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CONVERGENCE THEOREMS OF CONDITIONAL EXPECTATIONS BY USING CONTRACTIVE PROJECTIONS ON BANACH SPACES

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Abstract. For a given filter $\{\mathcal{G}_n\}$ of sub-algebras, the sequence of conditional expectations $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges strongly for any $X \in L^p(\Omega)$. We call it Lévy's theorem. In this paper, we show the more general condition of sub-algebras $\{\mathcal{G}_n\}$ such that the sequence of conditional expectations $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges strongly. It is an application of linear contractive projection theory on a Banach space by using nonlinear analytic methods.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable in L^1. Let $\{\mathcal{G}_n\}$ be any filtration of $\mathcal{F}$, and define $\mathcal{G}_\infty$ to be the minimal σ-algebra generated by $\{\mathcal{G}_n\}$. Then

$$\mathrm{E}[X|\mathcal{G}_n] \to \mathrm{E}[X|\mathcal{G}_\infty]$$

as $n \to \infty$, both P-almost surely and in L^1. If X be a random variable in L^p, $1 < p < \infty$, $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges in L^p. This result is usually called Lévy's upwards theorem. Similarly we have the Lévy's downwards theorem: Let $\{\mathcal{G}_n\}$ be any decreasing sequence of sub-sigma algebras of $\mathcal{F}$, and define $\mathcal{G}_\infty$ to be the intersection. Then

$$\mathrm{E}[X|\mathcal{G}_n] \to \mathrm{E}[X|\mathcal{G}_\infty]$$

as $n \to \infty$, both P-almost surely and in L^1. If X be a random variable in L^p, $1 < p < \infty$, $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges in L^p. Both Lévy's theorems are shown by using Doob's martingale convergence theorems: see [24]. In this paper, we show these theorems by using linear contractive projection theory on a Banach space without using martingale theory and we obtain more general condition such that the sequence of conditional expectations $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges in L^p.

Let E be a Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. We denote by $\text{s-}\liminf_{n\to\infty} C_n$ the set of limit points of $\{C_n\}$, that is, $x \in \text{s-}\liminf_{n\to\infty} C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbb{N}$ and $\{x_n\}$ converges strongly to x. Similarly, we denote by $\text{w-}\limsup_{n\to\infty} C_n$ the set of cluster points of $\{C_n\}$, that is, $y \in \text{w-}\limsup_{n\to\infty} C_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbb{N}$ and $\{y_{n_i}\}$ converges weakly to

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y. In general, we have

$$\liminf_{n\to\infty} C_n \subset \text{s-}\liminf_{n\to\infty} C_n \subset \text{w-}\limsup_{n\to\infty} C_n \subset \limsup_{n\to\infty} C_n :$$

see [20, 23]. Using these definitions, we define the Mosco convergence [4, 20, 23] of $\{C_n\}$. If C satisfies

$$\text{s-}\liminf_{n\to\infty} C_n = C = \text{w-}\limsup_{n\to\infty} C_n,$$

we say that $\{C_n\}$ Mosco converges to C and denote by

$$C = \text{M-}\lim_{n\to\infty} C_n.$$

In this case, C is a closed convex subset of E.

In 1984, Tsukada proved the following theorem for the metric projections in a Banach spaces.

Theorem 1.1 ([23]). *Let E be a Banach space whose dual space E^* has a Frechét differentiable norm and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. The following assertions are equivalent:*

(i) *$\{C_n\}$ Mosco converges to a nonempty subset of E;*
(ii) *there exists a nonempty closed convex subset C of E such that $d(x, C_n)$ tends to $d(x, C)$ as $n \to \infty$ for every $x \in E$;*
(iii) *$\{P_{C_n}x\}$ norm converges for any $x \in E$,*

where $d(x, C) = \inf_{y\in C} \|x - y\|$ and P_{C_n} is the metric projections of E onto C_n. In this case, we have $C = \text{M-}\lim_{n\to\infty} C_n$, and $\{P_{C_n}x\}$ norm converges to $P_C x$ for any $x \in E$.

