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A NOTE ON COLLECTIVELY COINCIDENCE RESULTS BETWEEN UPPER SEMICONTINUOUS KKM MAPS AND COMPACT DKT MAPS

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Abstract. We present collectively coincidence results between KKM type maps and compact DKT (or $HLPY$) type maps. Our argument is based on closed compact KKM self maps on an admissible convex set in a Hausdorff topological vector space.

1. Introduction

In this paper we use a fixed point theorem in the literature for KKM maps [3] to establish two collectively coincidence results between two different classes of set–valued maps defined on Hausdorff topological vector spaces. One class is the KKM type maps (which includes PK type maps, the Kakutani maps and the admissible maps of Gorniewicz) and the other class are DKT type maps (or $HLPY$ type maps). In addition the DKT type maps (or the $HLPY$ type maps) are compact maps which is quite different from results in the literature (see [10] and the references therein) where the compact maps are in the KKM class.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii) p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form

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$X \stackrel{p}{\leftarrow} \Gamma \stackrel{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map

and

(ii) $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [8]. A upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X,Y)$) provided there exists a selected pair (p,q) of ϕ. An example of an admissible map is a Kakutani map. A upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X,Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

We also discuss the following classes of maps in this paper. Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z,W)$ [9] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup \{\, int\, S^{-1}(w) : \; w \in W\}$; here $S^{-1}(w) = \{z \in Z : \; w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. These maps are related to the DKT maps in the literature and $F \in DKT(Z,W)$ [5] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the $\Phi^\star$ maps. We say $G \in \Phi^\star(Z,W)$ [2] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathcal{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class C of single valued continuous functions;

(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii) $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1,2,....\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

We say $F \in PK(X,Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

Next we describe a class of maps more general than the PK maps in our setting. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If S, $T : X \to 2^Y$ are two set valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mappping w.r.t. T. Now the set valued map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \to 2^Y$ w.r.t. T the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property (the intersection of each finite subfamily is nonempty). We let

$$KKM(X,Y) = \{T : X \to 2^Y \,|\, T \text{ has the } KKM \text{ property}\}.$$

Note $PK(X,Y) \subset KKM(X,Y)$ (see [4]). Next we recall the following results from the literature.

Theorem 1.1 ([4])**.** *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*

(i) *$T \in KKM(X,Y)$ iff $T|_\triangle \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X;*
(ii) *if $T \in KKM(X,Y)$ and $f \in C(Y,Z)$ then $f\,T \in KKM(X,Z)$;*
(iii) *if Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set valued map such that for each $f \in C(Y, \triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle, Y)$.*

Let I be an index set.

Theorem 1.2 ([10])**.** *Let X be a convex set in a Hausdorff topological vector space and $\{Y_i\}_{i\in I}$ be a family of Hausdorff topological spaces. Suppose $T_i \in KKM(X, Y_i)$ for each $i \in I$ and let $T : X \to 2^Y$ (here $Y = \prod_{i\in I} Y_i$) be defined by $T(x) = \prod_{i\in I} T_i(x)$ for $x \in X$. Then $T \in KKM(X,Y)$.*

In Section 2 we will make use of the following two properties [10]. Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i) If $T \in KKM(X,Y)$ then $G \equiv T|_C \in KKM(C,Y)$.
(ii) If $T \in KKM(X,Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X,Z)$.

Next we recall the following fixed point result for KKM maps. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1.3 ([3])**.** *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X,X)$ be a closed compact map. Then T has a fixed point in X.*

Theorem 1.4 ([10])**.** *Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Y a normal space. If $T \in KKM(X,Y)$ is a upper semicontinuous map with compact values and $f \in C(Y,X)$ then $T\,f \in KKM(Y,Y)$.*

2. Coincidence results

In this section we present coincidence results between two classes of set–valued maps. Throughout this section I and J will denote index sets.

