Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 9, Number 2, 2023, 149–174

A GENERALIZATION OF SAINT-VENANT EXNER EQUATIONS

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Abstract. We derive asymptotically an alternative model of the bedload sediment transport model also called Saint-Venant-Exner equations. Starting from the coupling fluid/particles with the Vlasov equations with an additional gravity term, the compressible Navier-Stokes equations with an anisotropic viscous tensor and a modified barotropic law, we derive first a Navier-Stokes equations like model. It takes account of the mixing of the sediments and fluid density. Then, taking advantages of the shallowness of the domain, we obtain the Saint-Venant equations naturally coupled with a bedload sediment transport equation. Finally, we show some stability results, for various solid transport flux; particularly for a diffusive and the so called Grass solid transport.

1. Introduction

The study of the sediment transport have a huge interest since they may damage environmental and human life. For instance, all sediment problems related to river beds have several consequences. The origin of sediment could be endogenous (this means that sediment are created from Aboriginal production) or exogenous (this means that sediment are provided by allochthonous material: soil erosion, decomposition of plant material or inputs of suspended matter, organic matter, nutrients and micro-pollutant emissions due to agricultural, industrial and domestic) and contributes to the evolution of the river bed and the modification of the fauna and flora. The increase of the bed may lead to flood and sediments affected have an impact on the quality of the water. Therefore, the analysis of sediment transport is important to predict and prevent natural disaster (or caused by human activities). Gravity and settling force between sediments and fluid (air or water) produce sediment transport, that can be classified in three categories: suspension, bedload and saltation. To this end, many mathematical models are proposed. The most often used is the so-called Saint-Venant-Exner (SVE) equations which describes the hydrodynamic part governed by the Saint-Venant equations and the bed-load morphodynamic part by a transport equation. Since, the bed is movable, these equations are simply coupled by the topography term. Then, (SVE) equations reads:

$$
\left\{ \begin{array}{l} \partial_t h + \operatorname{div}(q) = 0, \\ \partial_t q + \operatorname{div}(\dfrac{q \otimes q}{h}) + \nabla(gh^2/2) = -gh\nabla b \end{array} \right. \tag{1.1}
$$

2020 *Mathematics Subject Classification.* 35K51, 35Q30, 76D05.

Key words and phrases. Vlasov-Navier-Stokes equations, Fluid-particles interaction, Anisotropic viscosity, Saint-Venant-Exner, variable barotropic law, thin-layer.

with the general morphodynamic bed-load transport:

$$\partial_t b + \xi \mathrm{div}(q_b(h,q)) = 0, \tag{1.2}$$

where h denotes the water height, $q = hU$ the water discharge, with U the fluid velocity, q_b the sediment discharge (or the solid transport flux) (see Figure 1) and $\xi = 1/(1-\psi)$ the porosity of the sediment layer. The sediment discharge q_b which governes the topography elevation is a given empirical law. Among these laws (to cite only a few), we find in the literature:

- the Grass equation [11],
- The Meyer-Peter and Műller equation [16],
- others empirical laws are provided in [8].

This model have been numerically studied, for instance in [8, 18] and mathematically for viscous version (i.e., viscous Saint-Venant for the hydrodynamic part), for instance in [19, 6] using the well-known results [17, 12, 1, 2, 5, 15]. The aim of this paper is to present a way to derive asymptotically the (SVE) equations with viscous or non-viscous shallow water. The paper is organized as follows. In section 2, from a Vlasov-Compressible Navier-Stokes equations with an anisotropic viscous tensor and a variable barotropic law, we derive some equations for small relaxation time to modeling the fluid-particles interaction, where the particles are assumed small with respect to the domain geometry. Thereafter, we perform an asymptotic analysis close to those presented in [9, 13]. Contrary to these works, we study a compressible fluid, so we take advantage of the averaging of the horizontal velocity and the given pressure. Next, in Section 3, the asymptotic model is vertically averaged to obtain a model very closed to (SVE) equations with a minor difference which comes from the pressure law. Finally, in despite of the presssure law difference, we show that stability properties for the diffusive morphodynamic bed-load transport equation [6] remains true. This results also led in the case of the Grass equations, where an existence result is provided in [3].

2. Derivation of the AVACNS equations

This section is devoted to the derivation of the Averaged Vlasov-Compressible Navier-Stokes (AVACNS) equations to modeling the fluid-particles interaction which occurs during a sedimentation process. We assume that sedimentation (a set of particles) which evolves into a fluid (assumed to be water for instance) are governed by the so-called Vlasov equations, where a gravity term is included to take account the weight of particles. On the other hand, the fluid is governed by the compressible Navier-Stokes equations with a barotropic pressure law. The barotropic pressure law will be modified to take account the effect of sediments (see Equation (2.8)). A source term (principle of action reaction) is also added to deal with the effect of particles motion on the fluid. Finally, the classical viscosity $\nu(\rho)$ appearing in Navier-Stokes equations is replaced with an anisotropic viscous tensor. Finally, as we want to derive a Saint-Venant-Exner equations, we assume that the ratio H/L

is small, where H is a vertical characteristic length and L a horizontal one. The following section is very close to the one presented in [10].

2.1. **The AVACNS equations.** The evolution of particles interacting with a fluid is described through the density function $f(t, X, V) \geq 0$ where $X \in \Omega_0 \subset \mathbb{R}^3$ denotes the position of a particle with $V \in \mathbb{R}^3$ its kinetic speed in which

$$\Omega_0 = \big\{X = (x_1, x_2, x_3) \in \mathbb{R}^3; (x_1, x_2) \in [0, \mathfrak{L}_1] \times [0, \mathfrak{L}_2] \text{ and } 0 \leqslant x_3 \leqslant H(t, x_1, x_2)\big\}$$

denotes the domain and

$$\Omega(t) = \{X = (x_1, x_2, x_3) \in \mathbb{R}^3; (x_1, x_2) \in [0, \mathfrak{L}_1] \times [0, \mathfrak{L}_2] \\ \text{and } b(t, x_1, x_2) \leqslant x_3 \leqslant H(t, x_1, x_2)\}$$

the fluid-particles domain (see Figure 1). H is the total elevation of the free surface and b the topography induced by sediments store. The density function f satisfy the Vlasov equation with gravity:

$$\partial_t f + \text{div}_X(V\, f) + \text{div}_V((F + \vec{g})f) = r\Delta_V f. \tag{2.1}$$

Denoting $M = \rho_p \frac{4}{3}\pi a^3$ the mass of a particle where a is the radius (assumed constant, that is $a = a_0$ for all particles) and ρ_p the mass density of a particle, the quantity MF is the force acting on a particle which is proportional to the drag Stokes force given by:

$$F = \frac{6\pi\mu a}{M}(U - V), \tag{2.2}$$

where U is the fluid velocity and μ a characteristic viscosity (assumed constant) of the fluid.
The quantity $r > 0$ denotes the velocity diffusivity is given by the Einstein formula:

$$r = \frac{\nu_0 T}{M}\frac{6\pi\mu a}{M} = \frac{\nu_0 T}{M}\frac{9\mu}{2a^2\rho_p}, \tag{2.3}$$

where ν_0 is the Boltzmann constant, $T > 0$ is the temperature of the suspension assumed to be a constant. The term $r\Delta_V f$ describes then the Brownian motion of the particles.
The quantity $\vec{g}$ is the gravity vector $(0, 0, -g)^t$, also denoted by $-g\vec{k}$ with $\vec{k} = (0, 0, 1)^t$. On the other hand, the fluid is described by its velocity $U(t, X) = (u_1, u_2, u_3)(t, X)$ with $X = (x_1, x_2, x_3) \in \Omega(t)$, its density $\rho_w(t, X)$ and the macroscopic density of sediments $\rho_s = \int_{\mathbb{R}^3} f(t, X, \cdot)\, dv$, which satisfy the following Navier-Stokes equations:

$$\begin{cases} \partial_t \rho_w + \text{div}(\rho_w U) = 0, & (2.4) \\ \partial_t(\rho_w U) + \text{div}(\rho_w U \otimes U) = \text{div}\sigma(\rho, U) + \mathfrak{F}, & (2.5) \\ p = p(\rho) & (2.6) \end{cases}$$

where $\rho := \rho_w + \rho_s$ and $\sigma(\rho, U)$ is the total stress tensor:

$$-p(\rho)I_3 + 2\Sigma(\rho) : D(U) + \lambda(\rho)\text{div}(U)\, I_3,$$

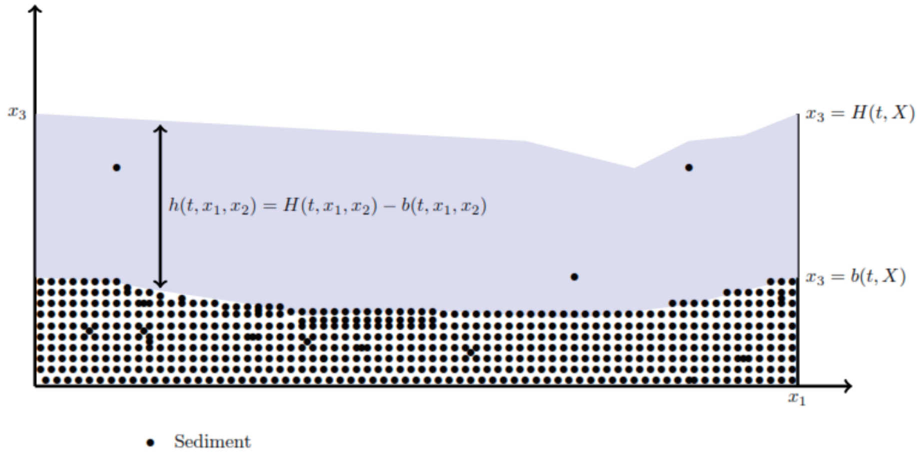

FIGURE 1. Domain $\Omega(t)$

where I_3 stands the identity matrix. The term $\Sigma(\rho)$ is the anisotropic matrix which take into account of sediment and allows to control the flow direction by playing with the magnitude of the viscosities μ_i. This matrix reads:

$$\begin{pmatrix} \mu_1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_1 & \mu_2 \\ \mu_3 & \mu_3 & \mu_3 \end{pmatrix}.$$

Then, the viscous tensor $\Sigma(\rho) : D(U)$ writes:

$$\begin{pmatrix} \mu_1(\rho)\partial_{x_1} u_1 & \frac{\mu_1(\rho)}{2}(\partial_{x_2} u_1 + \partial_{x_1} u_2) & \frac{\mu_2(\rho)}{2}(\partial_{x_3} u_1 + \partial_{x_1} u_3) \\ \frac{\mu_1(\rho)}{2}(\partial_{x_1} u_2 + \partial_{x_2} u_1) & \mu_1(\rho)\partial_{x_2} u_2 & \frac{\mu_2(\rho)}{2}(\partial_{x_3} u_2 + \partial_{x_2} u_3) \\ \frac{\mu_3(\rho)}{2}(\partial_{x_1} u_3 + \partial_{x_3} u_1) & \frac{\mu_3(\rho)}{2}(\partial_{x_2} u_3 + \partial_{x_3} u_2) & \mu_3(\rho)\partial_{x_3} u_3 \end{pmatrix}. \tag{2.7}$$

The pressure law is modified to take into account sediments density and it is given as follows:

$$p(\rho_w, \rho_s) = \frac{g}{\rho_p} \frac{h\,(\rho_w + \rho_s)^2}{4}, \tag{2.8}$$

where $h = H - b$ is the fluid height, given by the kinematic condition at the free-surface.

