of the proposed method is provided. And the some applications of suggested method is given in the Section 4. The graphical and numerical discussion is provided in Section 5. The last section contains some conclude notes.

2. Some preliminaries

Some fundamental concepts of fractional order derivatives, such as Riemann-Liouville's derivative, Caputo's derivative, and the new integral transform, will be covered in this part. Additionally, the new integral transformation of those derivatives will be discussed.

Definition 2.1. The $\beta^{th}(\beta > 0)$-order Riemann-Liouville [17] fractional integration of a function $\varphi(\chi)$ can be stated as below :

$$\mathcal{I}_{\chi}^{\beta}\varphi(\chi) = \begin{cases} \frac{1}{\Gamma(\beta)}\int\limits_{0}^{\chi}(\chi-\varsigma)^{\beta-1}\varphi(\varsigma)d\varsigma, & \beta > 0,\ \chi > 0 \\ \varphi(\chi),\ \text{for } \beta = 0 \end{cases} \tag{2.1}$$

Definition 2.2. The $\beta^{th}(\beta > 0)$-order Riemann-Liouville [30] fractional derivative of a function $\varphi(\chi)$ can be stated as below :

$$^{RL}\mathcal{D}_{\chi}^{\beta}\varphi(\chi) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{d\chi^m}\int\limits_{0}^{\chi}(\chi-\varsigma)^{m-\beta-1}\varphi(\varsigma)d\varsigma, \tag{2.2}$$

where m is a positive integer such that $m-1 < \beta \leq m$.

Definition 2.3. The $\beta^{th}(\beta > 0)$-order Caputo [30] fractional derivative of a function $\varphi(\chi)$ can be stated as below :

$$
{}^{C}\mathcal{D}_{\chi}^{\beta}\varphi(\chi) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int\limits_{0}^{\chi} (\chi-\varsigma)^{m-\beta-1} \frac{d^m}{d\varsigma^m}\varphi(\varsigma)d\varsigma, \text{ for } m-1 < \beta < m \\ \varphi(\chi), \text{ for } \beta = m, \ m \in \mathbb{N} \end{cases} \tag{2.3}
$$

Definition 2.4. Let the function $\varphi(\chi) \in \mho$, where

$$
\mho = \{\varphi(\chi) \mid \exists M, \ \eta_1, \ \eta_2 > 0, \ |\varphi(\chi)| < M \exp\left(\frac{|\chi|}{\eta_i}\right), \ if \ \chi \in (-1)^i \times [0, \infty)\}.
$$

Then the new integral transform [28] is defined as follows

$$
\mathbb{V}[\varphi(\chi)] = s \int\limits_{0}^{\infty} \varphi(\chi) e^{\frac{-s\chi}{u}} d\chi \tag{2.4}
$$

with the inverse of the new integral transform as

$$
\mathbb{V}^{-1}[\mathbb{V}[\varphi(\chi)]] = \varphi(\chi) = \lim_{t\to\infty} \frac{1}{2\pi i} \frac{1}{u} \int\limits_{c-it}^{c+it} \frac{1}{s} e^{\frac{s\chi}{u}} \mathbb{V}[\varphi(\chi)] ds \tag{2.5}
$$

Theorem 2.5 ([28]). *If $\varphi(\chi)$ is a piecewise-continuous function in $\chi \in [0, \eta]$, $\forall \eta \in \mathbb{R}$ and exponential order of $\varphi(\chi)$ is μ, then the new integral transform of $\varphi(\chi)$ exists for all $\frac{s}{u} > \mu$.*

Theorem 2.6. *If $\mathbb{V}[\varphi_1(\chi)]$ and $\mathbb{V}[\varphi_2(\chi)]$ are the new integral transform of $\varphi_1(\chi)$ and $\varphi_2(\chi)$ respectively [28], then*

$$
\mathbb{V}[x\varphi_1(\chi) + y\varphi_2(\chi)] = x\mathbb{V}[\varphi_1(\chi)] + y\mathbb{V}[\varphi_2(\chi)], \ \forall \ x, y \in \mathbb{R}. \tag{2.6}
$$