Theorem 2.1. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space Z_i, with $Y \equiv \prod_{i\in J} Y_i$ a paracompact subset of $Z \equiv \prod_{i\in J} Z_i$ and $X \equiv \prod_{i\in I} X_i$ is an admissible subset of $E \equiv \prod_{i\in I} E_i$. For each $i \in J$ suppose $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values and for each $j \in I$ suppose $G_j \in DKT(Y, X_j)$ (or alternatively, $G_j \in HLPY(Y, X_j)$). Also for each $j \in I$ suppose there exists a compact set $K_j \subseteq X_j$ with $G_j(Y) \subseteq K_j$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_i \in G_i(y)$ for all $i \in I$ (here x_i (respectively, y_j) is the projection of x (respectively, y) on X_i (respectively, Y_j)).*

Proof. Fix $j \in I$. Since Y is paracompact from [5] (or alternatively, from [9]) there exists a continuous (single valued) selection $g_j : Y \to X_j$ (i.e. $g_j \in C(Y, X_j)$) of G_j. Now let $g(y) = \prod_{i\in I} g_i(y)$ for $y \in Y$ and note $g \in C(Y, X)$.

Let $F(x) = \prod_{i\in J} F_i(x)$ for $x \in X$ and from Theorem 1.2 we note that $F \in KKM(X, Y)$ is a upper semicontinuous map with compact values. Also Theorem 1.1 guarantees that $g\,F \in KKM(X, X)$ is a upper semicontinuous compact map with compact values, so a closed map [1]. Now Theorem 1.3 guarantees a $x \in X$ with $x \in g\,F(x)$. Now let $y \in F(x)$ with $x = g(y)$. Note $y \in F(x)$ so $y_j \in F_j(x)$ for all $j \in J$. Also note $x = g(y) = \prod_{i\in I} g_i(y)$ so $x_i = g_i(y) \in G_i(y)$ for all $i \in I$. □

Remark 2.2. (i). In the proof of Theorem 2.1 we apply Theorem 1.3 on the map $g\,F$. It is also possible to consider $F\,g$ if we assume in addition in the statement of Theorem 2.1 that $\{Y_i\}_{i\in J}$ is a family of convex sets each in a Hausdorff topological vector space Z_i and Y is an admissible subset of Z. To see this note $g \in C(Y, X)$, $F \in KKM(X, Y)$ and Theorem 1.4 (note Y is normal since Hausdorff paracompact spaces are normal [6]) implies that $F\,g \in KKM(Y, Y)$ is a upper semicontinuous compact (note $F\,g(Y) \subseteq F(K)$) map with compact values. Then Theorem 1.3 guarantees a $y \in Y$ with $y \in F\,g(y)$.

(ii). In the statement of Theorem 2.1 we could replace "X is an admissible subset of E" with "$K = \prod_{i\in I} K_i$ is an admissible subset of E" if we assume in addition that K_j is convex for each $j \in I$. To see this fix $j \in I$. We claim $G_j \in DKT(Y, K_j)$ (or alternatively, $G_j \in HLPY(Y, K_j)$). Suppose $G_j \in DKT(Y, X_j)$. Then there exists a map $S_j : Y \to X_j$ with $co\,(S_j(y)) \subseteq G_j(y)$ for $y \in Y$, $S_j(y) \neq \emptyset$ for each $y \in Y$ and $S_j^{-1}(x)$ is open (in Y) for each $x \in X_j$. Note in particular we have $S_j^{-1}(x)$ is open (in Y) for each $x \in K_j$ and also note $S_j : Y \to K_j$ since $S_j(y) \subseteq co\,(S_j(y)) \subseteq G_j(y) \subseteq K_j$ for each $y \in Y$. Thus $G_j \in DKT(Y, K_j)$. Alternatively suppose $G_j \in HLPY(Y, X_j)$. Then there exists a map $S_j : Y \to X_j$ with $co\,(S_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $Y = \bigcup \{ int\, S_j^{-1}(w) : \ w \in X_j\}$. Note for any $y \in Y$ there exists a $w \in X_j$ with $y \in int\, S_j^{-1}(w)$ so $w \in S_j(y) \subseteq co\,(S_j(y)) \subseteq G_j(y) \subseteq K_j$. Thus $Y = \bigcup \{ int\, S_j^{-1}(w) : \ w \in K_j\}$ so $G_j \in HLPY(Y, K_j)$.

