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CONVERGENCE THEOREMS FOR FAMILIES OF MONOTONE NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Dedicated to the memory of Professor Kazimierz Goebel with deep gratitude.

Abstract. In this paper, we prove weak convergence theorems for finite noncommutative monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

1. Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}.$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. The fixed point theory for such mappings is rich and varied. It finds many applications in nonlinear functional analysis. The existence of fixed points for nonexpansive mappings in Banach and metric spaces has been investigated since the early 1960s (For example, see [6, 7, 8, 10, 14]).

In recent years, a new direction has been very active essentially after the publication of Ran and Reurings results [18]. They proved an analogue of the classical Banach contraction principle [5] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [13, 17, 26, 27]).

Mann [16] introduced an iteration process for approximation of fixed points of a mapping T in a Hilbert space as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$

for all $n \geq 1$, where $\{\alpha_n\}$ is sequences in $[0, 1]$. Later, Reich [15] discussed this iteration process in a uniformly convex Banach space whose norm is Frechet differentiable. Bin Dehaish and Khamsi [12] proved a weak convergence theorem of Mann's type [16] iteration for monotone nonexpansive mappings in Banach spaces

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endowed with a partial order (see also [1, 21]). Takahashi and Tamura [25] proved some weak convergence theorems for a pair of nonexpansive mappings in a Banach space by using the iteration process considered by Das and Debata [11]. Takahashi and Shimoji [24] introduced an iteration process, given by finite nonexpansive mappings, which generalizes Das and Debata's iteration, and then proved weak convergence theorems for finite nonexpansive mappings in a Banach space.

In this paper, we prove weak convergence theorems for finite noncommutative monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

2. Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x + z \preceq y + z,$$
$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. We will say that this Banach space $(E, \|\cdot\|, \preceq)$ is an *ordered Banach space*. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow) = \{z \in E : x \preceq z\}$ and $(\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow)$ and $(\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets $[a, \rightarrow) = \{x \in X; a \preceq x\}$ or $(\leftarrow, a] = \{x \in X; x \preceq a\}$. for any $a \in E$. As a direct consequence of this, the subset

$$[a, b] = \{x \in X; a \preceq x \preceq b\} = [a, \rightarrow) \cap (\leftarrow, b]$$

is also closed and convex for each $a, b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a partial order $\preceq$ compatible with the linear structure of E. We will say that this Banach space $(E, \|\cdot\|, \preceq)$ is an *ordered Banach space*. Let C be a nonempty subset of E. A mapping $T : C \rightarrow C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \rightarrow C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x, y \in C$ such that $x \preceq y$. For a mapping $T : C \rightarrow C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We write $x_n \rightarrow x$ (or $\lim_{n\rightarrow\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or $\text{w-}\lim_{n\rightarrow\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \; : \; \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex.

The following theorem was proved in [9].

Theorem 2.1 ([9]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element u in C and $\{x_n - Tx_n\}$ converges strongly to 0. Then, u is a fixed point of T.*

Lemma 2.2 ([20]). *Let E be a uniformly convex Banach space. Let b, c be real numbers with $0 < b \leq c < 1$. Let $\{t_n\}$ be a real sequence such that $b \leq t_n \leq c$ for $n = 1, 2, \dots$ and let $a \geq 0$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\overline{\lim}_{n\to\infty} \|x_n\| \leq a$, $\overline{\lim}_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Lemmas

Let C be a nonempty convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be finite mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \leq \alpha_i \leq 1$ for every $i = 1, 2, \dots, r$. Then, we define a mapping W of C into itself as follows (see [22, 24]):

$$\begin{aligned}
U_1 &= \alpha_1 T_1 + (1-\alpha_1)I, \\
U_2 &= \alpha_2 T_2 U_1 + (1-\alpha_2)I, \\
&\vdots \\
U_{r-1} &= \alpha_{r-1} T_{r-1} U_{r-2} + (1-\alpha_{r-1})I,
\end{aligned} \tag{3.1}$$

$$W = U_r = \alpha_r T_r U_{r-1} + (1 - \alpha_r)I.$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$ (see also [2]). Let $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr} (n = 1, 2, \dots)$ be real numbers such that $0 \le \alpha_{ni} \le 1$ for every $i = 1, 2, \dots, r$. Let $W_n (n = 1, 2, \dots)$ be the W-mappings generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n1}, \alpha_{n2}, \dots, \alpha_{nr}$. Now consider the following iteration process:

$$x_1 \in C, \quad x_{n+1} = W_n x_n \quad \text{for every} \quad n = 1, 2, \dots.$$

The following lemma is obvious from the definition of (3.1).

