Linear and Nonlinear Analysis
Volume 9, Number 2, 2023, 139–147
Yokohama Publishers
ISSN 2188-8167

EXISTENCE OF FIXED POINTS OF SET-VALUED UNIFORMLY LOCALLY CONTRACTIVE MAPPINGS

SIMEON REICH* AND ALEXANDER J. ZASLAVSKI

Abstract. In a 1961 paper by E. Rakotch it was shown that a uniformly locally contractive mapping on a complete metric space has a fixed point. In our recent work we have shown that for such a mapping, the fixed point problem is well posed and that inexact iterates of such a mapping converge to its unique fixed point, uniformly on bounded sets. In the present paper we show the existence of fixed points for set-valued uniformly locally strict contractions. This result is an extension of Nadler's classical theorem. For set-valued uniformly locally contractive mappings we establish the existence of approximate fixed points.

1. Introduction

For more than sixty years now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 3, 9, 10, 11, 14, 15, 16, 17, 18, 21, 24, 25, 26, 27, 31, 32] and references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this area including, in particular, studies of feasibility and common fixed points, which find important applications in engineering, medical and the natural sciences [4, 5, 6, 7, 12, 13, 29, 30, 31, 32].

In his 1961 paper, E. Rakotch [20] showed that a uniformly locally contractive mapping has a fixed point. This result was later improved in [22]. In our recent paper [28], we have shown that for a uniformly locally contractive mapping, the fixed point problem is well posed and that inexact iterates of such a mapping converge to its unique fixed point, uniformly on bounded sets. In the present paper we establish the existence of a fixed point for *set-valued* uniformly locally strict contractions. This result is an extension of Nadler's classical theorem [19]. In this connection, see also [8, 23, 10, 27]. For a set-valued uniformly locally contractive mapping, we show the existence of approximate fixed points.

2020 *Mathematics Subject Classification.* 47H09, 47H10, 54E50.

Key words and phrases. Complete metric space, fixed point, nonexpansive mapping, set-valued mapping.

*The first author was partially supported by the Israel Science Foundation (Grant No. 820/17), by the Fund for the Promotion of Research at the Technion (Grant 2001893) and by the Technion General Research Fund (Grant 2016723).

In this paper we use the following notations. Assume that (X,ρ) is a complete metric space. For each point $x \in X$ and each number $r > 0$, set

$$B(x,r) := \{y \in X : \ \rho(x,y) \le r\}.$$

For each point $x \in X$ and each nonempty set $A \subset X$, put

$$\rho(x,A) := \inf\{\rho(x,y) : \ y \in A\}.$$

For each pair of nonempty sets, define

$$H(A,B) := \max\{\sup\{\rho(a,B) : \ a \in A\}, \ \sup\{\rho(b,A) : \ b \in B\}\}.$$

Note that it is possible for $H(A,B)$ to be $+\infty$. We denote by 2^X and by $CL(X)$ the set of all subsets of X and the set of all closed subsets of X, respectively.

2. The first main result

Let $\Delta > 0$, $c \in [0,1)$ and let $T : X \to CL(X) \setminus \{\emptyset\}$ satisfy

$$H(T(x),T(y)) \le c\rho(x,y) \tag{2.1}$$

for each $x, y \in X$ satisfying $\rho(x,y) \le \Delta$. Such a mapping is called a set-valued uniformly locally strict contraction. We study the existence of fixed points of T, that is, points $x \in X$ satisfying $x \in T(x)$. In the case where T is a single-valued map, the existence of such fixed points was studied in [20, 22, 28]. In the case where (2.1) holds for each pair $x, y \in X$, a fixed point exists by Nadler's seminal theorem [19, 8, 10, 27].

Fix $c_0 \in (c,1)$. It is clear that the following auxiliary result is true.