Thus for each $i \in I$ since Y is paracompact from [5] (or alternatively, from [9]) there exists a selection $g_j \in C(Y, K_j)$ of G_j. Now let $g(y) = \prod_{i\in I} g_i(y)$ for $y \in Y$ and note $g \in C(Y, K)$. Now let $F(x) = \prod_{i\in J} F_i(x)$ for $x \in X$ and from Theorem 1.2 we note that $F \in KKM(X, Y)$ is a upper semicontinuous map with compact values. Also from Section 1 note $F\,(= F|_K) \in KKM(K, Y)$ is a upper semicontinuous map with compact values. Now Theorem 1.1 guarantees that $g\,F \in KKM(K, K)$ is a upper semicontinuous compact map with compact values. Thus Theorem 1.3 guarantees a $x \in K$ with $x \in g\,F(x)$, and we are finished.

Our next result replaces the condition that Y is paracompact in Theorem 2.1. In our next result $I = J$.

Theorem 2.3. *Let $\{X_i\}_{i\in I}$ be a family of convex sets, $\{Y_i\}_{i\in I}$ be a family of sets, each in a Hausdorff topological vector space E_i and $X \equiv \prod_{i\in I} X_i$ is an admissible subset of $E \equiv \prod_{i\in I} E_i$. For each $i \in I$ suppose $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values and for each $j \in I$ suppose $G_j \in DKT(Y, X_j)$ (or alternatively, $G_j \in HLPY(Y, X_j)$) where $Y \equiv \prod_{i\in I} Y_i$. Also for each $j \in I$ suppose there exists a compact set $K_j \subseteq X_j$ with $G_j(Y) \subseteq K_j$. Let $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$, $K = \prod_{i\in I} K_i$, $L(K)$ be the linear span of K (i.e. the smallest linear subspace of E that contains K) and assume $F(X) \subseteq L(K) \cap Y$ and $L(K) \cap Y$ is closed in both Y and $L(K)$. Then there exists a $x \in X$, a $y \in Y$ with $y_j \in F_j(x)$ for all $j \in I$ and $x_i \in G_i(y)$ for all $i \in I$.*

Proof. Note as in Theorem 2.1 that $F \in KKM(X, Y)$ is a upper semicontinuous map with compact values. Fix $j \in I$ and let $G_j \in DKT(Y, X_j)$ (or alternatively, $G_j \in HLPY(Y, X_j)$). We claim $G_j \in DKT(Y \cap L(K), X_j)$ (or alternatively, $G_j \in HLPY(Y \cap L(K), X_j)$). Suppose $G_j \in DKT(Y, X_j)$. Then there exists a map $S_j : Y \to X_j$ with $co\,(S_j(y)) \subseteq G_j(y)$ for $y \in Y$, $S_j(y) \neq \emptyset$ for each $y \in Y$ and $S_j^{-1}(x)$ is open (in Y) for each $x \in X_j$. Let $S_j\,(= S_j|_{Y\cap L(K)})$ be the restriction of S_j to $Y \cap L(K)$ and note for $x \in X_j$ we have

$$\begin{aligned} S_j^{-1}(x) &= \{z \in Y \cap L(K) : x \in S_j(z)\} \\ &= L(K) \cap \{z \in Y : x \in S_j(z)\} = L(K) \cap S_j^{-1}(x) \end{aligned}$$

which is open in $L(K)\cap Y$. Thus $G_j \in DKT(Y\cap L(K), X_j)$. Alternatively suppose $G_j \in HLPY(Y, X_j)$. Then there exists a map $S_j : Y \to X_j$ with $co\,(S_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $Y = \bigcup\{\,int\,S_j^{-1}(w) : w \in X_j\}$. Let $S_j = S_j|_{Y\cap L(K)}$. We now show $Y \cap L(K) = \bigcup\{\,int_{Y\cap L(K)}\,S_j^{-1}(w) : w \in X_j\}$. To see this first notice that

$$L(K)\cap Y = L(K)\cap\Big(\bigcup\{\,int\,S_j^{-1}(w) : w \in X_j\}\Big) = \bigcup\{L(K)\cap int\,S_j^{-1}(w) : w \in X_j\},$$