Theorem 2.7. *Let $\varphi^{(m)}(\chi) \in \mho$ is the $m-$the derivative of a function $\varphi(\chi)$ with respect to χ [28], then*

$$
\mathbb{V}[\varphi^{(m)}(\chi)] = \left(\frac{s}{u}\right)^m \mathbb{V}[\varphi(\chi)] - s\sum_{j=0}^{m-1} \left(\frac{s}{u}\right)^{m-j-1} \varphi^{(j)}(0) \tag{2.7}
$$

Theorem 2.8. *If $\mathbb{V}[\varphi_1(\chi)]$ and $\mathbb{V}[\varphi_2(\chi)]$ are the new integral transform of $\varphi_1(\chi)$ and $\varphi_2(\chi)$ respectively [28], then the new integral transform of the convolution of φ_1 and φ_2 is*

$$
\mathbb{V}[(\varphi_1 * \varphi_2)(\chi)] = \frac{1}{s}\mathbb{V}[\varphi_1(\chi)]\mathbb{V}[\varphi_2(\chi)] \tag{2.8}
$$

where,

$$
\varphi_1 * \varphi_2 = \int\limits_{0}^{\chi} \varphi_1(\varsigma)\varphi_2(\chi-\varsigma)d\varsigma.
$$

Definition 2.9. The new transform of Riemann-Liouville fractional integration can be given by [28]

$$\mathbb{V}[\mathcal{I}_{\chi}^{\beta}\varphi(\chi)] = \left(\frac{u}{s}\right)^{\beta}\mathbb{V}[\varphi(\chi)] \tag{2.9}$$

Definition 2.10. The new transform of Riemann-Liouville fractional derivative can be given by [28]

$$\mathbb{V}[{}^{RL}\mathcal{D}_{\chi}^{\beta}\varphi(\chi)] = \left(\frac{s}{u}\right)^{\beta}\mathbb{V}[\varphi(\chi)] - s\sum_{j=0}^{m-1}\left(\frac{s}{u}\right)^{m-j-1}\frac{d^j}{d\chi^j}\mathcal{I}_{\chi}^{m-\beta}\varphi(0+) \tag{2.10}$$

Definition 2.11. The new transform of Caputo fractional derivative operator can be given by [28]

$$\mathbb{V}[{}^{C}\mathcal{D}_{\chi}^{\beta}\varphi(\chi)] = \left(\frac{s}{u}\right)^{\beta}\mathbb{V}[\varphi(\chi)] - s\sum_{j=0}^{m-1}\left(\frac{s}{u}\right)^{\beta-j-1}\varphi^{(j)}(0+) \tag{2.11}$$

Some basic properties of the new integral transform: The following properties can be derived with the help of above theorems and definitions.

(1) $\mathbb{V}[c] = cs\left(\frac{u}{s}\right)$, $c \in \mathbb{R}$.
(2) $\mathbb{V}[\xi^{\beta}] = \Gamma(\beta+1)s\left(\frac{u}{s}\right)^{\beta+1}$, $\beta > -1$.

3. Analysis of the proposed method(HATT)

In this section, the brief analysis of the homotopy analysis transformation technique is discussed. This technique is a simple conjunction of the homotopy analysis method and the new integral transform. Let us consider the general temporal fractional non-linear partial differential equation in Caputo derivative sense as follows:

$${}^{C}\mathcal{D}_{\tau}^{\beta}\varphi(\chi,\tau) + \mathcal{N}[\varphi(\chi,\tau)] + \mathcal{R}[\varphi(\chi,\tau)] = \phi(\chi,\tau),\ \beta \in \lfloor m \rfloor,\ m \in \mathbb{N}. \tag{3.1}$$

Where ${}^{C}\mathcal{D}_{\tau}^{\beta}$ is a Caputo fractional derivative of β order w.r.t. τ, $\mathcal{N}[\varphi(\chi,\tau)]$ is the non-linear and $\mathcal{R}[\varphi(\chi,\tau)]$ is a remaining term of the differential equation and $\phi(\chi,\tau)$ is a source/known term.