Lemma 2.1. *Assume that $x, y \in X$, $\rho(x,y) \le \Delta$ and $x_0 \in T(x)$. Then there there exists a point $y_0 \in T(y)$ such that*

$$\rho(x_0,y_0) \le c_0\rho(x,y).$$

Theorem 2.2. *Assume that $x_0 \in X$,*

$$x_1 \in T(x_0) \tag{2.2}$$

and that there exist points $y_0, \dots, y_q \in X$, where q is a natural number, such that

$$y_0 = x_0, \ y_q = x_1 \tag{2.3}$$

and

$$\rho(y_i, y_{i+1}) \le \Delta, \ i = 0, \dots, q-1. \tag{2.4}$$

Let $\{y_i\}_{i=q+1}^{\infty}$ satisfy for each integer $i \ge 1$,

$$y_{i+q} \in T(y_i), \tag{2.5}$$

$$\rho(y_{i+1+q}, y_{i+q}) \le c_0\rho(y_i, y_{i+1}). \tag{2.6}$$

Then the following assertions hold.

1. The sequence $\{y_i\}_{i=0}^{\infty}$ *is well defined,*

$$\sum_{i=0}^{\infty} \rho(y_i, y_{i+1}) \leq \Delta q \sum_{i=0}^{\infty} c_0^i \leq \Delta q (1 - c_0)^{-1},$$

there exists

$$y_* = \lim_{i \to \infty} y_i,$$

$y_* \in T(y_*)$ *and for each integer* $p \geq 1$*, we have*

$$y_{pq} \in T^p(x_0)$$

and

$$\rho(y_{pq}, y_*) \leq \Delta q c_0^p (1 - c_0)^{-1}.$$

2. Assume that $\delta \in (0, \Delta]$*,* $\rho(x_0, x_1) \leq \delta$ *and* $q = 1$*. Then*

$$\rho(x_0, y_*) \leq \delta(1 - c_0)^{-1}.$$

Proof. By (2.2)–(2.4), we have

$$y_q = x_1 \in T(x_0) = T(y_0) \tag{2.7}$$

and

$$\rho(y_0, y_1) \leq \Delta. \tag{2.8}$$

Lemma 2.1 and (2.7) imply that there exists a point

$$y_{q+1} \in T(y_1) \tag{2.9}$$

such that

$$\rho(y_q, y_{q+1}) \leq c_0 \rho(y_0, y_1). \tag{2.10}$$

Assume that $p \geq q + 1$ is an integer and that $y_i \in X$, $i = q + 1, \ldots, p$, have been defined so that for each $i \in \{0, \ldots, p - q\}$,

$$y_{i+q} \in T(y_i) \tag{2.11}$$

and for each $i \in \{0, \ldots, p - q - 1\}$, we have

$$\rho(y_{i+q}, y_{i+q+1}) \leq c_0 \rho(y_i, y_{i+1}). \tag{2.12}$$

(In view (2.7), (2.9) and (2.10), our assumption does hold for $p = q + 1$.) Next, we define $y_{p+1} \in X$ as follows. In view of (2.11),

$$y_p \in T(y_{p-q}). \tag{2.13}$$

Lemma 2.1 and relations (2.4), (2.12) and (2.13) imply that there exists a point

$$y_{p+1} \in T(y_{p-q+1})$$

such that

$$\rho(y_p, y_{p+1}) \leq c_0 \rho(y_{p-q}, y_{p-q+1}).$$

Thus the assumption made regarding p also holds for $p+1$. Therefore the sequence $\{y_i\}_{i=0}^{\infty}$ is well defined by induction, and (2.11) and (2.12) hold for all integers $i \geq 0$. By (2.12), for each integer $p \geq 1$,

$$c_0 \sum_{i=0}^{q-1} \rho(y_{pq+i}, y_{pq+i+1}) \geq \sum_{i=0}^{q-1} \rho(y_{(p+1)q+i}, y_{(p+1)q+i+1}). \tag{2.14}$$

