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QUASICONJUGATE DUAL PROBLEMS FOR QUASICONVEX PROGRAMMING

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Abstract. In this paper, we study quasiconjugate dual problems for quasiconvex programming. We introduce three types of dual problems in terms of quasiconjugate functions. We show strong duality theorems for the dual problems. We discuss about our results and previous ones.

1. Introduction

In the research on optimization, dual problems play an important and essential role. In particular, dual problems in terms of conjugate functions have been investigated extensively, see [2–8, 10–19, 21–23]. In convex programming, Fenchel conjugate [3, 4], f^*, is known as the conjugate function. On the other hand, in quasiconvex programming, various types of quasiconjugate functions have been investigated. In [5], Greenberg and Pierskalla introduce λ-quasiconjugate, f^ν_λ, and study duality theorems. λ-quasiconjugate has an extra parameter λ, and also λ-semiconjugate [15, 16], f^θ_λ, has an extra parameter. For general quasiconvex functions, it is necessary to use extra parameters to obtain duality results. If we want to avoid these parameters, then we may restrict the class of quasiconvex functions. Some researchers study other conjugate functions in order to avoid these extra parameters. In [22], Thach introduces H-quasiconjugate, f^H, without extra parameter, and shows duality relationships between quasiconvex maximization and quasiconvex minimization. Additionally, Thach studies R-quasiconjugate functions, f^R, and duality theorems for R-evenly quasiconvex functions in [23]. Recently, we show dual problems in terms of Q-conjugate for quasiconvex programming in [19]. The Q-conjugate function of f is defined by

$$f^Q(v,t) = -\inf\{f(x) : \langle v, x\rangle \geq t\}.$$

It is known that $f^Q(v,t) = f^\nu_t - t$, $f^Q(v,1) = f^H$ on $\mathbb{R}^n \setminus \{0\}$, and $f^Q(v,-1) = f^R$. Although Q-conjugate function also has an extra parameter t, duality theorems in [19] are valid for general quasiconvex programing problems. In other words, by restricting the class of quasiconvex functions, we can obtain new duality theorems for quasiconvex programming.

In this paper, we consider the following optimization problem (P):

$$(P)\begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in A, \end{cases}$$

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where f is an extended real-valued quasiconvex function on $\mathbb{R}^n$, and A is a convex subset of $\mathbb{R}^n$. We study dual problems in terms of three types of quasiconjugate functions, H-quasiconjugate, O-quasiconjugate, and R-quasiconjugate. We introduce three dual problems, and show strong duality theorems. Additionally, we compare our results with previous ones.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we show duality theorems for quasiconvex programming in terms of quasiconjugate functions. In Section 4, we discuss about our results.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The inner product of two vectors v and x in $\mathbb{R}^n$ is denoted by $\langle v, x\rangle$. The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ \infty & otherwise. \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. A function f is said to be convex if for each x, $y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y).$$

The epigraph of f is defined as

$$\mathrm{epi} f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\}.$$

A function f is convex if and only if epi$f$ is convex. The Fenchel conjugate [3, 4] of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as

$$f^*(v) = \sup_{x \in \mathbb{R}^n} \{\langle v, x\rangle - f(x)\}.$$

A function f is said to be quasiconvex if for each x, $y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \leq \max\{f(x), f(y)\}.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\mathrm{lev}(f, \diamond, \alpha) = \{x \in \mathbb{R}^n : f(x) \diamond \alpha\}$$

for each $\alpha \in \mathbb{R}$. It is well known that f is quasiconvex if and only if lev$(f, \leq, \alpha)$ is convex for all $\alpha \in \mathbb{R}$.

We define the following families of open half spaces:

$$\begin{aligned} H &= \{\mathrm{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha \in \mathbb{R}\}, \\ H^+ &= \{\mathrm{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha > 0\}, \\ H^0 &= \{\mathrm{lev}(v, <, 0) : v \in \mathbb{R}^n\}, \\ H^- &= \{\mathrm{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha < 0\}. \end{aligned}$$

A subset A of $\mathbb{R}^n$ is said to be evenly (H-evenly, O-evenly, and R-evenly) convex if it is the intersection of a subfamily of H (H^+, H^0, H^-, respectively). We define the whole space and the empty set is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex by convention. A function f is said to be evenly (H-evenly, O-evenly, and R-evenly) quasiconvex if $\mathrm{lev}(f, \leq, \alpha)$ is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex for all $\alpha \in \mathbb{R}$.

We show the following two propositions without proof.