so $L(K) \cap Y \subseteq \bigcup\{\,int_{Y\cap L(K)}\,S_j^{-1}(w) : w \in X_j\}$ since for each $w \in X_j$ we have that $Y \cap int\,S_j^{-1}(w) = int\,S_j^{-1}(w)$ so $L(K) \cap int\,S_j^{-1}(w) = L(K) \cap Y \cap int\,S_j^{-1}(w) = (L(K) \cap Y) \cap int\,S_j^{-1}(w)$ with is open in $L(K) \cap Y$. On the other hand clearly $\bigcup\{\,int_{Y\cap L(K)}\,S_j^{-1}(w) : w \in X_j\} \subseteq L(K) \cap Y$. Thus $L(K) \cap Y = \bigcup\{\,int_{Y\cap L(K)}\,S_j^{-1}(w) : w \in X_j\}$ so $G_j \in HLPY(Y \cap L(K), X_j)$.

Next recall $L(K)$ is Lindelöf so paracompact [6, 7] and since $Y \cap L(K)$ is closed in $L(K)$ then $Y \cap L(K)$ is paracompact. Now from [5] (or alternatively, from [9]) there exists a selection $g_j \in C(Y \cap L(K), X_j)$ of G_j. Now let $g(y) = \prod_{i\in I} g_i(y)$ for $y \in Y \cap L(K)$ and note $g \in C(Y \cap L(K), X)$. Since $F(X) \subseteq Y \cap L(K)$ then from Section 1 we have that $F \in KKM(X, Y \cap L(K))$ is a upper semicontinuous map with compact values. Now Theorem 1.1 guarantees that $g\,F \in KKM(X, X)$ is a upper semicontinuous compact map with compact values, so a closed map. Now Theorem 1.3 guarantees a $x \in X$ with $x \in g\,F(x)$, and we are finished. □

Remark 2.4. In the proof of Theorem 2.3 we apply Theorem 1.3 on the map $g\,F$. It is also possible to consider $F\,g$ if we assume in addition in the statement of Theorem 2.3 that $\{Y_i\}_{i\in J}$ is a family of convex sets and $L(K) \cap Y$ is an admissible subset of E. To see this let g be as in Theorem 2.3 and note $F \in KKM(X, Y \cap L(K))$ and $g \in C(Y \cap L(K), X)$. Then Theorem 1.4 (note $Y \cap L(K)$ is normal since Hausdorff paracompact spaces are normal [6]) guarantees that $F\,g \in KKM(Y \cap L(K), Y \cap L(K))$ is a upper semicontinuous compact map with compact values, so a closed map. Now apply Theorem 1.3.

Now we relax some of the conditions in Theorem 2.1.

Theorem 2.5. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space Z_i, with $Y \equiv \prod_{i\in J} Y_i$ a paracompact subset of $Z \equiv \prod_{i\in J} Z_i$. For each $i \in J$, suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$, suppose $G_j : Y \to X_j$ and there exists a convex compact set K_j with $G_j(Y) \subseteq K_j \subseteq X_j$ and also there exists a map $S_j : Y \to X_j$ with $S_j(y) \subseteq G_j(y)$ for $y \in Y$, $S_j(y)$ has convex values for each $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in K_j$. Also assume $K \equiv \prod_{i\in I} K_i$ is an admissible subset of $E \equiv \prod_{i\in I} E_i$. Finally suppose for each $y \in Y$ there exists a $j \in I$ with $S_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$, a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$ (here x_i (respectively, y_j) is the projection of x (respectively, y) on X_i (respectively, Y_j)).*

Proof. Note $C_i = \{y \in Y : S_i(y) \neq \emptyset\}, i \in I$ is an open covering of Y and since Y is paracompact there exists a covering $\{D_i\}_{i\in I}$ of Y where D_i is closed in Y and $D_i \subset C_i$ for all $i \in I$. Now for each $i \in I$ let $M_i : Y \to K_i$ and $L_i : Y \to K_i$ be given by

$$M_i(y) = \begin{cases} G_i(y), & y \in D_i \\ K_i, & y \in Y\backslash D_i \end{cases} \quad \text{and} \quad L_i(y) = \begin{cases} S_i(y), & y \in D_i \\ K_i, & y \in Y\backslash D_i. \end{cases}$$