In view of (2.4) and (2.14), we have

$$\sum_{i=0}^{\infty} \rho(y_i, y_{i+1}) \leq \Delta q \sum_{i=0}^{\infty} c_0^i \leq \Delta q (1-c_0)^{-1},$$

the sequence $\{y_i\}_{i=0}^{\infty}$ converges and there exists

$$y_* = \lim_{j \to \infty} y_j. \tag{2.15}$$

It follows from (2.11) and (2.15) that

$$y_* = \lim_{i \to \infty} y_{iq},$$

$$y_{(i+1)q} \in T(y_{iq}),\ i = 0, 1, \ldots$$

and

$$y_* \in T(y_*).$$

Let $p \geq 1$ be an integer. By (2.12),

$$\begin{aligned} \rho(y_{pq}, y_*) &= \lim_{m \to \infty} \rho(y_{pq}, y_m) \\ &\leq \sum_{i=pq}^{\infty} \rho(y_i, y_{i+1}) \\ &\leq \sum_{i=pq}^{(p+1)q-1} \rho(y_i, y_{i+1}) (\sum_{i=0}^{\infty} c_0^i) \\ &\leq c_0^p \Delta q (1-c_0)^{-1}. \end{aligned}$$

This completes the proof of Assertion 1.

Now we prove Assertion 2. We have $q = 1$ by assumption. By Assertion 1, we have

$$\begin{aligned} \rho(x_0, y_*) &= \lim_{n \to \infty} \rho(y_0, y_n) \\ &\leq \sum_{i=0}^{\infty} \rho(y_i, y_{i+1}) \\ &\leq \rho(x_0, x_1) \sum_{i=0}^{\infty} c_0^i \\ &\leq \delta (1-c_0)^{-1}. \end{aligned}$$

Thus Assertion 2 is proved. This completes the proof of Theorem 2.2. □

3. The second main result

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Assume that $\phi : [0, \infty) \to [0, 1)$ is a decreasing function such that

$$\phi(t) < 1 \text{ for each } t > 0 \tag{3.1}$$

and that $T : X \to 2^X \setminus \{\emptyset\}$ satisfies

$$H(T(x), T(y)) \le \phi(\rho(x, y))\rho(x, y) \tag{3.2}$$

for each $x, y \in X$ satisfying $\rho(x, y) \le \Delta$. Such a mapping is called a set-valued uniformly locally contractive mapping. It is not difficult to see that the following auxiliary result is true.

Lemma 3.1. *Assume that $x, y \in X$, $\rho(x, y) \le \Delta$ and $x_1 \in T(x)$. Then there exists a point $y_1 \in T(y)$ such that*

$$\rho(x_1, y_1) \le 2^{-1}(1 + \phi(\rho(x, y))\rho(x, y)).$$

Theorem 3.2. *Assume that $x_0 \in X$,*

$$x_1 \in T(x_0)$$

and that there exist points $y_0, \dots, y_q \in X$, where q is a natural number, such that

$$y_0 = x_0,\ y_q = x_1 \tag{3.3}$$

and

$$\rho(y_i, y_{i+1}) \le \Delta,\ i = 0, \dots, q - 1. \tag{3.4}$$

Then there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that for each integer $i \ge 0$,

$$x_{i+1} \in T(x_i)$$

and

$$\lim_{i \to \infty} \rho(x_{i+1}, x_i) = 0.$$

Proof. By (3.3) and (3.4), we have

$$y_q = x_1 \in T(x_0) = T(y_0) \tag{3.5}$$

and

$$\rho(y_0, y_1) \le \Delta. \tag{3.6}$$

Lemma 3.2, and relations (3.5) and (3.6) imply that there exists a point

$$y_{q+1} \in T(y_1) \tag{3.7}$$

such that

$$\rho(y_q, y_{q+1}) \le 2^{-1}(1 + \phi(\rho(y_0, y_1)))\rho(y_0, y_1). \tag{3.8}$$