The first step is to apply new integral transform $\mathbb{V}$ to the above Eqn. (3.1). By doing this we can get simple differential equation in an integer order as follows:

$$\mathbb{V}\left[{}^{C}\mathcal{D}_{\tau}^{\beta}\varphi(\chi,\tau) + \mathcal{N}[\varphi(\chi,\tau)] + \mathcal{R}[\varphi(\chi,\tau)]\right] = \mathbb{V}[\phi(\chi,\tau)] \tag{3.2}$$

$$\mathbb{V}\left[{}^{C}\mathcal{D}_{\tau}^{\beta}\varphi(\chi,\tau)\right] + \mathbb{V}\left[\mathcal{N}[\varphi(\chi,\tau)] + \mathcal{R}[\varphi(\chi,\tau)]\right] = \mathbb{V}[\phi(\chi,\tau)] \tag{3.3}$$

$$\begin{aligned}&\left(\frac{s}{u}\right)^{\beta}\mathbb{V}[\varphi(\chi,\tau)] - s\sum_{j=0}^{m-1}\left(\frac{s}{u}\right)^{\beta-j-1}\varphi^{(j)}(\chi,0)\\ &+ \mathbb{V}\left[\mathcal{N}[\varphi(\chi,\tau)] + \mathcal{R}[\varphi(\chi,\tau)]\right] = \mathbb{V}[\phi(\chi,\tau)]\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\mathbb{V}[\varphi(\chi,\tau)] - s\left(\frac{u}{s}\right)^{\beta}\sum_{j=0}^{m-1}\left(\frac{s}{u}\right)^{\beta-j-1}\varphi^{(j)}(\chi,0)\\ &+\left(\frac{u}{s}\right)^{\beta}\mathbb{V}\left[\mathcal{N}[\varphi(\chi,\tau)]+\mathcal{R}[\varphi(\chi,\tau)]-\phi(\chi,\tau)\right]=0\end{aligned}\tag{3.5}$$

Now the second step is to solve above Eqn. (3.5) using the homotopy analysis method. For that let us consider the non-linear operator as follows:

$$\begin{aligned}\mathcal{NL}[\varphi(\chi,\tau;q)] &= \mathbb{V}[\varphi(\chi,\tau;q)] - s\left(\frac{u}{s}\right)^{\beta}\sum_{j=0}^{m-1}\left(\frac{s}{u}\right)^{\beta-j-1}\varphi^{(j)}(\chi,0;q)\\ &+\left(\frac{u}{s}\right)^{\beta}\mathbb{V}\left[\mathcal{N}[\varphi(\chi,\tau;q)]+\mathcal{R}[\varphi(\chi,\tau;q)]-\phi(\chi,\tau;q)\right]=0,\end{aligned}\tag{3.6}$$

where $q \in [0,1]$ is the homotopy parameter. Now by considering this non-linear operator $\mathcal{NL}[\varphi(\chi,\tau;q)]$, the deformation equation of zero order can be defined as follows:

$$(1-q)\mathcal{L}[\varphi(\chi,\tau;q)-\varphi_0(\chi,\tau)] = q\hbar\mathcal{NL}[\varphi(\chi,\tau;q)],\tag{3.7}$$

where $\mathcal{L}$ is the auxiliary linear operator, $\varphi_0(\chi,\tau)$ is the starting guess of the unknown function $\varphi(\chi,\tau)$ and $\hbar \neq 0$ is the auxiliary real parameter. If we put $q = 0$ and $q = 1$ in the above equation (3.7), it holds

$$\varphi(\chi,\tau;0) = \varphi_0(\chi,\tau) \text{ and } \varphi(\chi,\tau;1) = \varphi(\chi,\tau)\tag{3.8}$$

respectively. Thus we can say that as q goes from 0 to 1, the solution $\varphi(\chi,\tau;q)$ varies from the starting guess $\varphi_0(\chi,\tau)$ to the original solution of the problem (3.1). By expanding the $\varphi(\chi,\tau;q)$ in the Taylor series expansion form with respect to q, one can get

$$\varphi(\chi,\tau;q) = \sum_{k=0}^{\infty}\left[\frac{\partial^k\varphi(\chi,\tau;q)}{\partial q^k}\Big|_{q=0}\right]q^k\tag{3.9}$$