We claim for each $i \in I$ that $M_i \in \Phi^\star(Y, K_i)$. Let $i \in I$. First note $L_i(y) \neq \emptyset$ for $y \in Y$ since if $y \in D_i$ then $L_i(y) = S_i(y) \neq \emptyset$ since $D_i \subset C_i$ whereas if $y \in Y\backslash D_i$ then $L_i(y) = K_i$. Also if $y \in D_i$ then $L_i(y) = S_i(y) \subseteq G_i(y) = M_i(y)$ whereas if $y \in Y\backslash D_i$ we have $L_i(y) = K_i = M_i(y)$. Combining gives $L_i(y) \subseteq M_i(y)$ for $y \in Y$. Finally note if $x \in K_i$, we have

$$L_i^{-1}(x) \;=\; \{z \in Y : x \in L_i(z)\} = \{z \in Y\backslash D_i : x \in L_i(z) = K_i\} \cup \{z \in D_i : x \in L_i(z)\}$$

$$= (Y\backslash D_i) \cup \{z \in D_i : x \in S_i(z)\} = (Y\backslash D_i) \cup [D_i \cap \{z \in Y : x \in S_i(z)\}]$$
$$= (Y\backslash D_i) \cup \left[D_i \cap S_i^{-1}(x)\right] = Y \cap \left[(Y\backslash D_i) \cup S_i^{-1}(x)\right] = (Y\backslash D_i) \cup S_i^{-1}(x)$$

which is open in Y (note $S_i^{-1}(x)$ is open in Y and D_i is closed in Y). Thus for each $i \in I$ we have $M_i \in \Phi^\star(Y, K_i)$.

Now for $i \in I$ (since Y is paracompact) from [5] there exists a continuous (single valued) selection $g_i : Y \to K_i$ of L_i i.e. $g_i(y) \in L_i(y) \subseteq M_i(y)$ for $y \in Y$. Now let $g(y) = \prod_{i\in I} g_i(y)$ for $y \in Y$ and note $g \in C(Y, K)$. Also $F_i \in KKM(K, Y_i)$ for $i \in J$. Let $F(x) = \prod_{i\in J} F_i(x)$ for $x \in K$ and from Theorem 1.2 we note that $F \in KKM(K, Y)$ is a upper semicontinuous map with compact values. Also Theorem 1.1 guarantees that $g\,F \in KKM(K, K)$ is a upper semicontinuous compact map with compact values, so a closed map. Now Theorem 1.3 guarantees a $x \in K$ with $x \in g\,F(x)$. Now let $y \in F(x)$ with $x = g(y)$. Thus $y_j \in F_j(x)$ for all $j \in J$. Also note $x = g(y) = \prod_{i\in I} g_i(y)$ so $x_i = g_i(y) \in L_i(y) \subseteq M_i(y)$ for all $i \in I$. Next since $\{D_i\}_{i\in I}$ is a covering of Y there exists a $i_0 \in I$ with $y \in D_{i_0}$ so $x_{i_0} \in M_{i_0}(y) = G_{i_0}(y)$. □

Remark 2.6. In Theorem 2.5 one could replace "K is admissible" with "K is Schauder admissible", if we put appropriate assumptions so "$F \in KKM(K, Y)$" is replaced, by say, "$F \in PK(K, Y)$" (here F is in the proof of Theorem 2.5).

Theorem 2.5 can be extended to DKT and $HLPY$ type maps.