Proposition 2.1. *Let A be a subset of $\mathbb{R}^n$. Then, the following statements hold:*

(i) *if A is evenly convex, then A is convex,*
(ii) *if A is open or closed convex, then A is evenly convex,*
(iii) *if A is non-empty, then A is H-evenly convex if and only if A is evenly convex and contains 0,*
(iv) *if A is O-evenly convex, then for each $x \in A$ and $t > 0$, $tx \in A$,*
(v) *if A is R-evenly convex, then for each $x \in A$ and $t \geq 1$, $tx \in A$.*

Proposition 2.2. *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Then, the following statements hold:*

(i) *if f is evenly quasiconvex, then f is quasiconvex,*
(ii) *if f is lower semicontinuous (lsc) or upper semicontinuous (usc) quasiconvex, then f is evenly quasiconvex,*
(iii) *f is H-evenly quasiconvex if and only if f is evenly quasiconvex and $0 \in \mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$,*
(iv) *if f is O-evenly quasiconvex or R-evenly quasiconvex, then $0 \in \mathbb{R}^n$ is a global maximizer of f in $\mathbb{R}^n$,*
(v) *if f is O-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t > 0$, $f(x) = f(tx)$,*
(vi) *if f is R-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t \geq 1$, $f(x) \geq f(tx)$.*

Proposition 2.1 and Proposition 2.2 are elementary but important. Various researchers investigate evenly convex sets and evenly quasiconvex functions precisely, see [1, 2, 4, 9, 17, 18, 20, 21, 23] and references therein.

In quasiconvex programming, various types of quasiconjugate functions have been investigated. In [6], Martínez-Legaz introduces Q-conjugate of f as follows:

$$f^Q(v,t) = -\inf\{f(x) : \langle v, x\rangle \geq t\}.$$

In [22], Thach introduces H-quasiconjugate of f as follows:

$$f^H(v) = \begin{cases} -\inf\{f(x) : \langle v, x\rangle \geq 1\}, & v \neq 0, \\ -\sup\{f(x) : x \in \mathbb{R}^n\}, & v = 0. \end{cases}$$

In [11], Passy and Prisman introduce O-quasiconjugate of f as follows:

$$f^O(v) = -\inf\{f(x) : \langle v, x\rangle \geq 0\}.$$

In [22], Thach introduces R-quasiconjugate as follows:

$$f^R(v) = -\inf\{f(x) : \langle v, x\rangle \geq -1\}.$$

Although Q-conjugate and O-quasiconjugate are defined by other names, we denote f^Q and f^O for the sake of distinction.

In this paper, we consider the following quasiconvex programming problem (P):

$$(P)\begin{cases}\text{minimize } f(x),\\ \text{subject to } x\in A,\end{cases}$$

where f is an extended real-valued quasiconvex function on $\mathbb{R}^n$, and A is a convex subset of $\mathbb{R}^n$. In [19], we define the following dual problem (D) in terms of Q-conjugate for the primal problem (P):

$$(D)\begin{cases}\text{minimize } f^Q(v,t),\\ \text{subject to } (v,t)\in -\text{epi}\delta_A^*.\end{cases}$$

We denote

$$\text{val}(P)=\inf_{x\in A} f(x), \text{ and } \text{val}(D)=\inf_{(v,t)\in -\text{epi}\delta_A^*} f^Q(v,t).$$

In [19], we show the following strong duality theorem.

Theorem 2.3 ([19])**.** *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Then*

$$\text{val}(P)=-\text{val}(D).$$

Additionally, we introduce the following three dual problems:

$$(D)_1\begin{cases}\text{minimize } f^Q(v,1),\\ \text{subject to } (v,1)\in -\text{epi}\delta_A^*,\end{cases}$$

$$(D)_0\begin{cases}\text{minimize } f^Q(v,0),\\ \text{subject to } (v,0)\in -\text{epi}\delta_A^*,\end{cases}$$

$$(D)_{-1}\begin{cases}\text{minimize } f^Q(v,-1),\\ \text{subject to } (v,-1)\in -\text{epi}\delta_A^*.\end{cases}$$

We show the following corollary.

Corollary 2.4 ([19])**.** *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a nonempty convex subset of $\mathbb{R}^n$. Then*

$$\text{val}(P)=-\min\{\text{val}(D)_1,\text{val}(D)_0,\text{val}(D)_{-1}\}.$$

3. Quasiconjugate duality theorems

In this section, we study dual problems in terms of three types of quasiconjugate functions, By Corollary 2.4, val(P) is equal to at least one of val$(D)_1$, val$(D)_0$, and val$(D)_{-1}$. We show sufficient conditions for strong duality theorems between the primal problem and dual problems.