By choosing the properly values of $\mathcal{L}, \varphi_0(\chi,\tau)$ and $\hbar$, the above series (3.9) converges at $q = 1$. Now by taking k times derivative of Eqn. (3.7) w.r.t. q and dividing it by $k!$ (arranged $q = 0$), one can get the deformation equation of order k as below:

$$\mathcal{L}[\varphi_k(\chi,\tau) - \aleph_k\varphi_{k-1}(\chi,\tau)] = \hbar\Re_k[\overrightarrow{\varphi}_{k-1}]\tag{3.10}$$

where

$$\Re_k[\overrightarrow{\varphi}_{k-1}] = \frac{1}{(k-1)!}\frac{\partial^{k-1}\mathcal{NL}[\varphi(\chi,\tau;q)]}{\partial q^{k-1}}\Big|_{q=0}$$

and

$$\aleph_k = \begin{cases}0, & k \leq 1.\\ 1, & k > 1.\end{cases}$$

Here the k-th order deformation equation is a simple linear equation, so it can be solved easily. By solving that equation one can find $\varphi_1(\chi,\tau), \varphi_2(\chi,\tau), \varphi_3(\chi,\tau)$, etc.

4. Some applications of the proposed method

In this section, the application of the proposed method to some time-fractional non-linear partial differential equation is examined.

Example 4.1. Examined the time fractional Foam-drainage [1] problem as follows.

$$ {}^{C}\mathcal{D}_{\tau}^{\beta}\varphi(\chi,\tau)-\frac{1}{2}\varphi(\chi,\tau)\varphi_{\chi\chi}(\chi,\tau)+2\varphi^2(\chi,\tau)\varphi_x(\chi,\tau)-\varphi_x^2(\chi,\tau)=0,\ 0<\beta\leq 1,\ \tau\geq 0 \tag{4.1}$$

and the initial condition is

$$\varphi(\chi,0)=\begin{cases}-\sqrt{l}\tanh(\sqrt{l}\chi), & \chi\leq 0\\ 0, & \chi>0.\end{cases} \tag{4.2}$$

The exact solution of the problem for $\beta=1$

$$\varphi_{Exact}(\chi,\tau)=\begin{cases}-\sqrt{l}\tanh(\sqrt{l}(\chi-l\tau)), & \chi\leq l\tau\\ 0, & \chi>l\tau,\end{cases} \tag{4.3}$$

where l is a velocity constant.

By taking the initial guess $\varphi_0(\chi,\tau)$ as

$$\varphi_0(\chi,\tau)=-\sqrt{l}\tanh(\sqrt{l}\chi) \tag{4.4}$$

and applying our proposed method, discussed in above section, we will get

$$\varphi_0(\chi,\tau)=-\sqrt{l}\tanh(\sqrt{l}\chi) \tag{4.5}$$

$$\varphi_1(\chi,\tau)=\hbar l^2\left(\tanh(\sqrt{l}\chi)^2-1\right)\frac{\tau^\beta}{\Gamma(\beta+1)} \tag{4.6}$$

$$\begin{aligned}\varphi_2(\chi,\tau)&=(1+\hbar)\hbar l^2\left(\tanh(\sqrt{l}\chi)^2-1\right)\frac{\tau^\beta}{\Gamma(\beta+1)}\\ &-\frac{2\hbar^2 l^{\frac{7}{2}}\tanh(\sqrt{l}\chi)\left(\tanh(\sqrt{l}\chi)^2-1\right)}{\Gamma(\beta+1)}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}\end{aligned} \tag{4.7}$$