Theorem 2.7. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space Z_i, with $Y \equiv \prod_{i\in J} Y_i$ a paracompact subset of $Z \equiv \prod_{i\in J} Z_i$. For each $i \in J$, suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$, suppose $G_j : Y \to X_j$ and there exists a convex compact set K_j with $G_j(Y) \subseteq K_j \subseteq X_j$ and also there exists a map $T_j : Y \to K_j$ with $co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $T_j^{-1}(w)$ is open (in Y) for each $w \in K_j$. Also assume $K \equiv \prod_{i\in I} K_i$ is an admissible subset of $E \equiv \prod_{i\in I} E_i$. Finally suppose for each $y \in Y$ there exists a $j \in I$ with $T_j(y) \neq \emptyset$. Then there exists a $x \in X$, a $y \in Y$, a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$.*

Proof. For $j \in I$ let $S_j(y) = co\,(T_j(y))$ for $y \in Y$. Fix $j \in I$. Note $S_j(y)$ has convex values for each $y \in Y$ and note $S_j(y) \subseteq G_j(y)$ for $y \in Y$. In addition for $j \in I$ from [11, Lemma 5.1] we have that $S_j^{-1}(w)$ is open (in Y) for each $w \in K_j$. Finally note if $y \in Y$ then there exists a $j_0 \in I$ with $T_{j_0}(y) \neq \emptyset$ and so $\emptyset \neq T_{j_0}(y) \subseteq co\,(T_{j_0}(y)) = S_{j_0}(y)$. Now Theorem 2.5 guarantees that there exists a $x \in X$, a $y \in Y$ and a $i_0 \in I$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_{i_0} \in G_{i_0}(y)$. □

Theorem 2.8. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i and $\{Y_i\}_{i\in J}$ be a family of sets each in a Hausdorff topological space Z_i, with $Y \equiv \prod_{i\in J} Y_i$ a paracompact subset of $Z \equiv \prod_{i\in J} Z_i$. For each $i \in J$, suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ and $F_i \in KKM(X, Y_i)$ is upper semicontinuous with compact values. For each $j \in I$, suppose $G_j : Y \to X_j$ and there exists a convex* *compact set* K_j *with* $G_j(Y) \subseteq K_j \subseteq X_j$ *and also there exists a map* $T_j : Y \to K_j$ *with* $co\,(T_j(y)) \subseteq G_j(y)$ *for* $y \in Y$ *and* $Y = \bigcup_{j \in I} \bigcup \{\, int\, T_j^{-1}(w) :\ w \in K_j\}$*. Also assume* $K \equiv \prod_{i \in I} K_i$ *is an admissible subset of* $E \equiv \prod_{i \in I} E_i$*. Then there exists a* $x \in X$*, a* $y \in Y$*, a* $i_0 \in I$ *with* $y_j \in F_j(x)$ *for all* $j \in J$ *and* $x_{i_0} \in G_{i_0}(y)$*.*

Proof. For $j \in I$ let $R_j : Y \to K_j$ be given by

$$R_j(y) = \{z_j :\ y \in int\, T_j^{-1}(z_j)\},\ y \in Y$$

and let $S_j : Y \to K_j$ be given by

$$S_j(y) = co\,(R_j(y)) \ \text{ for } \ y \in Y.$$

Fix $j \in I$. Note $S_j(y)$ has convex values for each $y \in Y$ and also note that $R_j(y) \subseteq T_j(y)$ for $y \in Y$ since if $z_j \in R_j(y)$ then $y \in int\, T_j^{-1}(z_j) \subseteq T_j^{-1}(z_j) = \{w \in Y :\ z_j \in T_j(w)\}$ so $z_j \in T_j(y)$. Thus for $j \in I$ we have $S_j(y) = co\,(R_j(y)) \subseteq co\,(T_j(y)) \subseteq G_j(y)$ for $y \in Y$.

Now for $j \in I$ notice for $x_j \in K_j$ that $R_j^{-1}(x_j) = \{z :\ x_j \in R_j(z)\} = int\, T_j^{-1}(x_j)$ so $R_j^{-1}(x_j)$ is open (in Y) and so from [11, Lemma 5.1] we have that $S_j^{-1}(x_j)$ is open (in Y).

Now let $y \in Y$. Since $Y = \bigcup_{j \in I} \bigcup \{\, int\, T_j^{-1}(w) :\ w \in K_j\}$ there exists a $j \in I$ with $y \in int\, T_j^{-1}(w)$ for some $w \in K_j$ and so $w \in R_j(y)$ i.e. $R_j(y) \neq \emptyset$ and as a result $\emptyset \neq R_j(y) \subseteq co\,(R_j(y)) = S_j(y)$. Now Theorem 2.5 guarantees the result. □

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