Lemma 3.1 ([2])**.** *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \le \alpha_i \le 1$ for every $i = 1, 2, \dots, r$. Let $U_1, U_2, \dots, U_{r-1}$ and W be the mappings defined by (3.1). Then, $U_1, U_2, \dots, U_{r-1}$ and W are also nonexpansive.*

The following lemma was proved in [2].

Lemma 3.2 ([2])**.** *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r-1$ and $0 < \alpha_r \le 1$. Let W be the W-mapping of C into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Then, $F(W) = \bigcap_{i=1}^{r} F(T_i)$.*

In this section, we study approximate fixed point sequences and monotone sequences. Let C be a nonempty subset of an ordered Banach space E and let T be a mapping of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n \to \infty} \|x_n - T x_n\| = 0$$

(see also [15, 23]). Let $T_1, T_2, \dots, T_r$ be mappings of C into itself. A sequence $\{x_n\}$ in C is said to be an *approximate common fixed point sequence* of mappings $T_1, T_2, \dots, T_r$ of C if for every $k = 1, 2, \dots, r$,

$$\lim_{n \to \infty} \|x_n - T_k x_n\| = 0.$$

A sequence $\{x_n\}$ in E is said to be *monotone* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [12]).

Lemma 3.3. *Let E be an ordered Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \dots, T_r$ be monotone mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let $\alpha_{n,1}, \dots, \alpha_{n,r}$ $(n = 1, 2, \dots)$ be real numbers such that $0 < \alpha_{n,i} \le 1$ for every $i = 1, 2, ..., r$. Let $W_n (n = 1, 2, \dots)$ be W-mappings generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \dots, \alpha_{n,r}$. Then, $U_{n,1}, U_{n,2}, \dots, U_{n,r-1}$, $U_{n,r} = W_n$ $(n = 1, 2, \dots)$ and $T_2 U_{n,1}, T_3 U_{n,2}, \dots, T_r U_{n,r-1}$ $(n = 1, 2, \dots)$ are also monotone.*

Proof. We shall prove that $U_{n,1}, U_{n,2}, \ldots, U_{n,r-1}, W_n = U_{n,r}$ $(n = 1, 2, \ldots)$ are monotone by mathematical induction. Let n be an arbitrary positive integer. Let $x, y \in C$ with $x \succeq y$. Since T_1 is monotone and a partial order is compatible with the linear structure of E, we obtain that

$$\begin{aligned} U_{n,1}x &= \alpha_{n,1}T_1x + (1-\alpha_{n,1})x \\ &\succeq \alpha_{n,1}T_1y + (1-\alpha_{n,1})x \\ &\succeq \alpha_{n,1}T_1y + (1-\alpha_{n,1})y = U_{n,1}y \end{aligned}$$

Thus, $U_{n,1}$ is monotone. Let $k = 1, 2, \ldots, r-1$ and suppose $U_{n,k}x \succeq U_{n,k}y$. We shall prove that $U_{n,k+1}x \succeq U_{n,k+1}y$. We remark that n is an arbitrary positive integer. Since $T_i (i = 1, 2, \ldots, r)$ is also monotone and from the the assumption of mathematical induction, it follows that $T_{k+1}U_{n,k}x \succeq T_{k+1}U_{n,k}y$. Hence, we have $T_{k+1}U_{n,k}$ is also monotone. Then, since a partial order is compatible with the linear structure of E, we have

$$\begin{aligned} U_{n,k+1}x &= \alpha_{n,k+1}T_{k+1}U_{n,k}x + (1-\alpha_{n,k+1})x \\ &\succeq \alpha_{n,k+1}T_{k+1}U_{n,k}y + (1-\alpha_{n,k+1})x \\ &\succeq \alpha_{n,k+1}T_{k+1}U_{n,k}y + (1-\alpha_{n,k+1})y = U_{n,k+1}y \end{aligned}$$