Assume now that $p \ge q+1$ is an integer and that the points $y_i \in X$, $i = q+1, \dots, p$, have been defined so that for each $i \in \{0, \dots, p - q\}$, we have

$$y_{i+q} \in T(y_i) \tag{3.9}$$

and for each $i \in \{0, \dots, p-q-1\}$, we have

$$\rho(y_{i+q}, y_{i+q+1}) \le 2^{-1}(1 + \phi(\rho(y_i, y_{i+1})))\rho(y_i, y_{i+1}). \tag{3.10}$$

(In view (3.5), (3.7) and (3.8), our assumption does hold for $p = q + 1$.) Next, we define $y_{p+1} \in X$. In view of (3.9), we have

$$y_p \in T(y_{p-q}). \tag{3.11}$$

Lemma 3.1, and relations (3.4) and (3.9)-(3.11) imply that there exists a point

$$y_{p+1} \in T(y_{p-q+1}) \tag{3.12}$$

such that

$$\rho(y_p, y_{p+1}) \le 2^{-1}(1 + \phi(\rho(y_{p-q}, y_{p-q+1})))\rho(y_{p-q}, y_{p-q+1}).$$

Thus the assumption made for p also holds for $p+1$. Therefore the sequence $\{y_i\}_{i=0}^{\infty}$ is well defined by induction, and (3.9) and (3.10) hold for all integers $i \ge 0$.

Now let $\epsilon \in (0,1)$ be given. It follows from (3.4) and (3.10) that for each integer $n \ge q + 2$,

$$\begin{aligned}
\Delta q &\ge \sum_{i=1}^{q} \rho(y_{i-1}, y_i) \\
&\ge \sum_{i=1}^{q} \rho(y_{i-1}, y_i) - \sum_{i=nq+1}^{(n+1)q} \rho(y_{i-1}, y_i) \\
&= \sum_{j=0}^{n-1} \Big(\sum_{i=jq}^{(j+1)q-1} \rho(y_{i-1}, y_i) - \sum_{i=(j+1)q}^{(j+2)q-1} \rho(y_{i-1}, y_i) \Big) \\
&\ge \sum_{j=0}^{n-1} \sum_{i=jq}^{(j+1)q-1} 2^{-1}(1 - \phi(\rho(y_i, y_{i+1})))\rho(y_i, y_{i+1}) \\
&\ge (2q)^{-1}(1 - \phi(q^{-1}\epsilon))\epsilon \mathrm{Card}(\{j \in \{0, \dots, nq-1\} : \ \rho(y_i, y_{i+1}) \ge \epsilon/q\})
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Card}(\{j \in \{0, \dots, nq-1\} : \ \rho(y_i, y_{i+1}) \ge \epsilon/q\}) \\
&\qquad \le 2q^2\epsilon^{-1}(1 - \phi(q^{-1}\epsilon))\Delta.
\end{aligned}$$

Since the relation above holds for all sufficiently large natural numbers n we conclude that

$$\begin{aligned}
&\mathrm{Card}(\{j \in \{0, 1, \dots, \} : \ \rho(y_i, y_{i+1}) \ge \epsilon/q\}) \\
&\qquad \le 2q^2\epsilon^{-1}(1 - \phi(q^{-1}\epsilon))\Delta.
\end{aligned}$$

This implies that there exists a natural number n_0 such that for each integer $i \ge n_0 q$,

$$\rho(y_i, y_{i+1}) \le \epsilon/q.$$

When combined with (3.9), this implies that for each integer $n \geq n_0$,

$$\rho(y_{nq}, y_{(n+1)q}) \leq \sum_{i=nq}^{(n+1)q-1} \rho(y_i, y_{i+1}) \leq \epsilon$$

and

$$y_{(n+1)q} \in T(y_{nq}).$$

Since ϵ is an abitrary element of $(0,1)$, this completes the proof of Theorem 3.2. □

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[2] A. Betiuk-Pilarska and T. Domínguez Benavides, *Fixed points for nonexpansive mappings and generalized nonexpansive mappings on Banach lattices*, Pure Appl. Func. Anal. **1** (2016), 343–359.