The last term $\mathfrak{F}$ on the right hand side of Equation (2.5) is the effect of the particles motion on the fluid obtained by summing the contribution of all particles:

$$\mathfrak{F} = -\int_{\mathbb{R}^3} F\, f(t, X, V)\, dV + \rho_w \vec{g} = \frac{9\mu}{2a^2\rho_p} \int_{\mathbb{R}^3} (V - U) f(t, X, V)\, dV + \rho_w \vec{g}. \tag{2.9}$$

Finally, the model describing the fluid-particles interactions is given by the following unknowns (f, U, ρ_w) system, noted VACNS (Vlasov-Anisotropic Compressible Navier-Stokes):

(2.10)
$$\begin{cases} \partial_t f + \mathrm{div}_X(Vf) + \mathrm{div}_V\left(\left(\dfrac{6\pi\mu a}{M}(U-V)+\vec{g}\right)f\right) = \dfrac{\nu_0 T}{M}\dfrac{9\mu}{2a^2\rho_p}\Delta_V f, \\ \partial_t \rho_w + \mathrm{div}(\rho_w U) = 0, \\ \partial_t(\rho_w U) + \mathrm{div}(\rho_w U\otimes U) + \nabla p - 2\mathrm{div}\,(\Sigma : D(U)) = \nabla(\lambda(\rho_w)\mathrm{div}(U)) \\ \qquad +\dfrac{9\mu}{2a^2\rho_p}\displaystyle\int_{\mathbb{R}^3}(V-U)f(t,X,V)\,dV \\ \qquad -g\rho_w\vec{k} \end{cases}$$

where p is given by (2.8). The System (2.10) is then equipped with boundary conditions described in the following section.

2.2. **Boundary conditions.** Recall that the fluid domain is defined as follows:

$$\{X = (x_1, x_2, x_3) \in \mathbb{R}^3; (x_1, x_2) \in [0, \mathfrak{L}_1] \times [0, \mathfrak{L}_2] \\ \text{and } b(t, x_1, x_2) \leqslant x_3 \leqslant H(t, x_1, x_2)\},$$

where $x_3 = H$ represents the local water elevation from the surface $x_3 = 0$ to the free surface and $x_3 = b$ denotes the local sediment layer elevation from the surface $x_3 = 0$. We have to prescribe for fluid layer a free surface and bottom condition. Particularly, a bottom condition which describe how the sediment layer evolves through the constraints acting from the fluid and how incoming particles or outgoing particles increase or decrease the elevation of this interface. Nevertheless, kinematic boundary conditions as in [14] are not considered here.
The basic idea is the following: we assume the continuity of the velocity of displacement $\mathfrak{u}_\mathfrak{b}$ and the fluid velocity U on the bottom:

$$U \cdot n_b = \mathfrak{u}_\mathfrak{b} \cdot n_b, \text{ at } x_3 = b(t, x_1, x_2),$$

where $\mathfrak{u}_\mathfrak{b}$ is defined from the first moment of the density function of sediments as follows:

$$\mathfrak{u}_\mathfrak{b} = \frac{\displaystyle\int_{\partial\Omega_b^-} Vf(t,X,V)\,dV + \int_{\partial\Omega_b^+} Vf(t,X,V)\,dV}{\displaystyle\int_{\partial\Omega_b^-} f(t,X,V)\,dV + \int_{\partial\Omega_b^+} f(t,X,V)\,dV},$$

where $\partial\Omega_b^+$ is the set of incoming particles which tends to increase b and $\partial\Omega_b^-$ is the set of outgoing particles.

Free surface conditions. Neglecting the air viscosity, we prescribe a normal stress continuity condition on the free surface with surface tension at the air/fluid interface by:

$$\sigma(U)n_s = (\beta_0\,\kappa(t,X) - p_0)\,n_s \text{ at } x_3 = H(t,X),$$

where H is the total height as displayed on Figure 1. The quantity n_s is the unit outward normal to the free surface defined by

$$n_s = \frac{1}{\sqrt{1+|\nabla_{x_1,x_2}H|^2}}\begin{pmatrix}-\partial_{x_1}H\\ -\partial_{x_2}H\\ 1\end{pmatrix}.$$

β_0 is a capillary coefficient and κ is the mean curvature of the surface and p_0 the atmospheric pressure at the free surface. The free surface boundary conditions write:

$$\left\{\begin{array}{l}(p+\beta_0\kappa-p_0)\,\nabla_{x_1,x_2}H - \mu_1 D_{x_1,x_2}\begin{pmatrix}u_1\\u_2\end{pmatrix}\nabla_{x_1,x_2}H\\[2ex] +\mu_2\left(\partial_{x_3}\cdot\begin{pmatrix}u_1\\u_2\end{pmatrix}+\nabla_{x_1,x_2}u_3\right) - \lambda\mathrm{div}(U)\nabla_{x_1,x_2}H = 0,\\[2ex] -p-\beta_0\kappa+p_0-\mu_3\left(\partial_{x_3}\begin{pmatrix}u_1\\u_2\end{pmatrix}+\nabla_{x_1,x_2}u_3\right)\cdot\nabla_{x_1,x_2}H\\[2ex] +2\mu_3\partial_{x_3}u_3+\lambda\mathrm{div}(U) = 0\,.\end{array}\right. \tag{2.11}$$

Following [9, 13], we also introduce an indicator function ϕ which states that the fluid region is advected at the fluid velocity U:

$$\phi = \left\{\begin{array}{lll}1 & \text{if} & x_3\in[b(t,x_1,x_2),H(t,x_1,x_2)],\\ 0 & \text{otherwise}\,. & \end{array}\right. \tag{2.12}$$

Using Equations (2.4) and (2.12), the mass conservation writes

$$\partial_t(\rho_w\,\phi) + \mathrm{div}(\rho_w\,U\,\phi) = 0. \tag{2.13}$$

This indicator function is useful to obtain the so-called *free surface condition.*

Bottom conditions. we prescribe in a general way a wall-law condition which writes:

$$((\sigma(U)n_b)\cdot\tau_b)\,\tau_b = \mathfrak{K}(U) \text{ at } x_3 = b(t,x_1,x_2).$$

The unit outward normal vector is given by

$$n_b = \frac{1}{\sqrt{1+|\nabla_{x_1,x_2}b|^2}}\begin{pmatrix}\partial_{x_1}b\\ \partial_{x_2}b\\ -1\end{pmatrix}$$

and τ_b is any tangential component of the tangential basis at elevation $x_3 = b(t,x_1,x_2)$. The bottom conditions can also be written:

$$\sigma(U)n_b - (\sigma(U)n_b\cdot n_b)\,n_b = \mathfrak{K}(U) \text{ at } x_3 = b(t,x_1,x_2),$$

which writes for each components:
(2.14)

$$\begin{cases} -\mu_1 \left(\left(1 + |\nabla_{x_1,x_2} b|^2\right) I_2 - \nabla_{x_1,x_2} b \nabla_{x_1,x_2} b^t \right) D_{x_1,x_2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \nabla_{x_1,x_2} b \\ \quad + \mu_2 \left(\left(1 + |\nabla_{x_1,x_2} b|^2\right) I_2 - \nabla_{x_1,x_2} b \nabla_{x_1,x_2} b^t \right) \left(\partial_{x_3} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \nabla_{x_1,x_2} u_3 \right) \\ \quad - \mu_3 \left(\left(\partial_{x_3} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \nabla_{x_1,x_2} u_3 \right) \cdot \nabla_{x_1,x_2} b \right) \nabla_{x_1,x_2} b + 2\mu_3 \partial_{x_3} u_3 \, \nabla_{x_1,x_2} b \\ \quad = \left(1 + |\nabla_{x_1,x_2} b|^2\right)^{3/2} \begin{pmatrix} \mathfrak{K}_1(U) \\ \mathfrak{K}_2(U) \end{pmatrix}, \\ -\mu_3 \, |\nabla_{x_1,x_2} b|^2 \left(\partial_{x_3} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \nabla_{x_1,x_2} u_3 \right) \cdot \nabla_{x_1,x_2} b + 2\mu_3 \, |\nabla_{x_1,x_2} b|^2 \, \partial_{x_3} u_3 \\ \quad + \mu_2 \left(\partial_{x_3} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \nabla_{x_1,x_2} u_3 \right) \cdot \nabla_{x_1,x_2} b \\ \quad - \mu_1 \left(D_{x_1,x_2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \nabla_{x_1,x_2} b \right) \cdot \nabla_{x_1,x_2} b \\ \quad = \left(1 + |\nabla_{x_1,x_2} b|^2\right)^{3/2} \mathfrak{K}_3(U). \end{cases}$$

We complete the bottom conditions by the continuity condition of the velocity through out the interface separating the sediment and the fluid layer:

$$\partial_t b + \sqrt{1 + |\nabla_{x1,x_2} b|^2} U \cdot n_b = S \text{ at } x_3 = b(t, x_1, x_2), \tag{2.15}$$

where $S - \sqrt{1 + |\nabla_{x1,x_2} b|^2} U \cdot n_b$ governs the variation of b; as the formalism described (above) for outgoing and incoming particles.