Thus, $U_{n,k+1}$ is monotone. Therefore, we obtain that $U_{n,1}, U_{n,2}, \ldots, U_{n,r-1}$ and $U_{n,r} = W_n$ are also monotone by mathematical induction. Then, since $T_1, T_2, \ldots, T_r$ are monotone, we have that $T_2U_{n,1}, T_3U_{n,2}, \ldots, T_rU_{n,r-1}$ are also monotone. □

Lemma 3.4. *Let E be an ordered Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ $(n = 1, 2, \ldots)$ be real numbers such that $0 < \alpha_{n,i} \leq 1$ $(n = 1, 2, \ldots)$ for every $i = 1, 2, ..., r$. Fix $x \in C$ such that $T_kx \succeq x$ $(k = 1, 2, \ldots, r)$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$. Then,*

$$U_{n,1}x \succeq x,\ U_{n,2}x \succeq x, \ldots, U_{n,r-1}x \succeq x,\ W_nx \succeq x. \tag{3.2}$$

Proof. We note that for $c_1, c_2 \in C$ such that $c_1 \succeq c_2$,

$$c_1 \succeq \alpha c_1 + (1-\alpha)c_2 \succeq c_2$$

for each $\alpha \in [0, 1]$. This is true because all order intervals are convex and a partial order is compatible with the linear structure of E.

Fix $x \in C$ such that $T_kx \succeq x$ $(k = 1, 2, \ldots, r)$. Let $n \in \mathbb{N}$. We shall prove

$$U_{n,m}x \succeq x \quad (m = 1, 2, \ldots, r) \tag{3.3}$$

by mathematical induction. First, we shall prove $U_{n,1}x \succeq x$. Since a partial order is compatible with the linear structure of E, by the assumption of T_1, we have that

$$\begin{aligned} U_{n,1}x &= \alpha_{n,1}T_1x + (1-\alpha_{n,1})x \\ &\succeq \alpha_{n,1}x + (1-\alpha_{n,1})x = x. \end{aligned} \tag{3.4}$$

So, (3.3) is true for $m = 1$. Let $m = k \in \{1, 2, \ldots, r-1\}$ and suppose that

$$U_{n,k}x \succeq x.$$

Then, since T_{k+1} is monotone, from the assumption of mathematical induction and the assumption of T_{k+1}, we have

$$\begin{aligned} U_{n,k+1}x &= \alpha_{n,k+1}T_{k+1}U_{n,k}x + (1-\alpha_{n,k+1})x \\ &\succeq \alpha_{n,k+1}T_{k+1}x + (1-\alpha_{n,k+1})x \\ &\succeq \alpha_{n,k+1}x + (1-\alpha_{n,k+1})x = x. \end{aligned}$$

So, (3.3) is true for $m = k+1$. Therefore, we obtain

$$U_{n,m}x \succeq x \quad (m = 1, 2, \ldots, r) \tag{3.5}$$

by mathematical induction. □

As in the proof of Lemma 3.4, we have the following lemma.

Lemma 3.5. *Let E be an ordered Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ $(n = 1, 2, \ldots)$ be real numbers such that $0 < \alpha_{n,i} \leq 1$ $(n = 1, 2, \ldots)$ for every $i = 1, 2, ..., r$. Assume that $T_k x \succeq x$ $(k = 1, 2, \ldots, r)$ for every $x \in C$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$. Then,*

$$U_{n,1}x \succeq x,\ U_{n,2}x \succeq x, \ldots, U_{n,r-1}x \succeq x,\ W_n x \succeq x \text{ for every } x \in C. \tag{3.6}$$

The following lemma was essentially proved in [24]. For the sake of completeness, we prove it (see also [2]).

Lemma 3.6. *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ $(n = 1, 2, \ldots)$ be real numbers such that $a \leq \alpha_{n,i} \leq b$ for every $i = 1, 2, ..., r$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W_n x_n$$

for every $n = 1, 2, \ldots$. Then, the sequence $\{x_n\}$ is an approximate common fixed point sequence of $T_1, T_2, \ldots, T_r$, i.e., for each $k = 1, 2, \ldots, r$,

$$\lim_{n\to\infty} \|T_k x_n - x_n\| = 0.$$

Proof. For $x_1 \in C$ and $f \in \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, put $r = \|x_1 - f\|$ and set

$$X = \{u \in E : \|u - f\| \leq r\} \cap C.$$

Then, X is a nonempty bounded closed convex subset of C which is T_k-invariant for every $k = 1, 2, \ldots, r$ and contains x_1. So, without loss of generality, we may assume that C is bounded.