[3] R. E. Bruck, W. A. Kirk and S. Reich, *Strong and weak convergence theorems for locally nonexpansive mappings in Banach spaces*, Nonlinear Aalysis **6** (1982), 151–155.

[4] D. Butnariu, R. Davidi, G. T. Herman and I. G. Kazantsev, *Stable convergence behavior under summable perturbations of a class of projection methods for convex feasibility and optimization problems*, IEEE Journal of Selected Topics in Signal Processing **1** (2007), 540–547.

[5] Y. Censor, R. Davidi and G. T. Herman, *Perturbation resilience and superiorization of iterative algorithms*, Inverse Problems **26** (2010), 12 pp.

[6] Y. Censor and M. Zaknoon *Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review,* Pure Appl. Func. Anal. **3** (2018), 565–586.

[7] Y. Censor and A. J. Zaslavski, *Convergence and perturbation resilience of dynamic string-averaging projection methods*, Computational Optimization and Applications **54** (2013), 65–76.

[8] H. Covitz and S. B. Nadler, Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.

[9] F. S. de Blasi and J. Myjak, *Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach*, C. R. Acad. Sci. Paris **283** (1976), 185–187.

[10] F. S. de Blasi, J. Myjak, S. Reich and A. J. Zaslavski, *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set-Valued and Variational Analysis **17** (2009), 97–112.

[11] M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7–10.

[12] A. Gibali, *A new split inverse problem and an application to least intensity feasible solutions*, Pure Appl. Funct. Anal. **2** (2017), 243–258.

[13] A. Gibali, S. Reich and R. Zalas, *Outer approximation methods for solving variational inequalities in Hilbert space*, Optimization **66** (2017), 417–437.

[14] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.

[15] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, New York and Basel, 1984.

[16] J. Jachymski, *Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points*, Pure Appl. Funct. Anal. **2** (2017), 657–666.

[17] W. A. Kirk, *Contraction mappings and extensions*, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 1–34.

[18] R. Kubota, W. Takahashi and Y. Takeuchi, *Extensions of Browder's demiclosedness principle and Reich's lemma and their applications*, Pure Appl. Func. Anal. **1** (2016), 63–84.

[19] S. B. Nadler, Jr., *Multivalued contraction mappings,* Pacific Journal of Mathematics **30** (1969), 475–488.

[20] E. Rakotch, *On ϵ-contractive mappings*, Bull. Res. Council Israel **10F** (1961), 53–58.

[21] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459–465.

[22] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. **5** (1972), 26–42.

[23] S. Reich, *A fixed point theorem for locally contractive multi-valued functions*, Revue Roumaine Math. Pures Appl. **17** (1972), 569–572.

[24] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Analysis **15** (1990), 537–558.

[25] S. Reich and A. J. Zaslavski, *Generic aspects of metric fixed point theory*, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 557–575.

[26] S. Reich and A. J. Zaslavski, *Convergence to attractors under perturbations*, Commun. Math. Anal. **10** (2011), 57–63.

[27] S. Reich and A. J. Zaslavski, *Genericity in nonlinear analysis*, Developments in Mathematics, **34**, Springer, New York, 2014.

[28] S. Reich and A. J. Zaslavski, *Convergence and well-posedness properties of uniformly locally contractive mappings*, Topological Methods in Nonlinear Analysis **61** (2023), 761–773.

[29] W. Takahashi, *The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces*, Pure Appl. Funct. Anal. **2** (2017), 685–699.

[30] W. Takahashi, *A general iterative method for split common fixed point problems in Hilbert spaces and applications*, Pure Appl. Funct. Anal. **3** (2018), 349–369.

[31] A. J. Zaslavski, *Approximate Solutions of Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, Cham, 2016.

[32] A. J. Zaslavski, *Algorithms for Solving Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, Cham, 2018.

Manuscript received 23 September 2023
revised 14 December 2023

S. Reich
Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: sreich@technion.ac.il

A. J. Zaslavski
Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: ajzasl@technion.ac.il