So, the approximate solution of the above problem upto 3-terms can written as follows:

$$\begin{aligned}\varphi_{HATT}(\chi,\tau)&=-\sqrt{l}\tanh(\sqrt{l}\chi)+\hbar l^2\left(\tanh(\sqrt{l}\chi)^2-1\right)\frac{\tau^\beta}{\Gamma(\beta+1)}\\ &+(1+\hbar)\hbar l^2\left(\tanh(\sqrt{l}\chi)^2-1\right)\frac{\tau^\beta}{\Gamma(\beta+1)}\\ &-\frac{2\hbar^2 l^{\frac{7}{2}}\tanh(\sqrt{l}\chi)\left(\tanh(\sqrt{l}\chi)^2-1\right)}{\Gamma(\beta+1)}\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}.\end{aligned} \tag{4.8}$$

Example 4.2. Consider the one dimensional time fractional form of Burger equation [16] as follows.

$$ {}^{C}\mathcal{D}_{\tau}^{\beta}\varphi(\chi,\tau)+\varphi(\chi,\tau)\varphi_{\chi}(\chi,\tau)-\varphi_{\chi\chi}(\chi,\tau)=0,\ 0\le\chi\le 1,\ \tau\ge 0,\ 0<\beta\le 1 \tag{4.9}$$

and the initial condition is

$$\varphi(\chi,0)=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{4}\chi\right). \tag{4.10}$$

The exact solution [9] of the problem for $\beta=1$

$$\varphi(\chi,\tau)=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{4}\left(\chi-\frac{\tau}{2}\right)\right). \tag{4.11}$$

By taking the initial guess $\varphi_0(\chi,\tau)$ as

$$\varphi_0(\chi,\tau)=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{4}\chi\right) \tag{4.12}$$

and applying our proposed method(HATT), we will get

$$\varphi_0(\chi,\tau)=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{4}\chi\right) \tag{4.13}$$

$$\varphi_1(\chi,\tau)=\frac{\hbar}{16}\left(\tanh\left(\frac{1}{4}\chi\right)^2-1\right)\frac{\tau^{\beta}}{\Gamma(\beta+1)} \tag{4.14}$$

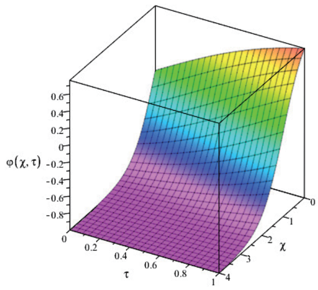
$$\begin{aligned}\varphi_2(\chi,\tau)=&\frac{(1+\hbar)\hbar}{16}\left(\tanh\left(\frac{1}{4}\chi\right)^2-1\right)\frac{\tau^{\beta}}{\Gamma(\beta+1)}\\&-\frac{\hbar^2}{64\Gamma(\beta+1)}\tanh\left(\frac{1}{4}\chi\right)\left(\tanh\left(\frac{1}{4}\chi\right)^2-1\right)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}\end{aligned} \tag{4.15}$$

So, the approximate solution of the above problem upto 3-terms can written as follows:

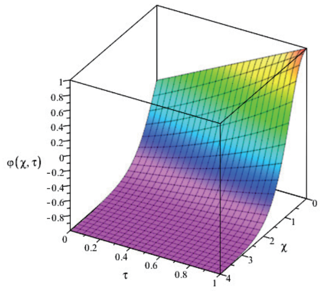
$$\begin{aligned}\varphi_{HATT}(\chi,\tau)=&\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{1}{4}\chi\right)+\frac{\hbar}{16}\left(\tanh\left(\frac{1}{4}\chi\right)^2-1\right)\frac{\tau^{\beta}}{\Gamma(\beta+1)}\\&+\frac{(1+\hbar)\hbar}{16}\left(\tanh\left(\frac{1}{4}\chi\right)^2-1\right)\frac{\tau^{\beta}}{\Gamma(\beta+1)}\\&-\frac{\hbar^2}{64\Gamma(\beta+1)}\tanh\left(\frac{1}{4}\chi\right)\left(\tanh\left(\frac{1}{4}\chi\right)^2-1\right)\frac{\tau^{2\beta}}{\Gamma(2\beta+1)}.\end{aligned} \tag{4.16}$$