2.3. **The non-dimensional VACNS equations.** We will perform an asymptotic analysis in the following framework: we want modelling, mainly,

- small relaxation time with respect to the time scale, and
- small particles with respect to the dimension of the domain.

To this end, we will write the non-dimensional System corresponding to (2.10). Thereby, let us introduce the characteristic length $\mathfrak{L}$ and the characteristic time scale $\mathfrak{T}$. Then, we write the characteristic fluid velocity as $\mathfrak{U} = \frac{\mathfrak{L}}{\mathfrak{T}}$. We define the relaxation time that we want small compared with $\mathfrak{T}$ by

$$\tau = \frac{M}{6\pi\mu a} = \frac{2a^2\rho_p}{9\mu} \tag{2.16}$$

and a fluctuation of the velocity by:

$$\sqrt{\theta} = \sqrt{\frac{\nu_0 T}{M}}$$

which will be compared to the characteristic velocity $\mathfrak{U}$. We also introduce the non-dimensional variables of time, space, kinematic speed, fluid velocity, pressure, density, viscosity and sediment density respectively:

$$\widetilde{t} = \frac{t}{\mathfrak{T}}, \quad \widetilde{X} = \frac{X}{\mathfrak{L}}, \quad \widetilde{V} = \frac{V}{\sqrt{\theta}}, \quad \widetilde{U} = \frac{U}{\sqrt{\theta}}, \quad \widetilde{p} = \frac{p}{\rho_0 \theta}, \quad \widetilde{\rho} = \frac{\rho}{\rho_0},$$

$$\widetilde{\mu_i}(\widetilde{\rho}) = \frac{\mu_i(\rho)}{\mu}, \quad \widetilde{\lambda}(\widetilde{\rho}) = \frac{\lambda(\rho)}{\mu} \quad \text{and} \quad \widetilde{f}(\widetilde{t}, \widetilde{X}, \widetilde{V}) = \rho_0 \sqrt{\theta}^3 f(\mathfrak{T}\widetilde{t}, \mathfrak{L}\widetilde{X}, \sqrt{\theta}\widetilde{V}).$$

With these dimensionless variables, System (2.10) writes:

$$\begin{cases} \frac{1}{\mathfrak{T}}\partial_{\widetilde{t}}\widetilde{f} + \frac{\sqrt{\theta}}{\mathfrak{L}}\widetilde{V}.\nabla_{\widetilde{X}}\widetilde{f} + \frac{1}{\sqrt{\theta}}\mathrm{div}_{\widetilde{V}}(\frac{\sqrt{\theta}}{\tau}(\widetilde{U} - \widetilde{V})\widetilde{f} - g\widetilde{f}\vec{k}) = \frac{1}{\tau}\Delta_{\widetilde{V}}\widetilde{f}, & (2.17) \\ \frac{1}{\mathfrak{T}}\partial_{\widetilde{t}}\widetilde{\rho_w} + \frac{\sqrt{\theta}}{\mathfrak{L}}\mathrm{div}(\widetilde{\rho_w}\widetilde{U}) = 0, & (2.18) \\ \frac{\sqrt{\theta}\rho_0}{\mathfrak{T}}\partial_{\widetilde{t}}(\widetilde{\rho_w}\widetilde{U}) + \frac{\theta\rho_0}{\mathfrak{L}}\mathrm{div}_{\widetilde{X}}(\widetilde{\rho_w}\widetilde{U}\otimes\widetilde{U}) - \\ \frac{\sqrt{\theta}\mu}{\mathfrak{L}^2}\left(2\mathrm{div}_{\widetilde{X}}(\widetilde{\Sigma} : D(\widetilde{U})) + \nabla_{\widetilde{X}}(\widetilde{\lambda}\mathrm{div}_{\widetilde{X}}(\widetilde{U}))\right) + \frac{\rho_0\,\theta}{\mathfrak{L}}\nabla_{\widetilde{X}}\widetilde{p} \\ = \frac{9\mu}{2a^2\,\rho_p}\rho_0\sqrt{\theta}\int_{\mathbb{R}^3}(\widetilde{V} - \widetilde{U})\widetilde{f}(\widetilde{t}, \widetilde{X}, \widetilde{V})\,d\widetilde{V} - \rho_0\, g\,\widetilde{\rho_w}\vec{k}. & (2.19) \end{cases}$$

Next, we write the dimensionless number B, C, E, F as follows:

$$B = \frac{\sqrt{\theta}}{\mathfrak{U}}, \quad C = \frac{\mathfrak{T}}{\tau}, \quad F = \frac{g\mathfrak{T}}{\sqrt{\theta}}, \quad E = \frac{2}{9}\left(\frac{a}{\mathfrak{L}}\right)^2\frac{\rho_p}{\rho_0}C. \tag{2.20}$$

Then, multiplying Equation (2.17) and Equation (2.18) by $\mathfrak{T}$, Equation (2.19) by $\dfrac{\mathfrak{T}}{\rho_0\sqrt{\theta}}$, using definition of τ in (2.16), with the dimensionless numbers (2.20), and dropping $\widetilde{\cdot}$, the previous system reads:

(2.21)

$$\begin{cases} \partial_t f + B\mathrm{div}_X(Vf) + C\mathrm{div}_V\left((U - V)f - \nabla_V f\right) = F\mathrm{div}_V(\vec{k}\,f), \\ \partial_t\rho_w + B\mathrm{div}(\rho_w U) = 0, \\ \partial_t(\rho_w U) + \mathrm{div}(\rho_w U\otimes U) + B\nabla p - 2\,E\,\mathrm{div}\,(\Sigma : D(U)) = E\,\nabla(\lambda\mathrm{div}(U)) \\ \qquad +C\int_{\mathbb{R}^3}(V - U)f(t, X, V)\,dV \\ \qquad -\rho_w F\vec{k}, \end{cases}$$

where $\vec{k} = (0,0,1)^t$.

Remark 2.1. Boundary conditions are invariant by this scaling.

2.4. **Formal asymptotic and the IVACNS equations.** Assume $\varepsilon = \dfrac{\tau}{\mathfrak{T}}$ "small" and let us consider the following asymptotic regime,

$$\frac{\rho_p}{\rho_0} = O(1), \quad B = O(1), \quad C = \frac{1}{\varepsilon}, \quad F = O(1), \quad E = O(1). \tag{2.22}$$

Let us give some explanation concerning the physical meanings of this asymptotic ordering:

- the assumption $\dfrac{\rho_p}{\rho_0} = O(1)$ means that weights of particles are not neglected.
- $B = O(1)$ means that the characteristic velocity $\mathfrak{U}$ of the fluid and the fluctuation of the velocity $\sqrt{\theta}$ are of the same order with respect to ε.
- as described in [10], the term C can be seen as the Knudsen number in fluid hydrodynamics and it corresponds to assume that the time scale of interactions are very fast.
- the term F can be seen as the inverse of the Froude number. Indeed, as $\sqrt{\theta} \approx \mathfrak{U} = \dfrac{\mathfrak{L}}{\mathfrak{T}}$, we can rewrite F as

$$F = \frac{g\mathfrak{T}}{\mathfrak{L}} = \frac{g\mathfrak{L}}{\mathfrak{U}^2}.$$

- $E = O(1)$ means that the particles size are smaller compared to the size of the domain.

Now, we will only interest to a solution at main order with respect to ε; we look for function f, U, p and ρ_w which admit the following asymptotic expansions:

$$\begin{aligned} f &= f^0 + \varepsilon f^1 + O(\varepsilon^2), \\ U &= U^0 + \varepsilon U^1 + O(\varepsilon^2), \\ p &= p^0 + \varepsilon p^1 + O(\varepsilon^2), \\ \rho_w &= \rho_w^0 + \varepsilon \rho_w^1 + O(\varepsilon^2). \end{aligned} \tag{2.23}$$

Then, plugging those expansions in System (2.21), we obtain:

At order $1/\varepsilon$:

$$\begin{cases} \mathrm{div}_V\big((U^0 - V)f^0 - \nabla_V f^0\big) &= 0, \\ \int_{\mathbb{R}^3} (V - U^0) f^0(t, X, V)\, dV &= 0. \end{cases} \tag{2.24}$$

At order 1:

(2.25)
$$\begin{cases} \partial_t f^0 + BV\nabla_X f^0 + \mathrm{div}_V\Big((U^0 - V)f^1 - \nabla_V f^1\Big) + \mathrm{div}_V(U^1 f^0) = F\nabla_V f^0, \\ \partial_t \rho_w^0 + B\mathrm{div}(\rho_w^0 U^0) = 0, \\ \partial_t(\rho_w^0 U^0) + B\mathrm{div}_X(\rho_w^0 U^0 \otimes U^0) + B\nabla_X p^0 = 2\,E\,\Big(\mathrm{div}(\Sigma^0 : D(U^0)) \\ +\nabla(\lambda(\rho_w^0)\mathrm{div}(U^0))\Big) + \int_{\mathbb{R}^3}(V - U^0)f^1(t,X,V)\,dV \\ - \int_{\mathbb{R}^3} U^1 f^0(t,X,V)\,dV - F\rho_w^0\vec{k}. \end{cases}$$

Defining the macroscopic density of particles ρ_s and the macroscopic speed V_m by the integrals:

$$\begin{pmatrix} \rho_s \\ \rho_s V_m \end{pmatrix} = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ V \end{pmatrix} f(t,X,V)\,dV \tag{2.26}$$

and assuming that ρ_s and V_m also admit an asymptotic expansions with respect to ε:

$$\rho_s = \rho_s^0 + \varepsilon\rho_s^1 + O(\varepsilon^2), \quad V_m = V_m^0 + \varepsilon V_m^1 + O(\varepsilon^2).$$