In 1999, Kimura and Takahashi proved the following theorem for the sunny nonexpansive retracts.

Theorem 1.2 ([17]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weak compact subset of E has the fixed point property for nonexpansive mappings. Let $\{C_n\}$ be a sequence of sunny nonexpansive retracts of E. If $C = \text{M-}\lim_{n\to\infty} C_n$ exists and nonempty, then C is also a sunny nonexpansive retract. In addition, if the duality mapping J is weakly sequentially continuous, then, for each $x \in E$, $Q_{C_n}x$ converges strongly to $Q_C x$, where Q_{C_n}, Q_C are sunny nonexpansive retractions of E onto C_n, C, respectively.*

In 2007, Ibaraki and Takahashi proved the following theorem for the sunny generalized nonexpansive retracts.

Theorem 1.3 ([15]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let $\{C_n\}$ be a sequence of sunny generalized nonexpansive retracts of E. Suppose that the normalized duality mapping $J : E \to E^*$ is weakly sequentially continuous. If $C = \text{M-}\lim_{n\to\infty} C_n$ exists and is nonempty, then C is a sunny generalized nonexpansive retract of E. Moreover, for each $x \in E$, the sequence $\{R_{JC_n}x\}$ converges strongly to $R_{JC}x$.*

In 2009, Honda and Takahashi showed the following characterization of a linear contractive projection in a Banach space by using the orthogonal decomposition of a Banach space.

Theorem 1.4 ([8]). *Let E be a strictly convex, reflexive and smooth Banach space, let Y^* be a closed linear subspace of the dual space E^* of E and let $J : E \to E^*$ be the normalized duality mapping. If the sunny generalized nonexpansive retraction R_{Y^*} is a quasi-nonexpansive projection of E onto $J^{-1}Y^*$, then it is a norm one linear projection and $J^{-1}Y^*$ is a closed linear subspace in E. Conversely, any norm one linear projection of E is a quasi-nonexpansive and sunny generalized nonexpansive retraction whose retract is $J^{-1}Y^*$, where Y^* is a closed linear subspace of E^*.*

There are many applications of the orthogonal decomposition of a Banach space: see [2, 3, 7, 8, 9, 10, 11, 12, 13, 14]. In this paper, we show weak and strong convergence theorems for linear contractive projections in a Banach space by using Tsukada's theorem (Theorem 1.1) and this theorem.

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space with the dual space E^*. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$.

A Banach space E is said to be strictly convex if

$$\left\| \frac{x+y}{2} \right\| < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Also, E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| > \varepsilon$. If a Banach space E is uniformly convex, E is strictly convex.

A Banach space E is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a reflexive Banach space. E is strictly convex if and only if E^* is smooth. E is smooth if and only if E^* is strictly convex.

The space E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S(E)$, the limit (2.1) is attained uniformly for $x \in S(E)$, where $S(E) = \{z \in E : \|z\| = 1\}$. The norm of E is said to be Fréchet differentiable if the limit (2.1) is attained if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of

E satisfying $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$ converges strongly to x. We know that a Banach space E is reflexive, strictly convex and has the Kadec-Klee property if and only if E^* has a Fréchet differentiable norm.

Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the (normalized) duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$. If E has a Fréchet differentiable norm, J is norm to norm continuous: see [22] for more details.

Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y.$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$$

is always a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C: see Alber [1], Kamimura and Takahashi [16].

Let D be a nonempty closed subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive [15] if $F(T)$ is nonempty and

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let P be a mapping from E onto C. Then P is said to be a retraction, or a projection if $Px = x$ for all $x \in C$. It is known that if a mapping T of E into E satisfies $T^2 = T$, then T is a projection of E onto $\{Tx \in E : x \in E\}$. A mapping T of E onto a nonempty subset M of E with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = M$. The mapping $T : E \to E$ is also said to be sunny if

$$T(Tx + t(x - Tx)) = Tx$$

whenever $x \in E$ and $t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi.

