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ON COINCIDENCE AND FIXED POINTS OF GENERAL MAPPINGS

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Abstract. In this paper, we establish two new fixed point theorem that guarantee the existence of a unique fixed point for mappings capable of admitting multiple fixed points. We adopt this approach to provide a new affirmative answer to Górnicki's problem appeared in [10]. Additionally, we propose sufficient conditions to ascertain the existence of coincidence points for a pair of self-mappings satisfying Caristi-type conditions.

1. Introduction and preliminaries

The Caristi-Kirk fixed point theorem, also known as the Caristi theorem, holds significant importance in fixed point theory [4, 5]. Its relevance extends across various mathematical domains, including convex minimization problems, variational inequalities, generalized differential calculus, critical point theory, normal solvability, and control theory, particularly through Ekeland's approach. Ekeland further contributed to the theorem by exploring equivalent characterizations [9]. The broader understanding and applications of the Caristi-Kirk theorem and its generalizations can be found in the works of [8, 16, 17] and related references.

In 1976, Caristi [4] proved the following well-known fixed point theorem:

Theorem 1.1. *Let (X, d) be a complete metric space and $f : X \to X$. If there exists a lower semicontinuous function $\phi : X \to [0, \infty)$ such that*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)), \quad \text{for all } x \in X, \tag{1.1}$$

then f has a fixed point.

If f is either continuous, orbitally continuous, or weakly orbitally continuous, the proof of Theorem 1.1 does not necessitate the lower semicontinuity assumption of the function ϕ. Pant et al. [12, 15] established various generalizations of Caristi's theorem in different settings.

The investigation of mappings allowing multiple fixed points without necessitating adherence to any contractive condition is currently an active area of research [3, 12, 15]. Bhakta and Basu [1] established a common fixed point theorem for Caristi-type mappings within a complete metric space, employing the concept of orbital continuity. Dien [7] initiated a study on common fixed point theorems by

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amalgamating the principles of Banach contractions and Caristi-Kirk mappings. In 2013, Turinici [17] provided a significant contribution by proving a comprehensive common fixed point theorem, encompassing a broad class of well-known fixed point theorems present in the existing mathematical literature.

Let (X, d) be a metric space and $f : X \to X$ be a mapping. The orbit of f at a point $x \in X$ is the set $O(f, x) = \{x, fx, f^2x, \cdots, f^nx, \cdots\}$. We now recall some weaker notions of continuity.

Definition 1.2 ([6])**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(f, x)$ for some $x \in X$, $x_n \to z$ implies $fx_n \to fz$ as $n \to \infty$.

Definition 1.3 ([12])**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is called weakly orbitally continuous if the set $\{y \in X : \lim_{i\to\infty} f^{m_i}y = u$ implies $\lim_{i\to\infty} ff^{m_i}y = fu\}$ is non-empty, whenever the set $\{x \in X : \lim_{i\to\infty} f^{m_i}x = u\}$ is non-empty for $u \in X$.

Definition 1.4 ([13])**.** Let (X, d) be a metric space and $f : X \to X$. A mapping $G : X \to \mathbb{R}$ is said to be f-orbitally lower semi-continuous at a point $z \in X$ if $\{x_n\}$ is a sequence in $O(f, x)$ for some $x \in X$, $\lim_{n\to\infty} x_n = z$ implies $G(z) \leq \liminf_{n\to\infty} G(x_n)$.

In [14], Nguyen demonstrated that the f-orbital lower semi-continuity of $x \to d(x, fx)$ is less stringent than orbital continuity and is independent from weak orbital continuity. For a thorough investigation and comparative analysis of different continuity assumptions, readers are requested to see Bisht [2].

Definition 1.5 ([11])**.** Two self-mappings f and g of a metric space (X, d) are called compatible if $\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some t in X.

In a recent work, Pant and Bisht [3] brought together and expanded upon the ideas introduced in [12, 15], leading to the development of unique fixed point theorems applicable to mappings without the necessity of satisfying any contractive condition.

In this paper, we consider new settings that ensure the existence of a unique fixed point for mappings that may have multiple fixed points. Additionally, we provide sufficient conditions to verify the existence of coincidence and common fixed points for a pair of self-mappings satisfying Caristi-type conditions.

2. Unique fixed point results

In 2020, Górnicki [10] mentioned that a mapping $f : (X, d) \to (X, d)$ satisfying the condition

$$d(fx, fy) < d(x, y) + K \cdot \{d(x, fx) + d(y, fy)\} \text{ for all } x, y \in X, x \neq y, \tag{2.1}$$

where $0 \leqslant K < \infty$, may not have a fixed point.

We start with the following fixed point result which provide an affirmative answer to the aforementioned Górnicki's problem.