Then Equations (2.24) provide:

$$f^0 = \frac{1}{\sqrt{2\pi}^3}\rho_s^0\, e^{-\frac{1}{2}\|U^0 - V\|^2} \text{ and },\tag{2.27}$$

$$V_m^0 = U^0.\tag{2.28}$$

and Equations (2.26)–(2.28) ensure the following identities:

$$\int_{\mathbb{R}^3}\begin{pmatrix} 1 \\ V \end{pmatrix}\partial_t f^0(t,X,V)\,dV = \begin{pmatrix} \partial_t\rho_s^0 \\ \partial_t(\rho_s^0\,U^0) \end{pmatrix},$$

$$\int_{\mathbb{R}^3}\begin{pmatrix} 1 \\ V \end{pmatrix}\mathrm{div}(Vf^0(t,X,V))\,dV = \begin{pmatrix} \mathrm{div}(\rho_s^0\,U^0) \\ \mathrm{div}(\rho_s^0\,U^0 \otimes U^0) + \nabla\rho_s^0 \end{pmatrix},$$

$$\begin{aligned} \int_{\mathbb{R}^3}\begin{pmatrix} 1 \\ V \end{pmatrix}\mathrm{div}_V\big((U^0 - V)f^1(t,X,V) \\ - \nabla_V f^1(t,X,V) - U^1 f^1(t,X,V) - Ff^0(t,X,V)\vec{k}\big)\,dV \\ = -\begin{pmatrix} 0 \\ \int_{\mathbb{R}^3}(U^0 - V)f^1 + U_1 f^0 + \rho_s^0\vec{k} \end{pmatrix}. \end{aligned}$$

Then, integrating the non-dimensional Vlasov Equations (2.25) against 1 and V provides:
(2.29)

$$\begin{cases} \partial_t \rho_s^0 + B\mathrm{div}_X(\rho_s^0 U^0) = 0, \\ \partial_t(\rho_s^0 U^0) + B\mathrm{div}_X(\rho_s^0\, U^0 \otimes U^0) + B\nabla_X(\rho_s^0) = \int_{\mathbb{R}^3}(U^0 - V) f^1(t,X,V)\,dV \\ \qquad\qquad + \int_{\mathbb{R}^3} U^1 f^0(t,X,V)\,dV + \rho_s^0 \vec{k}. \end{cases}$$

On the other hand, the dimensionless ACNS equations read:
(2.30)

$$\begin{cases} \partial_t \rho_w^0 + B\mathrm{div}_X(\rho_w^0 U^0) = 0, \\ \partial_t(\rho_w^0 U^0) + B\mathrm{div}_X(\rho_w^0 U^0 \otimes U^0) + B\nabla_X p^0 = 2\,E\,\Big(\mathrm{div}(\Sigma^0 : D(U^0)) \\ +\nabla(\lambda \mathrm{div}(U^0))\Big) + \displaystyle\int_{\mathbb{R}^3}(V - U^0) f^1(t,X,V)\,dV - \int_{\mathbb{R}^3} U^1 f^0(t,X,V)\,dV - F\rho_w^0 \vec{k}. \end{cases}$$

Then, denoting by ϱ

$$\varrho = \rho_w^0 + \rho_s^0$$

and adding Systems (2.29)-(2.30), we obtain:

$$\begin{cases} \partial_t \varrho + B\mathrm{div}_X(\varrho U^0) = 0, \\ \partial_t(\varrho U^0) + B\mathrm{div}_X(\varrho\, U^0 \otimes U^0) + B\nabla_X(p^0 + \rho_s^0) = 2\,E\,\Big(\mathrm{div}(\Sigma^0 : D(U^0)) \\ +\nabla(\lambda(\varrho)\mathrm{div}(U^0))\Big) - F\varrho\vec{k}. \end{cases} \tag{2.31}$$

Now, to return in the physical variables, we write:

$$\sqrt{\theta} U^0 = U, \quad \rho_0 \varrho = \rho$$

and we multiply the mass equation of System (2.31) by $\dfrac{\rho_0}{\mathfrak{T}}$, the momentum equation of System (2.31) by $\rho_0\sqrt{\theta}/\mathfrak{T}$ to obtain the following dimensional IVACNS unknowns (U, ρ) system (which means Integrated Vlasov Anisotropic Compressible Navier-Stokes):

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho U) = 0, \\ \partial_t(\rho U) + \mathrm{div}(\rho\, U \otimes U) + \nabla(p(\rho) + \theta\rho_s) = 2\,\Big(\mathrm{div}(\Sigma : D(U)) \\ +\nabla(\lambda(\rho)\mathrm{div}(U))\Big) - g\rho\vec{k}, \end{cases} \tag{2.32}$$

where the pressure law is a time and space variable barotropic law (as the one used in [4]):

$$p(\rho_w, \rho_s) = \frac{g}{\rho_p} \frac{h\,(\rho_w + \rho_s)^2}{4},$$

where $h = H - b$ is the fluid height, given by the kinematic condition at the free-surface.

3. Derivation of the generalized SVE equations

The model (2.32) is obtained from a coupling between the Vlasov equations with gravity and the Compressible Navier-Stokes equations with an anisotropic viscous tensor in a small relaxation time framework. The vertical average of the model (2.32), under the assumption that the ratio height/length of the domain is small, provides equations very close to SVE equations.

3.1. **The non-dimensional IVACNS system.** In a very closed way to the work done in [9, 13], we derive from model (2.32) an hydrostatic approximation. To this end, let us introduce a small parameter ε defined by:

$$\varepsilon = \frac{\mathfrak{L}}{L} = \frac{V_0}{U_0}$$

where $\mathfrak{L}$, L, V_0 and U_0 are respectively the characteristic scales for the vertical and horizontal dimensions of the fluid domain and the horizontal and the vertical characteristic velocities. We also introduce the characteristic time T defined by $\mathfrak{T} = \frac{L}{U_0}$ and the pressure units $\bar{p} = \overline{\rho} U_0^2$ where $\overline{\rho}$ is the characteristic density.
In the rest of the paper, we use the following notations:

$$x = (x_1, x_2), \quad y = x_3, \quad u = (u_1, u_2), \quad \text{and } v = u_3.$$

Then, we denote by

$$\widetilde{t} = \frac{t}{\mathfrak{T}}, \quad \widetilde{x} = \frac{x}{L}, \quad \widetilde{y} = \frac{y}{\mathfrak{L}}, \quad \widetilde{u} = \frac{u}{U_0}, \quad \widetilde{v} = \frac{v}{V_0},$$

$$\widetilde{p} = \frac{p}{\overline{p}}, \quad \widetilde{\rho} = \frac{\rho}{\overline{\rho}}, \quad \widetilde{H} = \frac{H}{\mathfrak{L}}, \quad \widetilde{b} = \frac{b}{\mathfrak{L}},$$

$$\widetilde{\lambda} = \frac{\lambda}{\overline{\lambda}}, \qquad \widetilde{\mu_j} = \frac{\mu_j}{\overline{\mu_j}}, j = 1, 2, 3$$

the dimensionless variables of time, space, fluid velocity, pressure, density, the total height, the bottom and the viscosities. With these notations, the Froude number F_r, the Reynolds number associated to the viscosity μ_i (i=1,2,3), Re_i and the Reynolds number associated to the viscosity λ, Re_λ read respectively:

$$F_r = \frac{U_0}{\sqrt{g\,\mathfrak{L}}}, \quad Re_i = \frac{\overline{\rho} U_0 L}{\overline{\mu_i}}, \quad Re_\lambda = \frac{\overline{\rho} U_0 L}{\overline{\lambda}}. \tag{3.1}$$

We also introduce $\overline{\mathfrak{K}}$ and $\overline{S}$, the characteristic quantities corresponding to the friction and the characteristic speed of the vertical variation of the moving bottom. In the sequel, in order to take into account the friction, we set $\overline{\mathfrak{K}}$ to $\overline{\rho}\, U_0 \varepsilon$ and $\overline{S}$ will make explicit later.

Then applying this scaling, the System (2.32) is written:

(3.2)