Lemma 2.1 ([15]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexisve Banach space E and let R be a retraction from E onto C. Then, the following are equivalent:*

(a) *R is sunny and generalized nonexpansive;*
(b) *$\langle x - Rx, Jy - JRx\rangle \leq 0$ for all $(x, y) \in E \times C$.*

Lemma 2.2 ([15]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.3 ([15]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then, the following hold:*

(a) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;*
(b) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

The following theorems were proved by Kohsaka and Takahashi.

Theorem 2.4 ([18]). *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Theorem 2.5 ([18]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following are equivalent.*

(1) *D is a sunny generalized nonexpansive retract of E;*
(2) *D is a generalized nonexpansive retract of E;*
(3) *JD is closed and convex.*

In this case, D is closed.

Let E be a smooth, strictly convex and reflexive Banach space, let J be the normalized duality mapping from E onto E^* and let C^* be a nonempty closed convex subset of E^*. From these theorems, we can define a unique sunny generalized nonexpansive retraction R_{C^*} of E onto $J^{-1}C^*$ as follows:

$$R_{C^*} = J^{-1}\Pi_{C^*}J,$$

where Π_{C^*} is the generalized projection from E^* onto C^*. If Y^* is a closed linear subspace of E^*, we also call R_{Y^*} a generalized conditional expectation and for any x, $z = R_{Y^*}x$ if and only if $z \in J^{-1}Y^*$ and

$$\langle x - z, y^* \rangle = 0$$

for any $y^* \in Y^*$: see [10].

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$$

is always nonempty and a singleton. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y \in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C: see [21, 22]. The following lemma is in [21, 22].

Lemma 2.6 ([21, 22])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z) \rangle \leq 0$ for all $y \in C$.*

Let E be a Banach space and let C be a nonempty closed convex subset of E. We call a mapping $T : C \to C$ nonexpansive if for any $x, y \in C$, we have

$$\|Tx - Ty\| \leq \|x - y\|.$$

A Banach space E is said to have the fixed point property for nonexpansive mappings, if any nonexpansive mapping $T : C \to C$ have a fixed point for an arbitrary nonempty weakly compact convex subset $C \subset E$. Let C be a nonempty closed convex subset of a Banach space E, let T be a mapping from C into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be quasi-nonexpansive if $F(T)$ is nonempty and

$$\|Tx - u\| \leq \|x - u\|$$

for all $x \in C$ and $u \in F(T)$.

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

In a reflexive Banach space E, we have $Y_\perp = Y^\perp$ for an arbitrary nonempty subset $Y \subset E$.

By using a sunny generalized retraction and a metric projection, we introduced the orthogonal decomposition to a Banach space as follows.

Theorem 2.7 ([3, 10])**.** *Let E be a reflexive, strictly convex and smooth Banach space, let I be the identity operator of E onto itself and let $J : E \to E^*$ be the normalized duality mapping. Let Y^* be a closed linear subspace of the dual space E^* and let R_{Y^*} be the sunny generalized nonexpansive retraction onto $J^{-1}Y^*$. Then, the mapping $I - R_{Y^*}$ is the metric projection of E onto $Y^*_\perp$. Conversely, let Y be a closed linear subspace of E and let P_Y be the metric projection of E onto Y. Then, the mapping $I - P_Y$ is the sunny generalized nonexpansive retraction $R_{Y^\perp}$ onto $J^{-1}Y^\perp$, i.e., $I - P_Y = R_{Y^\perp}$.*

From this theorem, we obtain that, when E is a reflexive, strictly convex and smooth Banach space, any linear contractive projections $P : E \to E$: i.e. $\|P\| = 1$, are sunny generalized nonexpansive retractions (Theorem 1.4). If a closed linear subspace Y of a Banach space E is the range of a linear contractive projection of E, we call Y a 1-complemented subspace of E.

3. Strong converegnce theorems

By using this orthogonal decomposition of a Banach space (Theorem 2.7), we obtain the following theorem.