Theorem 2.1. *Let f be a self-mapping of a complete metric space (X, d) satisfying the (2.1). Suppose $\phi : X \to [0, \infty)$ is a function such that*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)), \quad x \in X. \tag{2.2}$$

Then f has a unique fixed point provided that f is weakly orbitally continuous or $x \to d(x, fx)$ is f-orbitally lower semicontinuous.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X by $x_1 = f(x_0)$, $x_2 = f(x_1)$, …, $x_n = f(x_{n-1})$, that is, $x_n = f^n x_0$. Then, using (2.2), we have

$$d(x_0, x_1) = d(x_0, f(x_0)) \leq \phi(x_0) - \phi(f(x_0)) = \phi(x_0) - \phi(x_1).$$

Continuing this and adding, we get

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \phi(x_0).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$ and $\lim_{n\to\infty} f^p x_n = z$ for each $p \geq 1$.

Suppose f is weakly orbitally continuous. Since the sequence $f^n x_0$ is convergent for each x_0 in X, weak orbital continuity of f implies that there exists $y_0 \in X$ such that $f^n y_0 \to t$ and $f^{n+1} y_0 \to ft$ for some $t \in X$. This implies that $t = ft$ and, hence, t is a fixed point of f.

Next, assume that $x \to d(x, fx)$ is f-orbitally lower semi-continuous. Since $\{x_n\} \subset O(f, x)$ and $x_n \to z$ satisfying $d(x_n, fx_n) = d(f^n x_0, f^{n+1} x_0) \to 0$ as $n \to \infty$, by the f-orbital lower semi-continuity of $x \to d(x, fx)$, we have

$$d(z, fz) \leq \liminf_{n\to\infty} d(x_n, fx_n) = 0.$$

Thus, z is a fixed point of f.

Assume that w is also a fixed point of f and $w \neq z$. Then, using (2.1), we get

$$\begin{aligned} d(z, w) &= d(f(z), f(w)) \\ &< d(z, w) + K\{d(z, f(z)) + d(w, f(w))\} \\ &= d(z, w), \end{aligned}$$

a contradiction. Hence, f possesses a unique fixed point. $\square$

The following fixed point theorem is an easy consequence of Theorem 2.2:

Theorem 2.2. *Let f be a self-mapping of a complete metric space (X, d) satisfying the condition*

$$d(f(x), f(y)) < \max\{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}\}, \quad x, y \in X. \tag{2.3}$$

whenever the right hand side is non-zero. Suppose $\phi : X \to [0, \infty)$ is a function such that (2.2) holds. Then f has a unique fixed point provided that $x \to d(x, fx)$ is f-orbitally lower semicontinuous.

The following example illustrates Theorem 2.2.

Example 2.3. Let $X = [0, 2]$ and d be the Euclidean metric. Let $f : X \to X$ be defined by

$$f(x) = \begin{cases} \frac{1+x}{2} & \text{if } x \leq 1, \\ \frac{2-x}{2} & \text{if } x > 1. \end{cases}$$

Then f satisfies all the conditions of Theorem 2.2 and has a unique fixed point $x = 1$ at which f is discontinuous. Also, f satisfies (2.2) with $\phi : X \to [0, \infty)$ defined by

$$\phi(x) = \begin{cases} 1 - x & \text{if } x < 1, \\ 1 + x & \text{if } x \geq 1. \end{cases}$$

.

3. Coincidence point results

We start with the following result:

Theorem 3.1. *Let f and g be continuous self-mappings of a complete metric space (X, d) such that $f(X) \subseteq g(X)$. Suppose $\phi : X \to [0, \infty)$ is such that for all x in X,*

$$d(f(x), g(x)) \leq \phi(g(x)) - \phi(f(x)). \tag{3.1}$$

If f and g are compatible, then f and g have a coincidence point.

Proof. Let x_0 be any point in X. Define a sequence $\{y_n\}$ in X recursively by $y_n = f(x_n) = g(x_{n+1})$. Then

$$d(y_0, y_1) = d(f(x_1), g(x_1)) \leq \phi(g(x_1)) - \phi(f(x_1)).$$

Thus,

$$d(y_0, y_1) \leq \phi(y_0) - \phi(y_1).$$

Similarly,

$$\begin{aligned} d(y_1, y_2) &\leq \phi(y_1) - \phi(y_2), \\ d(y_2, y_3) &\leq \phi(y_2) - \phi(y_3), \\ &\cdots \\ d(y_{n-1}, y_n) &\leq \phi(y_{n-1}) - \phi(y_n), \\ d(y_n, y_{n+1}) &\leq \phi(y_n) - \phi(y_{n+1}). \end{aligned}$$

Adding these inequalities, we get

$$d(y_0, y_1) + d(y_1, y_2) + \ldots + d(y_n, y_{n+1}) \leq \phi(y_0) - \phi(y_{n+1}) \leq \phi(y_0).$$

Making $n \to \infty$, we obtain

$$\sum_{n=0}^{\infty} d(y_n, y_{n+1}) \leq \phi(y_0).$$

This implies that $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists $t \in X$ such that $\lim_{n\to\infty} y_n = t$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$.