Let $x_1 \in C$ and $w \in \bigcap_{i=1}^{r} F(T_i)$. By the definition of $\{x_n\}_{n=1}^{\infty}$, we have,

$$\begin{aligned}
\|x_{n+1} - w\| &= \|U_{n,r}x_n - w\| \\
&= \|\alpha_{n,r}(T_r U_{n,r-1}x_n - w) + (1-\alpha_{n,r})(x_n - w)\| \\
&\le \alpha_{n,r}\|U_{n,r-1}x_n - w\| + (1-\alpha_{n,r})\|x_n - w\| \\
&\le \alpha_{n,r}\alpha_{n,r-1}\|U_{n,r-2}x_n - w\| + (1-\alpha_{n,r}\alpha_{n,r-1})\|x_n - w\| \\
&\le \alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}\|U_{n,r-3}x_n - w\| + (1-\alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2})\|x_n - w\| \\
&\;\vdots \\
&\le \alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}\cdots\alpha_{n,2}\|U_{n,1}x_n - w\| \\
&\qquad + (1-\alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}\cdots\alpha_{n,2})\|x_n - w\| \\
&= \alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}\cdots\alpha_{n,2}\|\alpha_{n,1}T_1(x_n - w) + (1-\alpha_{n,1})(x_n - w)\| \\
&\qquad + (1-\alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}\cdots\alpha_{n,2})\|x_n - w\| \\
&\le \|x_n - w\|
\end{aligned}$$

and hence $\lim_{n\to\infty}\|x_n - w\|$ exists. Put $c = \lim_{n\to\infty}\|x_n - w\|$ and fix k with $1 \le k \le r-1$. Then, we have

$$\overline{\lim_{n\to\infty}}\|U_{n,k}x_n - w\| \le \lim_{n\to\infty}\|x_n - w\| = c.$$

Further, since

$$\|x_n - w\| \le \|U_{n,k}x_n - w\| + \frac{\|x_n - w\| - \|x_{n+1} - w\|}{\alpha_{n,r}\alpha_{n,r-1}\alpha_{n,r-2}\cdots\alpha_{n,k+1}}$$

we have,

$$c \le \underline{\lim}_{n\to\infty}\|U_{n,k}x_n - w\|$$

and hence $\lim_{n\to\infty}\|U_{n,k}x_n - w\| = c$. Therefore, we have

$$\lim_{n\to\infty}\|\alpha_{n,k}T_kU_{n,k-1}x_n + (1-\alpha_{n,k})x_n - w\| = c.$$

for all $k = 1, 2, \ldots r$, where $U_{n,0} = I$. By Lemma 2.2, we have

$$\lim_{n\to\infty}\|T_kU_{n,k-1}x_n - x_n\| = 0.$$

If $k = 1$, we have

$$\lim_{n\to\infty}\|T_1x_n - x_n\| = 0.$$

For any k with $2 \le k \le r$, from

$$\begin{aligned}
\|T_kx_n - x_n\| &\le \|T_kx_n - T_kU_{n,k-1}x_n\| + \|T_kU_{n,k-1}x_n - x_n\| \\
&\le \|x_n - U_{n,k-1}x_n\| + \|T_kU_{n,k-1}x_n - x_n\| \\
&= \alpha_{n,k-1}\|x_n - T_kU_{n,k-2}x_n\| + \|T_kU_{n,k-1}x_n - x_n\|,
\end{aligned}$$

we have $\lim_{n\to\infty}\|T_kx_n - x_n\| = 0$. □

Lemma 3.7. *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Assume that $T_k x \succeq x (k = 1, 2, \ldots, r)$ for every $x \in C$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r}$ $(n = 1, 2, \ldots)$ be real numbers such that $0 < \alpha_{n,i} \leq 1$ $(n = 1, 2, \ldots)$ for every $i = 1, 2, ..., r$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W_n x_n$$

for every $n = 1, 2, \ldots$. Then, the sequence $\{x_n\}$ is monotone.