Theorem 3.1. *Let E be a reflexive and strictly convex Banach space with a Frechét differentiable norm and let $\{M_n\}$ be a sequence of closed linear subspaces of E. If $\{M_n\}$ converges to a closed linear subspace M of E in the sense of Mosco ($M =$ M-$\lim_{n\to\infty} M_n$) and $P_{M_n}x$ converges to $P_M x$ strongly for any $x \in E$, then the sequence of annihilators $\{M_n^\perp\}$ converges to a closed linear subspace $M^\perp$ in the sense of Mosco ($M^\perp =$ M-$\lim_{n\to\infty} M_n^\perp$).*

Proof. First, We shall show

$$\text{w-}\limsup M_n^\perp \subset M^\perp.$$

Let $\{x^*_{n_k}\}$ be a sequence such that $x^*_{n_k} \in M^\perp_{n_k}$ and $\{x^*_{n_k}\}$ convereges to an element of E^* weakly as k goes to infinity, i.e.

$$x^*_{n_k} \rightharpoonup x^* \in E^*.$$

Since s-$\liminf M_n = M$, for any $x \in M$, there exists a sequence $\{x_n\}$, $x_n \in M_n$ which converges to x strongly as n goes to infinity. We have for any $k \in \mathbb{N}$,

$$\langle x^*_{n_k}, x_{n_k}\rangle = 0$$

and

$$\lim_{k\to\infty} \langle x^*_{n_k}, x_{n_k}\rangle = \langle x^*, x\rangle = 0.$$

Then we obtain $x^* \in M^\perp$.

Next, We shall show

$$\text{s-}\liminf M_n^\perp \supset M^\perp.$$

Let $x^* \in M^\perp$ and let $J^{-1}(x^*) = x \in E$. From Theorem 2.7, there exists an element y in E such that

$$x = y - P_M y.$$

Let $x_n = y - P_{M_n}y$. We have that $J(x_n) = J(y - P_{M_n}y) \in M_n^{\perp}$ and

$$\lim_{n\to\infty} J(x_n) = \lim_{n\to\infty} J(y - P_{M_n}y) = J(y - P_M y) = J(x) = x^*$$

by the continuity of J. This means $x^* \in$ s-lim inf $M_n^{\perp}$. □

By using Theorem 1.1, we obtain the following strong convergence theorem.

Theorem 3.2. *Let E be a Banach space and let $\{M_n^*\}$ be a sequence of closed linear subspaces of E^*. We assume that E and E^* have Frechét differentiable norms.*

If $\{M_n^*\}$ converges to a closed linear subspace M^* of E^* in the sense of Mosco ($M^* = \text{M-lim}_{n\to\infty} M_n^*$) , then for each $x \in E$, the sequence $\{R_{M_n^*}x\}$ converges to $R_{M^*}x$ strongly.

Conversely, if for each $x \in E$ the sequence $\{R_{M_n^*}x\}$ converges to some element of E strongly, then $\{M_n^*\}$ converges to a closed linear subspace M^* of E^* in the sense of Mosco ($M^* = \text{M-lim}_{n\to\infty} M_n^*$) and the limit of $\{R_{M_n^*}x\}$ is $R_{M^*}x$

Proof. Since E and E^* have Frechét differentiable norms, E is a reflexive and strictly convex Banach space with a Frechét differentiable norm. If $\{M_n^*\}$ converges to a closed linear subspace M^* of E^* in the sense of Mosco, from Theorem 1.1, $\{P_{M_n^*}x^*\}$ norm converges to $P_{M^*}x^*$ for any $x^* \in E^*$. Hence, from Theorem 3.1, the sequence $\{(M_n^*)_\perp\}$ convereges to $(M^*)_\perp$ in the sense of Mosco. From Theorem 1.1, for any $x \in E$, we have the sequence $\{P_{(M_n^*)_\perp}x\}$ convereges strongly to $P_{(M^*)_\perp}x$. Since $R_{M_n^*}x = x - P_{(M_n^*)_\perp}x$ from Theorem 2.7, the sequence $\{R_{M_n^*}x\}$ converges strongly to $R_{M^*}x$.