Suppose that f and g are compatible. Then, $\lim_{n\to\infty} d(fg(x_n), gf(x_n)) = 0$. Considering the continuity of f and g, we obtain $ft = gt$, and as a result, f and g have a coincidence point. □

Example 3.2. Let $X = [0, 1]$ equipped with the Euclidean metric. Define $f, g : X \to X$ by

$$f(x) = 1 - \sqrt{1 - x^2}, \quad g(x) = \sqrt{2x - x^2}, \quad x \in X.$$

Let $\phi : X \to [0, \infty)$ be defined by $\phi(x) = x$ for each x in X. Then f and g satisfy the conditions of Theorem 3.1 and have two coincidence points $x = 0$ and $x = 1$.

Example 3.3. Let $X = [0, \infty)$ and d be the Euclidean metric. Define $f, g : X \to X$ by

$$f(x) = \frac{1 + x}{2}, \quad g(x) = x^2, \quad x \in X.$$

Let us define $\phi : X \to [0, \infty)$ by

$$\phi(x) = \begin{cases} 1 - x & \text{if } x \leq 1, \\ x & \text{if } x > 1. \end{cases}$$

Then f and g are continuous mappings which satisfy all the conditions of Theorem 3.1 and have a unique coincidence point $x = 1$.

4. Conclusion

In this paper, we have established conditions ensuring the presence of a unique fixed point for mappings that may exhibit multiple fixed points. This holds significance as a unique fixed point often simplifies analysis and yields more meaningful results in various applications. By adopting this approach, we have presented a novel affirmative solution to Górnicki's problem. Furthermore, we have introduced sufficient conditions that confirm the existence of coincidence points for a pair of self-mappings adhering to Caristi-type conditions.

References

[1] P. S. Bhakta and T. Basu, *Some fixed point theorems on metric spaces*, J. Indian Math. Soc. **45** (1981), 399–404.

[2] R. K. Bisht, *An overview of the emergence of weaker continuity notions, various classes of contractive mappings and related fixed point theorems*, J. Fixed Point Theory Appl. (2023) **25**:11.

[3] R. K. Bisht and R. P. Pant, *Unique fixed point results for general mappings*, Journal of Nonlinear and Convex Analysis **24** (2023), 2629–2642.

[4] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[5] J. Caristi and W. A. Kirk, *Geometric fixed point theory and inwardness conditions*, in:The Geometry of Metric and Linear Spaces Michigan State Univ., 1974, Lecture Notes Math., vol. 490, Springer, Berlin, 1975, pp. 74–83.

[6] Lj. B. Ćirić, *On contraction type mappings*, Math. Balkanica **1** (1971), 52–57.

[7] N. H. Dien, *Some remarks on common fixed point theorems*, J. Math. Anal. Appl. **187** (1994), 76–90.

[8] W. S. Du, *Some generalizations of fixed point theorems of Caristi type and Mizoguchi–Takahashi type under relaxed conditions*, Bull Braz Math Soc, New Series **50** (2019), 603–624.

[9] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (New Series) **1** (1979), 443–474.

[10] J. Górnicki, *On some mappings with a unique fixed point*, J. Fixed Point Theory Appl. (2020) **22**: 8.

[11] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Sci., **9** (1986), 771–779.

[12] M. C. Joshi, Abhijit Pant, and R. P. Pant, *Caristi type and Meir-Keeler type fixed point theorems*, Filomat **33** (2019), 3711–3721.

[13] T. L. Hicks and B. E. Rhoades, *A Banach type fixed-point theorem*, *Math. Japon.* **24** (1979/80) 327–330.

[14] L. V. Nguyen, *On fixed points of asymptotically regular mappings*, *Rend. Circ. Mat. Palermo, II. Ser* **70** (2021), 709–719.

[15] R. P. Pant, V. Rakočević, D. Gopal, A. Pant and M. Ram, *A general fixed point theorem*, Filomat **35** (2021), 4061–4072.

[16] M. Turinici, *Functional versions of the Caristi-Kirk theorem*, Revista Union Mat. Argentina **50** (2009), 89–97.

[17] M. Turinici, *Common fixed points for Banach-Caristi contractive pairs*, arXiv:1303.0235v2 [math.GN] 15 Nov 2013.

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