5. Results and discussion

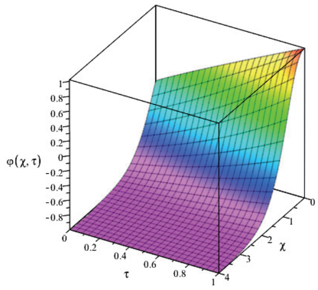
In this section, the numerical and graphical analysis provided for above two examples. There are two main benefits of the using proposed method. The first one is that the method contains the new integral transform which converts the fractional differential equation to the simple differential equation. And the second one is that it contains the embedding parameter $\hbar$ which is very helpful to control



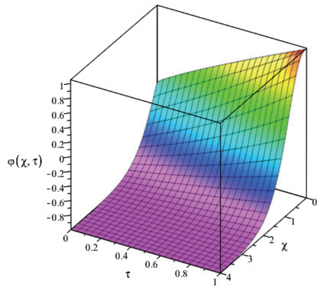
(a) Exact Solution



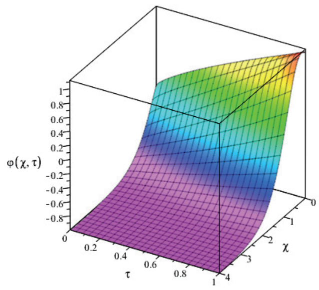
(b) HATT Solution for $\beta = 1$



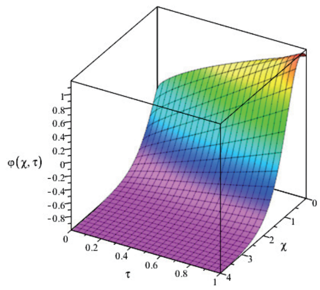
(c) HATT Solution for $\beta = 0.95$



(d) HATT Solution for $\beta = 0.90$



(e) HATT Solution for $\beta = 0.80$

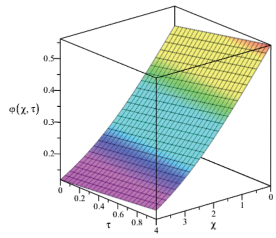


(f) HATT Solution for $\beta = 0.70$

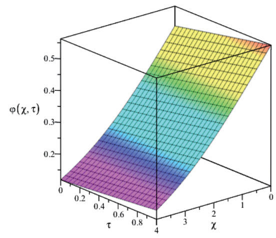
FIGURE 1. Graphs of exact solution and HATT approximate solution for different order β.

the convergence of the series solution.

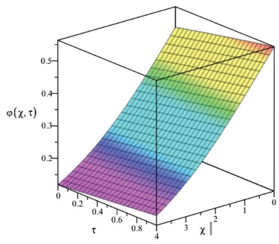
Figure (1) is all about Ex. (4.1). It contains the graphs of exact solution and the approximate solution derived by HATT for various order β. Graph of exact one can be found in the subfigure (1a) and subfigures (1b)-(1f) display the HATT approximate solutions for different values of β like $\beta = 1,\ \beta = 0.95,\ \beta = 0.90,\ \beta = 0.80,\ \beta = 0.70$ respectively. Table (1) shows the numerical solutions for different order β and compared with exact one for $\beta = 1$ for Example (4.1). In Table (2), the comparison of approximate solution obtained using HATT with other method is provided.