Conversely, if $\{R_{M_n^*}x\}$ converges to an element $y \in E$ strongly, $P_{(M_n^*)_\perp}x = x - R_{M_n^*}x$ converges strongly. From Theorem 1.1, $\{(M_n^*)_\perp\}$ convereges to a nonempty closed convex subset M of E in the sense of Mosco and $\{P_{(M_n^*)_\perp}x\}$ converges to $P_M x = x - y$ strongly. Since $M = \text{s-lim inf}_{n\to\infty}(M_n^*)_\perp$, M is a closed linear subspace of E. From Theorem 2.7, $\{((M_n^*)_\perp)^\perp\}$ converges to $M^\perp$ in the sense of Mosco. Since M_n^* are closed linear subspaces of E^*, we have $((M_n^*)_\perp)^\perp = M_n^*$ and $M^\perp = M^*$: see [19]. In this case, we have $y = x - P_M x = x - P_{M^*_\perp}x = R_{M^*}x$. □

By using Theorem 1.4, we obtain the following corollary, immediately.

Corollary 3.3. *Let E be a Banach space and let P_n, $n \in \mathbb{N}$ be linear contractive projections of E whose retracts are M_n. We assume that E and E^* have Frechét differentiable norms.*

If $\{JM_n\}$ converges to a closed linear subspace JM of E^* in the sense of Mosco ($JM = \text{M-lim}_{n\to\infty} JM_n$) , then M is a 1-complemented subspace of E and for each $x \in E$, the sequence $\{P_n x\}$ converges strongly to Px, where P is a linear contractive projection of E whose retract is M.

Conversely, if for each $x \in E$ the sequence $\{P_n x\}$ converges to some element of E strongly, then $\{JM_n\}$ converges to a closed linear subspace JM of E^* in the sense of Mosco ($JM = \text{M-lim}_{n\to\infty} JM_n$) and the limit of $\{P_n x\}$ is Px, where P is a linear contractive projection of E whose retract is M.

4. Applications

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let E be the real $L^p(\Omega),\ 1 < p < \infty$. We can see that E and E^* are real Banach spaces which have Frechét differentiable norms: see [6]. The normalized duality mapping $J : E \to E^*$ is defined as for $x(\omega) \in E$

$$Jx(\omega) = |x(\omega)|^{p-1} \frac{\text{sign } x(\omega)}{\|x\|^{p-2}} :$$

see [5]. If $x \in E$ is a measurable function with respect to a sub-algebra $\mathcal{G}$ of $\mathcal{F}$, $Jx \in E^*$ is also a $\mathcal{G}$ measurable function. Let M^p be a closed linear subspace of E which consists of all $\mathcal{G}$ measurable functions in E and let M^q be a closed linear subspace of E^* which consists of all $\mathcal{G}$ measurable functions in E^*. We have $JM^p = M^q$. The conditional expectation $\mathrm{E}[x|\mathcal{G}]$ of $x \in E$ with respect to a sub-algebra $\mathcal{G}$ is a linear contractive projection of E onto M^p. From Corollary 3.3, we obtain the following theorem.

Theorem 4.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}_n,\ n \in \mathbb{N}$ be sub-algebras of $\mathcal{F}$, $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$ and M_n^q be a closed linear subspace of real $L^q(\Omega)$ which consists of all $\mathcal{G}_n$ measurable functions in real $L^q(\Omega)$.*

If $\{M_n^q\}$ converges to a closed linear subspace M^q of real $L^q(\Omega)$ in the sense of Mosco ($M^q = \text{M-lim}_{n\to\infty} M_n^q$), then for each real valued random variable $X \in L^p(\Omega)$, the sequence of conditional expectations $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges to some random variable in real $L^p(\Omega)$ in the L^p-norm.

Conversely, if the sequence of conditional expectations $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges to some random variable in real $L^p(\Omega)$ in the L^p-norm, $\{M_n^q\}$ converges to a closed linear subspace M^q of real $L^q(\Omega)$ in the sense of Mosco ($M^q = \text{M-lim}_{n\to\infty} M_n^q$) .