$$\left\{\begin{array}{l}
\dfrac{1}{\mathfrak{T}}\partial_{\widetilde{t}}\widetilde{\rho} + \dfrac{U_0}{L}\mathrm{div}_{\widetilde{x}}\,(\widetilde{\rho}\,\widetilde{u}) + \dfrac{V_0}{\mathfrak{L}}\partial_{\widetilde{y}}\,(\widetilde{\rho v}) = 0,\\[2ex]
\dfrac{\overline{\rho}\,U_0}{\mathfrak{T}}\partial_{\widetilde{t}}\,(\widetilde{\rho}\,\widetilde{u}) + \dfrac{\overline{\rho}\,U_0^2}{L}\mathrm{div}_{\widetilde{x}}\,(\widetilde{\rho}\,\widetilde{u}\otimes\widetilde{u}) + \dfrac{\overline{\rho}\,U_0\,V_0}{\mathfrak{L}}\partial_{\widetilde{y}}\,(\widetilde{\rho}\,\widetilde{v}\widetilde{u}) + \nabla_{\widetilde{x}}(\dfrac{\bar{p}}{L}\widetilde{p}(\widetilde{\rho}) + \dfrac{\bar{\rho}\theta}{L}\rho_s) =\\[2ex]
\qquad \dfrac{\overline{\mu_1}\,U_0}{L^2}\mathrm{div}_{\widetilde{x}}\,(\mu_1 D_{\widetilde{x}}(\widetilde{u})) + \dfrac{\overline{\mu_2}\,U_0}{\mathfrak{L}^2}\partial_{\widetilde{y}}\left(\widetilde{\mu_2}\,\partial_{\widetilde{y}}\widetilde{u}\right) + \dfrac{\overline{\mu_2}\,V_0}{L\,\mathfrak{L}}\partial_{\widetilde{y}}\,(\widetilde{\mu_2}\,\nabla_{\widetilde{x}}\widetilde{v})\\[2ex]
\qquad\qquad + \dfrac{\overline{\lambda}\,U_0}{L^2}\nabla_{\widetilde{x}}\left(\widetilde{\lambda}\,\mathrm{div}_{\widetilde{x}}\,(\widetilde{u})\right) + \dfrac{\overline{\lambda}\,V_0}{L\,\mathfrak{L}}\nabla_{\widetilde{x}}\left(\widetilde{\lambda}\,\partial_{\widetilde{y}}\widetilde{v}\right),\\[2ex]
\dfrac{\overline{\rho}\,V_0}{\mathfrak{T}}\partial_{\widetilde{t}}\,(\widetilde{\rho}\,\widetilde{v}) + \dfrac{\overline{\rho}\,U_0\,V_0}{L}\mathrm{div}_{\widetilde{x}}\,(\widetilde{\rho}\,\widetilde{u}\,\widetilde{v}) + \dfrac{\overline{\rho}\,V_0^2}{\mathfrak{L}}\partial_{\widetilde{y}}\left(\widetilde{\rho}\,\widetilde{v}^2\right) + \partial_{\widetilde{y}}(\dfrac{\bar{p}}{\mathfrak{L}}\widetilde{p}(\widetilde{\rho}) + \dfrac{\bar{\rho}\theta}{\mathfrak{L}}\rho_s) =\\[2ex]
\qquad -g\,\overline{\rho}\widetilde{\rho} + \dfrac{\overline{\mu_3}\,U_0}{L\mathfrak{L}}\mathrm{div}_{\widetilde{x}}\left(\widetilde{\mu_3}\,\partial_{\widetilde{y}}\widetilde{u}\right) + \dfrac{\overline{\mu_3}\,V_0}{L^2}\mathrm{div}_{\widetilde{x}}\,(\widetilde{\mu_3}\,\nabla_{\widetilde{x}}\widetilde{v}) + 2\dfrac{\overline{\mu_3}\,V_0}{\mathfrak{L}^2}\partial_{\widetilde{y}}(\widetilde{\mu_3}\partial_{\widetilde{y}}\widetilde{v})\\[2ex]
\qquad\qquad + \dfrac{\overline{\lambda}\,U_0}{L\,\mathfrak{L}}\partial_{\widetilde{y}}\left(\widetilde{\lambda}\,\mathrm{div}_{\widetilde{x}}\,(\widetilde{u})\right) + \dfrac{\overline{\lambda}\,V_0}{\mathfrak{L}^2}\partial_{\widetilde{y}}\left(\widetilde{\lambda}\,\partial_{\widetilde{y}}\widetilde{v}\right),
\end{array}\right.$$

where the free surface boundary conditions (2.11) are written:

$$(3.3)\qquad \left\{\begin{array}{l}
\dfrac{\bar{p}\,\mathfrak{L}}{L}\,(\widetilde{p} + \beta\widetilde{\kappa} - \widetilde{p_0})\,\nabla_{\widetilde{x}}\widetilde{H} - \dfrac{\overline{\mu_1}\,U_0\,\mathfrak{L}}{L^2}\widetilde{\mu_1}D_{\widetilde{x}}\,(\widetilde{u})\,\nabla_{\widetilde{x}}\widetilde{H}\\[2ex]
\qquad + \overline{\mu_2}\left(\widetilde{\mu_2}\left(\dfrac{U_0}{\mathfrak{L}}\partial_{\widetilde{y}}\widetilde{u} + \dfrac{V_0}{L}\nabla_{\widetilde{x}}\widetilde{v}\right)\right) - \dfrac{\overline{\lambda}\,U_0\,\mathfrak{L}}{L^2}\widetilde{\lambda}\mathrm{div}_{\widetilde{x}}(\widetilde{u})\nabla_{\widetilde{x}}\widetilde{H}\\[2ex]
\qquad\qquad - \dfrac{\overline{\lambda}\,V_0}{L}\widetilde{\lambda}\partial_{\widetilde{y}}(\widetilde{v})\nabla_{\widetilde{x}}\widetilde{H} = 0,\\[2ex]
\bar{p}(-\widetilde{p} - \beta\widetilde{\kappa} + \widetilde{p_0}) - \overline{\mu_3}\left(\widetilde{\mu_3}\left(\dfrac{U_0}{L}\partial_{\widetilde{y}}\widetilde{u} + \dfrac{V_0\,\mathfrak{L}}{L^2}\nabla_{\widetilde{x}}\widetilde{v}\right)\right)\cdot\nabla_{\widetilde{x}}\widetilde{H}\\[2ex]
\qquad\qquad + 2\,\dfrac{\overline{\mu_3}\,V_0}{\mathfrak{L}}\widetilde{\mu_3}\partial_{\widetilde{y}}\widetilde{v} + \dfrac{\overline{\lambda}\,U_0}{L}\widetilde{\lambda}\mathrm{div}_{\widetilde{x}}(\widetilde{u}) + \dfrac{\overline{\lambda}\,V_0}{\mathfrak{L}}\widetilde{\lambda}\partial_{\widetilde{y}}(\widetilde{v}) = 0
\end{array}\right.$$

and the corresponding bottom boundary conditions (2.14):

$$
(3.4)\quad \begin{cases}
-\dfrac{\overline{\mu_1}\,U_0\,\mathfrak{L}}{L^2}\widetilde{\mu_1}\left(\left(1+\left|\dfrac{\mathfrak{L}}{L}\nabla_{\widetilde{x}}\widetilde{b}\right|^2\right)I_2-\dfrac{\mathfrak{L}^2}{L^2}\nabla_{\widetilde{x}}\widetilde{b}\nabla_{\widetilde{x}}\widetilde{b}^t\right)D_{\widetilde{x}}(\widetilde{u})\nabla_{\widetilde{x}}\widetilde{b} \\[2ex]
+\overline{\mu_2}\,\widetilde{\mu_2}\left(\left(1+\left|\dfrac{\mathfrak{L}}{L}\nabla_{\widetilde{x}}\widetilde{b}\right|^2\right)I_2-\dfrac{\mathfrak{L}^2}{L^2}\nabla_{\widetilde{x}}\widetilde{b}\nabla_{\widetilde{x}}\widetilde{b}^t\right)\left(\dfrac{U_0}{\mathfrak{L}}\partial_{\widetilde{y}}\widetilde{u}+\dfrac{V_0}{L}\nabla_{\widetilde{x}}\widetilde{v}\right) \\[2ex]
-\overline{\mu_3}\,\widetilde{\mu_3}\left(\left(\dfrac{\mathfrak{L}\,U_0}{L^2}\partial_{\widetilde{y}}\widetilde{u}+\dfrac{V_0\,\mathfrak{L}^2}{L^3}\nabla_{\widetilde{x}}\widetilde{v}\right)\cdot\nabla_{\widetilde{x}}\widetilde{b}\right)\nabla_{\widetilde{x}}\widetilde{b}+2\dfrac{V_0\,\overline{\mu_3}}{L}\,\widetilde{\mu_3}\partial_{\widetilde{y}}\widetilde{v}\,\nabla_{\widetilde{x}}\widetilde{b} \\[2ex]
\qquad =\left(1+\left|\dfrac{\mathfrak{L}}{L}\nabla_{\widetilde{x}}\widetilde{b}\right|^2\right)^{3/2}\overline{\mathfrak{K}}U_0\begin{pmatrix}\mathfrak{K}_1(\widetilde{u})\\ \mathfrak{K}_2(\widetilde{u})\end{pmatrix}, \\[2ex]
-\overline{\mu_3}\mu_3\left|\nabla_{\widetilde{x}}\widetilde{b}\right|^2\left(\dfrac{\mathfrak{L}^2\,U_0}{L^3}\partial_{\widetilde{y}}\widetilde{u}+\dfrac{\mathfrak{L}^3\,V_0}{L^2}\nabla_{\widetilde{x}}\widetilde{v}\right)\cdot\nabla_{\widetilde{x}}\widetilde{b}+2\dfrac{\mathfrak{L}\,V_0\,\overline{\mu_3}}{L^2}\mu_3\left|\nabla_{\widetilde{x}}\widetilde{b}\right|^2\partial_{\widetilde{y}}\widetilde{v} \\[2ex]
+\overline{\mu_2}\dfrac{\mathfrak{L}}{L}\,\widetilde{\mu_2}\left(\dfrac{U_0}{\mathfrak{L}}\partial_{\widetilde{y}}\widetilde{u}+\dfrac{V_0}{L}\nabla_{\widetilde{x}}\widetilde{v}\right)\cdot\nabla_{\widetilde{x}}\widetilde{b}-\overline{\mu_1}\dfrac{U_0\,\mathfrak{L}^2}{L^3}\,\widetilde{\mu_1}\left(D_{\widetilde{x}}(\widetilde{u})\nabla_{\widetilde{x}}\widetilde{b}\right)\cdot\nabla_{\widetilde{x}}\widetilde{b} \\[2ex]
\qquad =\left(1+\left|\dfrac{\mathfrak{L}}{L}\nabla_{\widetilde{x}}\widetilde{b}\right|^2\right)^{3/2}\overline{\mathfrak{K}}V_0\mathfrak{K}_3(\widetilde{v}).
\end{cases}
$$

Finally, the continuity condition for the velocity at the bottom (2.15) is written:

$$
(3.5)\qquad \frac{\mathfrak{L}}{\mathfrak{T}}\partial_{\widetilde{t}}\widetilde{b}+\frac{U_0\,\mathfrak{L}}{L}\widetilde{u}(\widetilde{t},\widetilde{x},\widetilde{b})\cdot\nabla_{\widetilde{x}}\widetilde{b}-V_0\widetilde{v}(\widetilde{t},\widetilde{x},\widetilde{b})=\overline{S}\widetilde{S}.
$$