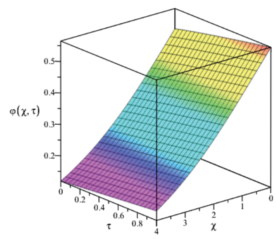
(a) Exact Solution



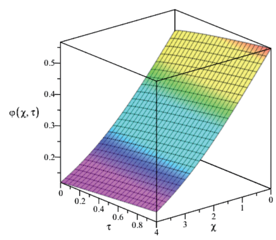
(b) HATT Solution for $\beta = 1$



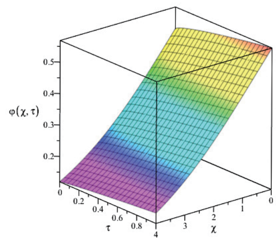
(c) HATT Solution for $\beta = 0.95$



(d) HATT Solution for $\beta = 0.90$

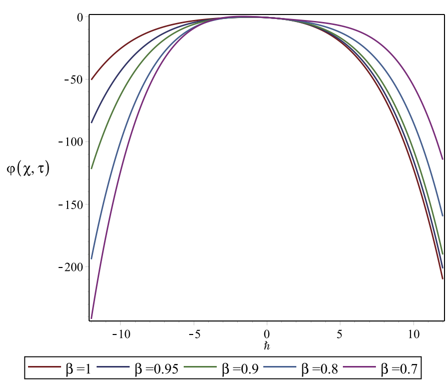


(e) HATT Solution for $\beta = 0.80$

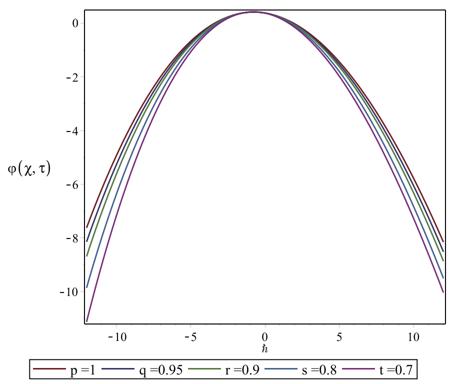


(f) HATT Solution for $\beta = 0.70$

FIGURE 2. Graphs of exact solution and HATT approximate solution for different order β.



(a) $\hbar$-curves for various values of β for Example (4.1).



(b) $\hbar$-curves for various values of β for Example (4.2).

FIGURE 3. $\hbar$-curves for Example (4.1) and (4.2).

TABLE 1. Numerical data for different order β and compared with exact one for $\beta = 1$ at $l = 1$ and $\hbar = -1.04$ for Example (4.1).

χ	τ	φ_{HATT} at $\beta = 0.70$	φ_{HATT} at $\beta = 0.80$	φ_{HATT} at $\beta = 0.90$	φ_{HATT} at $\beta = 0.95$	φ_{HATT} at $\beta = 1$	φ_{Exact} at $\beta = 1$
1	0.02	-0.729332	-0.740955	-0.748332	-0.750959	-0.753067	-0.753066
	0.04	-0.706779	-0.724473	-0.736286	-0.740668	-0.744277	-0.744277
	0.06	-0.686061	-0.708773	-0.724374	-0.730290	-0.735222	-0.735222
	0.08	-0.666200	-0.693368	-0.712415	-0.719748	-0.725897	-0.725897
	0.10	-0.646809	-0.678061	-0.700333	-0.709007	-0.716298	-0.716298
2	0.02	-0.958494	-0.960514	-0.961781	-0.962229	-0.962587	-0.962587
	0.04	-0.954526	-0.957658	-0.959716	-0.960472	-0.961090	-0.961090
	0.06	-0.950827	-0.954903	-0.957654	-0.958685	-0.959534	-0.959534
	0.08	-0.947241	-0.952172	-0.955566	-0.956856	-0.957917	-0.957917
	0.10	-0.943709	-0.949435	-0.953441	-0.954980	-0.956237	-0.956237

TABLE 2. Comparison of absolute error for our proposed method with other method for Example (4.1) at $\beta = 1,\ l = 1$ and $\hbar = -1.04$.