If $\{\mathcal{G}_n\}$ is a filtration, then $\{M_n^q\}$ is a monotone sequence and converges to a closed linear subspace of $L^q(\Omega)$ in the sense of Mosco: see [20]. While the sequence $\{\mathrm{E}[X|\mathcal{G}_n]\}$ is a L^p bounded martingale: see [24], $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges to some random variable in $L^p(\Omega)$ in the L^p-norm: see [24]. From this Theorem, if $\{\mathcal{G}_n\}$ is a decreasing sequence of sub-sigma algebras, $\{\mathrm{E}[X|\mathcal{G}_n]\}$ converges to some random variable in $L^p(\Omega)$ in the L^p-norm.

References

[1] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, 15–50.

[2] Ya. I. Alber, *Generalized Projections, Decompositions, and the Pythagorean-Type Theorem in Banach Spaces*, Appl. Math. Lett. **11** (1998), 115–121.

[3] Ya. I. Alber, *James orthogonality and orthogonal decompositions of Banach spaces*, J. Math. Anal. Appl. **312** (2005), 330–342.

[4] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer Academic, Dordrecht, 1993.

[5] I. Cioranescu, *Geometry of Banach spaces, Duality Mappings and Nonlinear Problems* Kluwer Academic Publishers, Dordecht, 1990.

[6] M. Fabian, P. Habala, P. Hájek, V. M. Santalucía, J. Pelant, V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics, Springer-Verlag, New York, 2001.

[7] T. Honda, *A nonlinear analytic approach to the splitting theorem of Jacobs-de Leeuw-Glicksberg*, Nonlinear Anal. **75** (2012), no. 16, 6002–6008.

[8] T. Honda and W. Takahashi, *Norm One Projections and Generalized Conditional Expectations*, Sci. Math. Jpn. **69** (2009), 303–313.

[9] T. Honda and W. Takahashi, *Nonlinear projections and orthogonal decompositions in Banach spaces*, in Nonlinear Analysis and Convex Analysis (S. Akashi, Y. Kimura and T. Tanaka. Eds.), 53–64, Yokohama Publ., Yokohama, 2010.

[10] T. Honda and W. Takahashi, *Nonlinear projections and generalized conditional expectations in Banach spaces*, Taiwanese J. Math., **15** (2011), no. 5, 2169–2193.

[11] H. Honda, T. Ibaraki and W. Takahashi, *Duality theorems and convergence theorems for nonlinear mappings in Banach spaces and applications*, Int. J. Math. Stat. **6** (2010), 46–64.

[12] H. Honda, W. Takahashi and J. C. Yao, *Nonexpansive retractions onto closed convex cones in Banach spaces*, Taiwanese J. Math., **14** (2010), no. 3B, 1023–1046.

[13] H. Honda, W. Takahashi and J. C. Yao, *Strong convergence theorems and nonlinear analytic methods for linear contractive mappings in Banach spaces*, J. Nonlinear Convex Anal. **10** (2010), no. 3, 547–566.

[14] H. Hudzik, Y. Wang and R. Sha, *Orthogonally complemented subspaces in Banach spaces* , Numer. Funct. Anal. Optim. **29** (2008), 779–790.

[15] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[16] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[17] Y. Kimura and W. Takahashi, *Strong convergence of sunny nonexpansive retractions in Banach spaces*, PanAmer. Math. J. **9** (1999), 1–6.

[18] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 197–209.

[19] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer–Verlag, Berlin–Heidelberg–New York, 1998.

[20] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (1969), 510–585.

[21] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer–Verlag, Berlin–Heidelberg–New York, 1970.

[22] W. Takahashi, *Nonlinear Functional Analysis – Fixed Point Theory and Its Applications*, Yokohama Publishers, 2000.

[23] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory **40** (1984), 301–309.

[24] D. Williams, *Probability with martingales*, Cambridge University Press, 1991.

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