Proof. As in the proof of Lemma 3.6, without loss of generality, we may assume that C is bounded. Let $n \in \mathbb{N}$. Since $T_k u \succeq u$ $(k = 1, 2, \ldots, r)$ for every $u \in C$, we have

$$\begin{aligned} x_{n+1} &= W_n x_n = U_{n,r} x_n = \alpha_{n,r} T_r U_{n,r-1} x_n + (1 - \alpha_{n,r}) x_n \\ &\succeq \alpha_{n,r} U_{n,r-1} x_n + (1 - \alpha_{n,r}) x_n \\ &= \alpha_{n,r} \{\alpha_{n,r-1} T_{r-1} U_{n,r-2} x_n + (1 - \alpha_{n,r-1}) x_n\} + (1 - \alpha_{n,r}) x_n \\ &\succeq \alpha_{n,r} \alpha_{n,r-1} U_{n,r-2} x_n + (1 - \alpha_{n,r} \alpha_{n,r-1}) x_n \\ &\vdots \\ &\succeq \alpha_{n,r} \alpha_{n,r-1} \ldots \alpha_{n,2} U_{n,1} x_n + (1 - \alpha_{n,r} \alpha_{n,r-1} \ldots \alpha_{n,2}) x_n \\ &= \alpha_{n,r} \alpha_{n,r-1} \ldots \alpha_{n,1} T_1 x_n + (1 - \alpha_{n,r} \alpha_{n,r-1} \ldots \alpha_{n,1}) x_n \\ &\succeq \alpha_{n,r} \alpha_{n,r-1} \ldots \alpha_{n,1} x_n + (1 - \alpha_{n,r} \alpha_{n,r-1} \ldots \alpha_{n,1}) x_n = x_n. \end{aligned}$$

Hence, we have that $\{x_n\}$ is monotone. □

Lemma 3.8. *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Fix $u \in C$ such that $T_k u \succeq u (k = 1, 2, \ldots, r)$. Let $\alpha_1, \ldots, \alpha_r$ be real numbers such that $0 < \alpha_i \leq 1$ for every $i = 1, 2, ..., r$. Let W be a W-mapping generated by $T_1, T_2, \ldots, T_r$ and $\alpha_1, \ldots, \alpha_r$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_1$$

for every $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ is monotone.

Proof. As in the proof of Lemma 3.6, without loss of generality, we may assume that C is bounded. We shall prove that

$$x_{n+1} \succeq x_n \ (n = 1, 2, \ldots) \tag{3.7}$$

by mathematical induction. By Lemma 3.4, we have

$$x_2 = W x_1 \succeq x_1.$$

Thus, (3.7) is true for $n = 1$. Let $k \in \mathbb{N}$ and suppose

$$x_{k+1} \succeq x_k.$$

Since T_rU_{r-1} is monotone, it follows from the assumption of mathematical induction that

$$\begin{aligned} x_{k+2} &= W^{k+1}x_1 = Wx_{k+1} = U_rx_{k+1} \\ &= \alpha_rT_rU_{r-1}x_{k+1} + (1-\alpha_r)x_{k+1} \\ &\succeq \alpha_rT_rU_{r-1}x_k + (1-\alpha_r)x_{k+1} \\ &\succeq \alpha_rT_rU_{r-1}x_k + (1-\alpha_r)x_k = x_{k+1}, \end{aligned}$$

where $U_0 = I$. So, by mathematical induction, we obtain that $x_{n+1} \succeq x_n$ for every $n = 1, 2, \ldots$. □

4. Main results

Theorem 4.1. *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \ldots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^r F(T_i)$ is nonempty. Assume that $T_kx \succeq x$ $(k = 1, 2, \ldots, r)$ for every $x \in C$. Let a, b be real numbers with $0 < a \le b < 1$. Let $\alpha_{n,1}, \ldots, \alpha_{n,r} (n = 1, 2, \ldots)$ be real numbers such that $a \le \alpha_{n,i} \le b$ for every $i = 1, 2, ..., r$. Let $W_n (n = 1, 2, \ldots)$ be W-mappings generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \ldots, \alpha_{n,r}$,*

$$\begin{aligned} U_{n,1} &= \alpha_{n,1}T_1 + (1-\alpha_{n,1})I, \\ U_{n,2} &= \alpha_{n,2}T_2U_{n,1} + (1-\alpha_{n,2})l, \\ &\vdots \\ U_{n,r-1} &= \alpha_{n,r-1}T_{r-1}U_{n,r-2} + (1-\alpha_{n,r-1})I \\ W = U_{n,r} &= \alpha_{n,r}T_rU_{n,r-1} + (1-\alpha_{n,r})I. \end{aligned}$$