Using the definition of the dimensionless numbers (3.1), setting $\rho_p=\bar{\rho}$, dropping $\widetilde{\cdot}$, multiplying the mass equation of System (3.2) by $\mathfrak{T}$, the momentum equation for u of System (3.2) by $\frac{\mathfrak{T}}{\bar{\rho}\,U_0}$ and the momentum equation for v of System (3.2) by $\frac{\mathfrak{T}}{\bar{\rho}\,V_0}$, we get the non-dimensional version of System (2.32) as follows:

$$
(3.6)\quad \begin{cases}
\partial_t \rho + \operatorname{div}_x(\rho\, u) + \partial_y(\rho v) = 0, \\
\partial_t(\rho\, u) + \operatorname{div}_x(\rho\, u \otimes u) + \partial_y(\rho\, v u) + \nabla_x\left(p(\rho) + \dfrac{\theta \mathfrak{T}^2}{L^2}\rho_s\right) = \\
\qquad \frac{1}{Re_1}\operatorname{div}_x(\mu_1 D_x(u)) + \frac{1}{Re_2}\partial_y\left(\mu_2\left(\frac{1}{\varepsilon^2}\partial_y u + \nabla_x v\right)\right) + \frac{1}{Re_\lambda}\nabla_x(\lambda \operatorname{div}(u)), \\
\partial_t(\rho\, v) + \operatorname{div}_x(\rho\, u\, v) + \partial_y\left(\rho\, v^2\right) + \partial_y\left(\dfrac{1}{\varepsilon^2}p(\rho) + \dfrac{\theta \mathfrak{T}^2}{\mathfrak{L}^2}\rho_s\right) = \\
\qquad -\frac{1}{\varepsilon^2}\frac{1}{F_r^2}\rho + \dfrac{1}{Re_3}\operatorname{div}_x\left(\mu_3\left(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v\right)\right) + \dfrac{2}{\varepsilon^2 Re_3}\partial_y(\mu_3 \partial_y v) \\
\qquad + \dfrac{1}{\varepsilon^2\, Re_\lambda}\partial_y(\lambda \operatorname{div}(u)).
\end{cases}
$$

Multiplying the first Equation of (3.3) by $\dfrac{L}{\bar{p}\,\mathfrak{L}}$, the second one by $\dfrac{1}{\bar{p}}$ and dropping $\widetilde{\cdot}$, we get the corresponding free surface conditions:

$$
(3.7)\quad \begin{cases}
(p + \beta\kappa - p_0)\nabla_x H - \dfrac{F_r^2}{Re_1}\mu_1 D_x(u)\nabla_x H \\
\qquad + \dfrac{F_r^2}{Re_2}\left(\mu_2\left(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v\right)\right) - \dfrac{F_r^2}{\varepsilon\, Re_\lambda}\lambda \operatorname{div}_x(u)\nabla_x H \\
\qquad - \dfrac{F_r^2}{Re_\lambda}\lambda \partial_y(v)\nabla_x H = 0, \\
(-p - \alpha\kappa + p_0) - \dfrac{F_r^2}{Re_3}\left(\mu_3\left(\partial_y u + \varepsilon^2 \nabla_x v\right)\right)\cdot \nabla_x H \\
\qquad + 2\dfrac{F_r^2}{Re_3}\mu_3 \partial_y v + \dfrac{F_r^2}{Re_\lambda}\lambda\left(\operatorname{div}_x(u) + \partial_y(v)\right) = 0.
\end{cases}
$$

Multiplying the first equation and the last equation of system (3.4) respectively by $\dfrac{L}{\overline{\rho}\, U_0^2\, \mathfrak{L}}$ and $\dfrac{1}{\overline{\rho}\, U_0^2}$, and dropping $\widetilde{\cdot}$, the conditions on the moving bottom read:

$$(3.8)\qquad \begin{cases} -\dfrac{\mu_1}{Re_1}\left(\left(1+\varepsilon^2\left|\nabla_x b\right|^2\right) I_2 - \varepsilon^2 \nabla_x b \nabla_x b^t\right) D_x(u)\nabla_x b \\[2ex] +\dfrac{\mu_2}{Re_2}\left(\left(1+\varepsilon^2\left|\nabla_x b\right|^2\right) I_2 - \varepsilon^2 \nabla_x b \nabla_x b^t\right)\left(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v\right) \\[2ex] -\dfrac{\mu_3}{Re_3}\left(\left(\partial_y u + \varepsilon^2 \nabla_x v\right)\cdot \nabla_x b\right)\nabla_x b + 2\dfrac{1}{Re_3}\mu_3 \partial_y v\, \nabla_x b \\[2ex] \qquad = \left(1+\varepsilon^2\left|\nabla_x b\right|^2\right)^{3/2}\begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u)\end{pmatrix}, \\[2ex] -\dfrac{\mu_3}{Re_3}\left|\nabla_x b\right|^2\left(\varepsilon^2\partial_y u + \varepsilon^4 \nabla_x v\right)\cdot\nabla_x b + 2\varepsilon^2\dfrac{\mu_3}{Re_3}\left|\nabla_x b\right|^2 \partial_y v \\[2ex] +\dfrac{\mu_2}{Re_2}\left(\partial_y u + \varepsilon^2 \nabla_x v\right)\cdot\nabla_x b - \varepsilon^2\dfrac{\mu_1}{Re_1}\left(D_x(u)\nabla_x b\right)\cdot\nabla_x b \\[2ex] \qquad = \varepsilon^2\left(1+\varepsilon^2\left|\nabla_x b\right|^2\right)^{3/2}\mathfrak{K}_3(v). \end{cases}$$

Multiplying Equation (3.5) by $\dfrac{\mathfrak{T}}{\mathfrak{L}}$ and dropping $\widetilde{\cdot}$, the continuity condition reads:

$$(3.9)\qquad \partial_t b + u(t,x,b)\cdot\nabla_x b - v(t,x,b) = \frac{\overline{S}}{\varepsilon\, U_0} S.$$

Applying the hydrostatic approximation which consists to drop all the term of second order ε (where $\frac{\theta \mathfrak{T}^2}{L^2} = \varepsilon^2$) of System (3.6) with the boundary conditions

(3.7)- (3.8)- (3.9), we obtain the following system :

$$
(3.10)\qquad
\begin{cases}
\partial_t \rho + \mathrm{div}_x\,(\rho\, u) + \partial_y\,(\rho v) = 0, \\[2mm]
\partial_t\,(\rho\, u) + \mathrm{div}_x\,(\rho\, u \otimes u) + \partial_y\,(\rho\, v u) + \nabla_x (p(\rho) + \varepsilon^2 \rho_s) \\[2mm]
\qquad = \dfrac{1}{Re_1}\mathrm{div}_x\,(\mu_1 D_x(u)) + \dfrac{1}{Re_2}\partial_y \left(\mu_2 \left(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v\right)\right) \\[2mm]
\qquad\quad + \frac{1}{Re_\lambda}\nabla_x\,(\lambda\,\mathrm{div}(u))\,, \\[2mm]
\partial_t(\rho v) + \ \mathrm{div}_x(\rho u v) + \partial_y(\rho v^2) + \partial_y\big(\dfrac{1}{\varepsilon^2}p(\rho) + \rho_s\big) = -\dfrac{1}{\varepsilon^2}\dfrac{1}{F_r^2}\rho \\[2mm]
\qquad + \dfrac{1}{Re_3}\mathrm{div}_x\Big(\mu_3\Big(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v\Big)\Big) + \dfrac{2}{\varepsilon^2 Re_3}\partial_y(\mu_3 \partial_y v) \\[2mm]
\qquad\quad + \dfrac{1}{\varepsilon^2 Re_\lambda}\partial_y(\lambda\,\mathrm{div}(u)),
\end{cases}
$$

with the free surface conditions

$$
(3.11)\qquad
\begin{cases}
(p + \beta\kappa - p_0)\,\nabla_x H - \dfrac{F_r^2}{Re_1}\mu_1 D_x\,(u)\,\nabla_x H + \\[2mm]
\dfrac{F_r^2}{Re_2}\left(\mu_2\left(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v\right)\right) - \dfrac{F_r^2}{\varepsilon\, Re_\lambda}\lambda \mathrm{div}_x(u)\nabla_x H \\[2mm]
-\dfrac{F_r^2}{Re_\lambda}\lambda\partial_y(v)\nabla_x H = 0, \\[2mm]
(-p - \alpha\kappa + p_0) - \dfrac{F_r^2}{Re_3}\,(\mu_3 \partial_y u)\cdot\nabla_x H \\[2mm]
+2\,\dfrac{F_r^2}{Re_3}\mu_3\partial_y v + \dfrac{F_r^2}{Re_\lambda}\lambda\,(\mathrm{div}_x(u) + \partial_y(v)) = O(\varepsilon^2),
\end{cases}
$$

the moving bottom conditions

$$
(3.12)\qquad
\begin{cases}
-\dfrac{\mu_1}{Re_1}D_x(u)\nabla_x b + \dfrac{\mu_2}{Re_2}\left(\dfrac{1}{\varepsilon^2}\partial_y u + \nabla_x v + \nabla_x b^\perp \left(\nabla_x b^\perp\right)^t \cdot \partial_y u\right) \\[2mm]
-\dfrac{\mu_3}{Re_3}\,(\partial_y u\cdot\nabla_x b)\,\nabla_x b + 2\dfrac{1}{Re_3}\mu_3 \partial_y v\,\nabla_x b = \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{pmatrix} + O(\varepsilon^2), \\[2mm]
\dfrac{\mu_2}{Re_2}\partial_y u\cdot\nabla_x b = O(\varepsilon^2)
\end{cases}
$$

and the equation of the continuity

$$\partial_t b + u(t,x,b)\cdot\nabla_x b - v(t,x,b) = \frac{\overline{S}}{\varepsilon\, U_0} S. \tag{3.13}$$

Let us remark that from Equations (3.10)–(3.12), we deduce:

$$\begin{aligned} \partial_y\left(\frac{\mu_2}{Re_2}\partial_y u\right) &= O(\varepsilon^2), & \left(\frac{\mu_\lambda}{Re_\lambda}\mathrm{div}_x u\right)_{|y=H} &= O(\varepsilon), \\ \left(\frac{\mu_2}{Re_2}\partial_y u\right)_{|y=H} &= O(\varepsilon^2), & \left(\frac{\mu_2}{Re_2}\partial_y u\right)_{|y=b} &= O(\varepsilon^2). \end{aligned} \tag{3.14}$$

3.2. **The approximation of the IVACNS system at main order.** As we seek for the main order approximation, assuming that the unknowns admits an asymptotic expansion with respect to ε as $u = u^0 + \varepsilon u^1$:

- we drop all term of $O(\varepsilon)$ in System (3.10),
- we use the hydrostatic approximation to write explicitely the pressure,
- and state the asymptotic regime:

$$\frac{\mu_i(\rho)}{Re_i} = \varepsilon^{i-1}\nu_i(\rho),\ i = 1,2,3 \text{ and } \frac{\lambda(\rho)}{Re_\lambda} = \varepsilon^2\gamma(\rho). \tag{3.15}$$

to obtain the system:

$$\left\{\begin{aligned} &\partial_t\rho + \mathrm{div}_x(\rho\, u) + \partial_y(\rho v) = 0,\\ &\partial_t(\rho\, u) + \mathrm{div}_x(\rho\, u\otimes u) + \partial_y(\rho\, v u) + \nabla_x p(\rho)\\ &\qquad\qquad = \mathrm{div}_x(\nu_1 D_x(u)) + \partial_y\left(\nu_2\partial_y u^1\right)\\ &h\rho(t,x,y) = 2(H-y), \end{aligned}\right. \tag{3.16}$$

where we denote again u^0 by u.