χ	τ	Absolute error by RPS [1]	Absolute error by HATT
1.0	0.02	6.6627×10^{-6}	8.2427×10^{-7}
	0.04	9.0798×10^{-5}	6.5549×10^{-6}
	0.06	3.8406×10^{-4}	2.1982×10^{-5}
	0.08	1.0024×10^{-3}	5.1754×10^{-5}
3.0	0.02	7.2440×10^{-10}	5.2061×10^{-11}
	0.04	2.4547×10^{-8}	4.2262×10^{-9}
	0.06	1.9773×10^{-7}	1.4413×10^{-8}
	0.08	8.8589×10^{-7}	3.4523×10^{-7}
5.0	0.02	2.4314×10^{-13}	1.0133×10^{-14}
	0.04	8.2420×10^{-12}	8.4632×10^{-13}
	0.06	6.6377×10^{-11}	2.6669×10^{-12}
	0.08	2.9735×10^{-10}	6.4613×10^{-11}

Similarly, the graphical analysis of Ex. (4.2) given in Figure (2). Subfigure (2a) displays the graph of exact solution and subfigures (2b)-(2f) contain the graphs of approximate solution derived using HATT for different order β. Table (3) shows the absolute error analysis and Table (4) contains the numerical solutions for different order β and compared with exact solution for $\beta = 1$. In the last Figure (3), $\hbar$-curves for different order β of Example (4.1) and (4.2) are given. From Figure (**??**), the valid region for $\hbar$-values of Ex. (4.1) for $\beta = 1$ can be found as $-3 \leq \hbar \leq 2$. Similarly, we can find the valid region for other fractional order β as segment of curve which is parallel to horizontal line. Also, the valid region for $\hbar$- values of Ex. (4.2) for $\beta = 1$ is an interval $-1.5 \leq \hbar \leq 1$.

TABLE 3. Absolute error for Example (4.2) at $\hbar = -1$.

χ	τ	$\lvert\varphi_{Exact} - \varphi_{HATT}\rvert$
1.0	0.02	2.0209×10^{-9}
	0.04	1.6162×10^{-8}
	0.06	5.4422×10^{-8}
	0.08	1.2920×10^{-7}
2.0	0.02	7.8794×10^{-10}
	0.04	6.0196×10^{-9}
	0.06	2.0095×10^{-8}
	0.08	4.8014×10^{-8}
5.0	0.02	8.6945×10^{-8}
	0.04	6.8432×10^{-9}
	0.06	2.2920×10^{-8}
	0.08	5.4303×10^{-8}

TABLE 4. Numerical data for different order β and compared with exact one for $\beta = 1$ for Example (4.2) at $\hbar = -1$.

χ	τ	φ_{HATT} at $\beta = 0.70$	φ_{HATT} at $\beta = 0.80$	φ_{HATT} at $\beta = 0.90$	φ_{HATT} at $\beta = 0.95$	φ_{HATT} at $\beta = 1$	φ_{Exact} at $\beta = 1$
1	0.02	0.381736	0.380305	0.379349	0.379000	0.378716	0.378716
	0.04	0.384369	0.382360	0.380919	0.380362	0.379894	0.379894
	0.06	0.386625	0.384214	0.382411	0.381691	0.381072	0.381072
	0.08	0.388669	0.385951	0.383855	0.382999	0.382252	0.382252
	0.10	0.390569	0.387605	0.385266	0.384293	0.383433	0.383433
2	0.02	0.272461	0.271258	0.270456	0.270163	0.269925	0.269926
	0.04	0.274680	0.272985	0.271773	0.271305	0.270912	0.270912
	0.06	0.276588	0.274547	0.273026	0.272421	0.271900	0.271900
	0.08	0.278321	0.276013	0.274242	0.273520	0.272892	0.272892
	0.10	0.279935	0.277413	0.275431	0.274610	0.273885	0.273885

6. CONCLUSION

In this study, the non-linear fractional partial differential equations were examined using a novel analytical technique termed the homotopy analysis transform technique (HATT), which is a straightforward combination of the new integral transform and the homotopy analysis method. We have studied two issues and produced approximate analytical solutions using the suggested methodology. We can see that compared to other methods described in the literature, the approximations answers obtained using HATT look better. HATT has been discovered to be user-friendly and computation-efficient.

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