Suppose $x_1 \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = W_nx_n$$

for every $n = 1, 2, \ldots$. Then, the sequence $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \ldots, T_r$.

Proof. Since E is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. Let z_1, z_2 be two weak cluster-points of $\{x_n\}$. Then, there exists two subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$, respectively. It follows from Lemmas 3.6 that for every $k = 1, 2, \ldots, r$,

$$\lim_{i\to\infty} \|x_{n_i} - T_kx_{n_i}\| = 0 \quad \text{and} \qquad \lim_{j\to\infty} \|x_{n_j} - T_kx_{n_j}\| = 0.$$

It follows from Theorem 2.1 that $z_1, z_2 \in \bigcap_{i=1}^r F(T_i) \neq \emptyset$. Next, we show $z_1 = z_2$ (see also [12]). Fix $k \ge 1$. By Lemma 3.7, we obtain that $\{x_n\}$ is monotone. Then, since the order interval $[x_k, \rightarrow)$ is weakly closed, we conclude that $z_i \in [x_k, \rightarrow)$ for $i = 1, 2$. So, we see that z_i is an upper bound for $\{x_n\}$ for $i = 1, 2$. Then, we also obtain that $\{x_n\} \subset (\leftarrow, z_i]$ for $i = 1, 2$. It follows from the same reason that

$z_j \in (\leftarrow, z_i]$ for $i, j = 1, 2$. So, we have $z_1 = z_2$. Therefore, we obtain that $\{x_n\}$ converges weakly to a point of $\bigcap_{i=1}^{r} F(T_i) \neq \emptyset$. □

By Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let E be an ordered uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T_1, T_2, \dots, T_r$ be finite monotone nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{r} F(T_i)$ is nonempty. Fix $x_1 = u \in C$ such that $T_k x_1 \succeq x_1$ $(k = 1, 2, \dots, r)$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\alpha_{n,1}, \dots, \alpha_{n,r} (n = 1, 2, \dots)$ be real numbers such that $a \leq \alpha_{n,i} \leq b$ for every $i = 1, 2, ..., r$. Let W be a W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \dots, \alpha_r$. Suppose $x_1 = u \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = W^n x_1$$

for every $n = 1, 2, \dots$. Then, the sequence $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \dots, T_r$.

Proof. We remark that $x_{n+1} = W^n x_1 = W x_n$. By Lemma 3.8, we have that $\{x_n\}$ is monotone. By Lemma 3.6, we obtain that for each $k = 1, 2, \dots, r$,

$$\lim_{n \to \infty} \|T_k x_n - x_n\| = 0.$$

Then, as in the proof of Theorem 4.1, we have that $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \dots, T_r$. □

By Theorem 4.1, we get some convergence theorems (see also [21]).

Theorem 4.3. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let T and S be monotone nonexpansive mappings of C into itself such that $F(T) \cap F(S) \neq \emptyset$. Assume that $x \preceq Tx$ and $x \preceq Sx$ for each $x \in C$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequence of real numbers such that $a < \alpha_n < b$ and $a < \beta_n < b$ for each $n = 1, 2, \dots$, respectively. Suppose $x_1 \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n S[\beta_n T x_n + (1 - \beta_n) x_n] + (1 - \alpha_n) x_n$$

for every $n = 1, 2, \dots$. Then, $\{x_n\}$ converges weakly to a common fixed point of T and S.

Theorem 4.4. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$. Let a, b be real numbers with $0 < a \leq b < 1$. Let $\{\alpha_n\}$ be the sequence of real numbers such that $a < \alpha_n < b$ for each $n \in \mathbb{N}$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n$$

for every $n = 1, 2, \dots$. Then, $\{x_n\}$ converges weakly to a fixed point of T.

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