Setting $\kappa = p_0 = 0$, the free surface boundary conditions reduce to:

$$\left\{\begin{aligned} &-\nu_1 D_x(u)\nabla_x H + \left(\nu_2\partial_y u^1\right) = 0,\\ &\nabla_x(h\rho) = 0 \end{aligned}\right. \tag{3.17}$$

and the moving bottom conditions are written:

$$\left\{\begin{aligned} &-\nu_1 D_x(u)\nabla_x b + \nu_2\partial_y u^1 = \begin{pmatrix}\mathfrak{K}_1(u)\\ \mathfrak{K}_2(u)\end{pmatrix} + O(\varepsilon),\\ &\nu_2\partial_y u\cdot\nabla_x b = O(\varepsilon),\\ &\partial_t b + u(t,x,b)\cdot\nabla_x b - v(t,x,b) = \frac{\overline{S}}{\varepsilon\, U_0} S \end{aligned}\right. \tag{3.18}$$

We have from Estimates (3.14) and from the asymptotic ordering (3.15) that:

$$\partial_y \left(\nu_2 \partial_y u\right) = O(\varepsilon), \quad \left(\nu_2 \partial_y u\right)_{|y=H} = O(\varepsilon), \quad \left(\nu_2 \partial_y u\right)_{|y=b} = O(\varepsilon).$$

which imply:

$$u(t,x,y) = \overline{u}(t,x) + O(\varepsilon)$$

whenever $\nu_2(\rho)$ is of order $O(\varepsilon^\delta)$, $\forall \delta \leqslant 0$ otherwise the *motion by slice* is not guaranteed. On the other hand, if $\overline{S} = V$, the following continuity equation writes:

$$\partial_t b + u(t,x,b) \cdot \nabla_x b - v(t,x,b) = S. \tag{3.19}$$

Equations close to SVE equations are then obtained by setting $S = 0$.

3.3. **The non-dimensional system vertically averaged.** Now, in order to derive equations closed to SVE model, we have to average vertically the System (3.16) with boundary conditions (3.17)–(3.19).

For any function f, we note the mean value of f over the vertical as

$$h(t,x)\overline{f}(t,x) = \int_b^H f \, dy.$$

We first note that: from the equation $h\rho = 2(H-y)$(the hydrostatic equation), we deduce that

$$\int_b^H \rho \, dy = \frac{1}{h} \int_b^H h\rho \, dy = \frac{2}{h} \int_b^H (H-y) \, dy = h \tag{3.20}$$

and the mean pressure is written:

$$\int_b^H h\rho^2 \, dy = \frac{4}{3} h^2. \tag{3.21}$$

Next, using

- Leibniz formulas,
- bottom conditions (3.18)-(3.19),
- $u = \overline{u} + O(\varepsilon)$,
- and equation (3.20),

the averaged mass equation reads:

$$\partial_t h + \operatorname{div}(h\bar{u}) = 0. \tag{3.22}$$

Similarly, integrating the horizontal momentum equations in System (3.16) for $b \leqslant y \leqslant H$ gives:

$$\begin{aligned}
&\partial_t(h\,\overline{u}) + \mathrm{div}_x(h\,\overline{u}\otimes\overline{u}) + \frac{1}{3\,F_r^2}\nabla_x(h^2)\\
&+\Big(\rho\,u\,(\partial_t b + u\cdot\nabla_x b - w)\Big)_{|z=b}\nabla_x b\\
&-\Big(\rho\,u\,(\partial_t H + u\cdot\nabla_x H - w)\Big)_{|z=H}\nabla_x H\\
&= \mathrm{div}_x\Big(\int_b^H D(u-\overline{u})\,dz + \overline{(\nu_1)}hD(\overline{u})\Big)\\
&+\Big(\frac{\nu_2}{\varepsilon}\partial_z u^1 - \nu_1 D_x(u)\nabla_x b\Big)_{|z=b}\\
&+\Big(\nu_1 D(\bar{u})\nabla_x H - \frac{\nu_2}{\varepsilon}\partial_z u^1\Big)_{|z=H}
\end{aligned}.$$

Using boundary conditions (3.17-3.18-3.19), and $u = \overline{u} + O(\varepsilon)$ we finally obtain:

$$\partial_t(h\bar{u}) + \mathrm{div}(h\bar{u}\otimes\bar{u}) + \frac{1}{3\,F_r^2}\nabla h^2 = -\frac{h}{F_r^2}\nabla b + \mathrm{div}(hD(\bar{u})) - \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{pmatrix}. \tag{3.23}$$

Coming back to the physical variables, the presented SVE like model reads:

$$\left\{\begin{array}{lcl}
\partial_t h + \mathrm{div}(h\bar{u}) & = & 0,\\
\partial_t(h\bar{u}) + \mathrm{div}(h\bar{u}\otimes\bar{u}) + gh\nabla\left(\dfrac{h}{3} + b\right) & = & \mathrm{div}(hD(\bar{u})) - \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u) \end{pmatrix},\\
\partial_t b + u\nabla_x b & = & v.
\end{array}\right. \tag{3.24}$$

Remark 3.1. Averaging System (3.16) with $S \neq 0$ modifies the hydrodynamic part of the flow by adding a source term to the mass equation ($-2S$) and to the momentum equations ($-2uS$). Source terms $-2S$, $-2uS$ also modify the mass conservation equation for h as for b.

Those equations called averaged IVACNS equations are very close to the usual viscous or non viscous (if we drop the viscosity in Equations (3.24)) SVE equations (1.1)-(1.2). Namely, if we set $u\nabla_x b - v = \xi\mathrm{div}(q_b(h,u))$, we obtain a very close model with the own difference which comes from the coefficient $\frac{1}{3}$. Let us also note that stability results obtained in [19, 6] hold also here, by adapting there proofs as illustrated in the following sections.

4. Example 1: the viscous sedimentation model of [6]

In this section we illustrate Model (3.24) with the new morphodynamic bed-load transport introduced by Zabsonre *et al* [6], slightly modified in the sense that we introduce coefficient α, β. The main difference from the model [6] and Model (3.24) comes from the coefficient 1/3 of the pressure term which is classicaly 1/2. Then, the Model (3.24) with the morphodynamic bed-load transport [6] reads:

$$
\left\{
\begin{array}{lcl}
\partial_t h + \operatorname{div}(hu) & = & 0, \\
\partial_t(hu) + \operatorname{div}(hu \otimes u) + gh\nabla\left(\dfrac{h}{3} + b\right) & = & 2\nu \operatorname{div}(hD(u)), \\
\partial_t b + \operatorname{div}(\alpha hu\,|u|^r - \beta\nu\nabla b I) & = & 0
\end{array}
\right. \tag{4.1}
$$

where I is the identity matrix. The bed-load transport term $u\nabla_x b - v$ writes $\operatorname{div}(\alpha hu\,|u|^r - \beta\nu\nabla b)$ for some α and β satisfying relations (4.4), with $r \in [0, 1/2]$. We also set the friction term in (3.24) to zero. The System (4.1) is equipped with the initial data:

$$
L^2(\Omega) \ni h_{|t=0} = h_0 \geqslant 0, \quad b_{|t=0} = b_0 \in L^2(\Omega), \quad hu_{|t=0} = m_0 \tag{4.2}
$$

and

$$
|m_0|^2 / h_0 \in L^1(\Omega), \quad \nabla\sqrt{h_0} \in L^2(\Omega)^2 \tag{4.3}
$$

where $\Omega = \mathcal{T}^2$ is the torus. Then the main result is a straightforward consequence to the one presented in [6],i.e:

Theorem 4.1. *Let α, β and $\gamma = \gamma(\alpha, \beta)$, $\delta = \delta(\beta)$ (called stability coefficients) such that*

$$
\begin{array}{l}
0 < \beta < 2,\ \alpha > 0 \\
\phi(\beta) = \dfrac{2}{2-\beta} > 0, \\
\gamma(\alpha, \beta) = 3\alpha\phi(\beta) > 0, \\
\delta(\beta) = \phi(\beta) - 1 > 0.
\end{array} \tag{4.4}
$$

Let (h_n, u_n, b_n) be a sequence of weak solutions of (4.1) with initial conditions (4.2)-(4.3), in the following sense: $\forall r \in [0, 1/2]$:

- *System (4.1) holds in $(\mathcal{D}'((0,T)\times\Omega))^4$ with (4.2-4.3),*
- *Energy (4.5), Entropy (4.7) and the following regularities are satisfied:*

$$
\begin{array}{ll}
\sqrt{hu} \in L^\infty(0,T;(L^2(\Omega))^2), & \sqrt{h}\nabla u \in L^2(0,T;(L^2(\Omega))^4), \\
h^{1/(r+2)}u \in L^\infty(0,T;(L^{r+2}(\Omega))^2), & h/3 + b \in L^\infty(0,T;L^2(\Omega)), \\
\nabla(h/3) + \nabla b \in L^2(0,T;(L^2(\Omega))^2), & \nabla\sqrt{h} \in L^\infty(0,T;(L^2(\Omega))^2), \\
h^{1/r}D(u)^{2/r} \in L^r(0,T;(L^r(\Omega))^4). &
\end{array}
$$

If $h_0^n \geqslant 0$ and the sequence $(h_0^n, u_0^n, m_0^n) \to (h_0, u_0, m_0)$ converges in $L^1(\Omega)$ then, up to a subsequence, the sequence (h_n, u_n, m_n) converges strongly to a weak solution of (4.1) and satisfies Energy (4.5), Entropy (4.7) inequalities.

Energy and entropy estimates. Estimates for shallow water equations are generally obtained by the energy and entropy inequality where

- energy inequality is obtained by multiplying the momentum equation by u and integrating by parts,
- entropy inequality is obtained by, let's say, the BD-entropy [1].

Although System (3.24) is composed by a shallow water type equations, the steps to obtain useful estimates described above are not sufficient since the equation for b, the coefficient $1/3$, α and β introduce a lot of terms where their signs are unknown. Nevertheless, useful estimates are obtained by combining the BD-entropy and the energy with the factor $u(1+\gamma|u|^r)$ for some $\gamma(\alpha,\beta)$, instead of u. We obtain the following inequality:

Proposition 4.2. *Let (h,u,b) be a regular solution of (4.1) and γ, δ satisfying condition (4.4). Then we have:*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{h|u|^2}{2}+\frac{\gamma(\alpha,\beta)}{r+2}h|u|^{r+2}+g\phi(\beta)\left(\sqrt{\frac{3}{2}}b+\sqrt{\frac{1}{6}}h\right)^2+\delta(\beta)h\frac{|\psi|^2}{2}\,dx\\
&+2\nu\int_\Omega h\left(1+(1-2r)|u|^r\right)|D(u)|^2+\delta(\beta)|A(u)|^2\,dx\\
&+g\nu\int_\Omega\left|\nabla\left(\sqrt{3\phi(\beta)\beta}b+\sqrt{2/3\delta(\beta)}h\right)\right|^2dx\leqslant 0
\end{aligned}\tag{4.5}$$

where $\psi = u+2\nu\nabla\ln h$.

Proof of Proposition 4.2: We only draw the outline of the proof. We multiply the momentum equation by $u+\gamma u|u|^r$ and using the mass equation for h and b and integrate by parts to obtain:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{h|u|^2}{2}+\frac{\gamma}{r+2}h|u|^{r+2}\,dx+2\nu\int_\Omega h|D(u)|^2-\gamma\mathrm{div}(hD(u))\cdot u|u|^r\,dx\\
&+g\int_\Omega \partial_t h^2/6+b\partial_t h+h\gamma/(3\alpha)\partial_t b+\gamma/(2\alpha)\partial_t b^2\,dx\\
&+g\nu\int_\Omega \beta\gamma/(3\alpha)\nabla b\cdot\nabla h+\beta\gamma/\alpha|\nabla b|^2\,dx=0
\end{aligned}\tag{4.6}$$

As the sign of terms in (4.6) are unknown, we have to get more additional information to conclude which is done with the BD-entropy. We first take the gradient of the mass equation, then we multiply by 2ν and write the terms ∇h as $h\nabla\ln h$ to obtain:

Lemma 4.3. *Let (h, u, b) be a regular solution of (4.1). Then the following equality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h\,|\psi|^2\;dx + \int_\Omega 2\nu\,|A(u)|^2\;dx$$
$$+\int_\Omega g/6\partial_t h^2 + 2g\nu/3\,|\nabla h|^2 + gb\partial_t h + 2g\nu\nabla b\cdot\nabla h\,dx = 0 \tag{4.7}$$

Proof. **(Proof of Lemma 4.3):** We first take the gradient of the mass equation of (4.1), then we multiply by 2ν and write the terms ∇h as $h\nabla\ln h$ to obtain:

$$\partial_t\left(2\nu h\nabla\ln h\right) + \mathrm{div}\left(2\nu h\nabla\ln h\otimes u\right) + \mathrm{div}\left(2\nu h\nabla^t u\right) = 0 \tag{4.8}$$

Next, we sum Equation (4.8) with the momentum equation of System (4.1) to get the equation:

$$\partial_t\left(h\,\psi\right) + \mathrm{div}\left(\psi\otimes hu\right) + h\nabla(h/3+b) + 2\nu\mathrm{div}\left(hA(u)\right) \tag{4.9}$$

where $\psi = u + 2\nu\nabla\ln h$ the BD multiplier and $2A(u) = \nabla u - \nabla^t u$ the vorticity tensor. The BD-entropy is then obtained by multiplying the previous equation by ψ and integrating by parts:

- $$\int_\Omega \partial_t\left(h\,\psi\right) + \mathrm{div}\left(\psi\otimes hu\right)\;dx = \frac{d}{dt}\int_\Omega h\frac{|\psi|^2}{2}\,dx, \tag{4.10}$$
- remarking that
  $$\int_\Omega 2\nu\mathrm{div}\left(hA(u)\right)\nabla\ln h\,dx = 0,$$
  then the integral reduces to
  $$\int_\Omega 2\nu\mathrm{div}\left(hA(u)\right)u\,dx = 2\nu\int_\Omega h\,|A(u)|^2, \tag{4.11}$$
- we have also:

$$\begin{aligned}\int_\Omega h\nabla h\psi\,dx &= \int_\Omega h\nabla hu\,dx + 2\nu\int_\Omega h\nabla h\nabla\ln h\,dx\\ &= \frac{d}{dt}\int_\Omega h^2/2\,dx + 2\nu\int_\Omega |\nabla h|^2\;dx,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\int_\Omega h\nabla b\psi\,dx &= \int_\Omega h\nabla bu\,dx + 2\nu\int_\Omega h\nabla b\nabla\ln h\,dx\\ &= \int_\Omega b\partial_t h\,dx + 2\nu\int_\Omega \nabla b\cdot\nabla h\,dx.\end{aligned} \tag{4.13}$$

Finally, gathering results (4.10)–(4.13) provides the following entropy equality:

$$\frac{d}{dt}\int_\Omega h\frac{|\psi|^2}{2} + g\frac{h^2}{6}\,dx + 2\nu\int_\Omega h\,|A(u)|^2 + g\frac{2\nu}{3}\,|\nabla h|^2\;dx$$
$$+\int_\Omega b\partial_t h + 2\nu\nabla h\cdot\nabla b\,dx = 0. \tag{4.14}$$

Next, computing (4.6) + δ(4.14), setting (4.4), and remarking that:

$$\int_\Omega \operatorname{div}(hD(u))\cdot u\,|u|^r\,dx = -\int_\Omega h\,|u|^r\,|D(u)|^2\,dx - r\int_\Omega (hD(u)u\cdot\nabla)\,u\cdot u\,|u|^{r-2}\,dx$$

and

$$\left|\int_\Omega (hD(u)u\cdot\nabla)\,u\cdot u\,|u|^{r-2}\,dx\right| \leqslant 2\int_\Omega h\,|u|^r\,|D(u)|^2\,dx,$$

we obtain the inequality (4.5). □

Proof. **Proof of Theorem 4.1:** Thanks to Inequality (4.5), Theorem 4.1 is a straightforward consequence of the result in [6]. □

5. Example 2: the Grass sedimentation model

If we assume that the morphodynamic bed-load transport equation is given by:

$$\xi q_b = hu$$

which means that the sediment layer level evolves as the fluid height. Thus, Model (3.24) reduces to :

$$\left\{\begin{array}{lcl} \partial_t h + \operatorname{div}(hu) & = & 0 \\ \partial_t(hu) + \operatorname{div}(hu\otimes u) + gh\nabla\left(\dfrac{h}{3}+b\right) & = & 2\nu\operatorname{div}(hD(u)) - \begin{pmatrix} \mathfrak{K}_1(u) \\ \mathfrak{K}_2(u)\end{pmatrix} \\ \partial_t b + \operatorname{div}(hu) & = & 0 \end{array}\right. \tag{5.1}$$

thanks to the Grass bedload transport equation. Then, from the mass equation for h and the solid transport equation for b, we have:

$$b(t,x) = h(t,x) - b_0(x) \tag{5.2}$$

for some given data b_0. A very close model has been mathematically studied by Bresch *et al* [3] and they show an existence result under the regularity assumption on $b_0 > 0$. We also have, for the same model, results of Toumbou *et al* [19] where the viscous term is Δu instead of $\operatorname{div}(hD(u))$. We show that in spite of the pressure term $h^2/3$, if we add a friction term $r_0u+r_1u\,|u|$ (that we do not write for simplicity in the below inequalities but required for stability), the existence result [3] holds for Model (5.1). Indeed, the energy equality is given by:

Proposition 5.1. *Let (h,u,b) be a regular solution of (5.1), then the inequality holds:*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h\,|u|^2\,dx + g\frac{h^2}{6} + g\frac{b_0^2}{2}\,dx + 2\nu\int_\Omega h\,|D(u)|^2\,dx \leqslant \int_\Omega g\frac{b_0^2}{2}\,dx \tag{5.3}$$

The BD-entropy is given by:

Proposition 5.2. *Let (h, u, b) be a regular solution of (5.1), then the inequality holds:*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega h\,|\psi|^2 + g\frac{h^2}{6}\,dx \\ &+\int_\Omega 2\nu\,|A(u)|^2\,dx + 2g\nu\int_\Omega \frac{5}{3}\,|\nabla h|^2 \leqslant \int_\Omega g\frac{b_0^2}{2} + g\nu\,|\nabla b_0|^2\,dx \end{aligned} \tag{5.4}$$

Then it is sufficient to have $b_0 \in L^2(0,T;L^2(\Omega)$ to apply [3] to obtain the existence result.

6. Conclusion

We have presented a way to derive a SVE like equations with viscous and non-viscous shallow water equations. This is done in the Vlasov-Compressible Navier-Stokes coupling framework, where the particles are assumed small with respect to the dimension of the domain and the relaxation time is also assumed small with respect to the time scale. As the usual SVE equations have already studied numerically and mathematically, now our study provide a derivation of an alternative of SVE equations for sedimentation. The derivation of the sedimentation is done thanks to a modified barotropic law. In [7], changing the barotropic law, we derive in a very close way, a sedimentation model for closed pipes where we mathematically show the stability as in [19, 6] for instance.

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Manuscript received 13 March 2023
revised 11 April 2023

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