At first, we show a strong duality theorem for $(D)_1$ under H-evenly quasiconvexity.

Theorem 3.1. *Let f be an usc H-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$. Then*

$$\mathrm{val}(P) = -\mathrm{val}(D)_1.$$

Proof. By Corollary 2.4,

$$\mathrm{val}(P) = -\min\{\mathrm{val}(D)_1, \mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\} \geq -\mathrm{val}(D)_1.$$

Assume that $\mathrm{val}(P) > -\mathrm{val}(D)_1$. Since $\mathrm{val}(P) = \inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^n} f(x)$, there exists $\alpha \in \mathbb{R}$ such that $\mathrm{val}(P) > \alpha > -\mathrm{val}(D)_1$ and $\mathrm{val}(P) > \alpha > \inf_{x \in \mathbb{R}^n} f(x)$. This shows that $\mathrm{lev}(f, <, \alpha)$ is open convex and $0 \in \mathrm{lev}(f, <, \alpha)$ since f is usc H-evenly quasiconvex. By the separation theorem between A and $\mathrm{lev}(f, <, \alpha)$, there exists $(v_0, t_0) \in \mathbb{R}^{n+1}$ such that for each $x \in A$ and $z \in \mathrm{lev}(f, <, \alpha)$,

$$\langle v_0, x \rangle \geq t_0 > \langle v_0, z \rangle .$$

Since $0 \in \mathrm{lev}(f, <, \alpha)$, $t_0 > 0$. Hence,

$$\left\langle \frac{v_0}{t_0}, x \right\rangle \geq 1 > \left\langle \frac{v_0}{t_0}, z \right\rangle$$

Let $v^* = \frac{v_0}{t_0}$, then $(v^*, 1) \in -\mathrm{epi}\delta_A^*$ and

$$-f^Q(v^*, 1) = \inf\{f(y) : \langle v^*, z \rangle \geq 1\} \geq \alpha > -\mathrm{val}(D)_1 \geq -f^Q(v^*, 1).$$

This is a contradiction. This completes the proof. $\square$

Next, we show a strong duality theorem for $(D)_0$ under O-evenly quasiconvexity.

Theorem 3.2. *Let f be an usc O-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then*

$$\mathrm{val}(P) = -\mathrm{val}(D)_0.$$

Proof. By Corollary 2.4,

$$\mathrm{val}(P) = -\min\{\mathrm{val}(D)_1, \mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\} \geq -\mathrm{val}(D)_0.$$

Assume that $\mathrm{val}(P) > -\mathrm{val}(D)_0$. Then, there exists $\alpha \in \mathbb{R}$ such that $\mathrm{val}(P) > \alpha > -\mathrm{val}(D)_0$. Since $(0, 0) \in -\mathrm{epi}\delta_A^*$,

$$\alpha > -\mathrm{val}(D)_0 \geq -f^Q(0, 0) = \inf_{x \in \mathbb{R}^n} f(x).$$

This shows that $\mathrm{lev}(f, <, \alpha)$ is nonempty. Furthermore, $\mathrm{lev}(f, <, \alpha)$ is open convex since f is usc O-evenly convex. By the separation theorem between A and $\mathrm{lev}(f, <, \alpha)$, there exists $(v_0, t_0) \in \mathbb{R}^{n+1}$ such that for each $x \in A$ and $z \in \mathrm{lev}(f, <, \alpha)$,

$$\langle v_0, x \rangle \geq t_0 > \langle v_0, z \rangle$$

For $x_0 \in A$, assume that $\langle v_0, x_0 \rangle < 0$. This shows that for each $z \in \mathrm{lev}(f, <, \alpha)$,

$$0 > \langle v_0, x_0 \rangle \geq t_0 > \langle v_0, z \rangle .$$

Let $t = \frac{\langle v_0, x_0 \rangle}{2\langle v_0, z \rangle}$, then $t > 0$ and $tz \in \mathrm{lev}(f, <, \alpha)$ by Proposition 2.1 (iv). However,

$$\langle v_0, tz \rangle = \frac{\langle v_0, x_0 \rangle}{2} > \langle v_0, x_0 \rangle .$$

This is a contradiction. Hence for each $x \in A$, $\langle v_0, x\rangle \geq 0$.

Next, assume that there exists $z_0 \in \mathrm{lev}(f, <, \alpha)$ such that $\langle v_0, z_0\rangle \geq 0$. If $\langle v_0, z_0\rangle > 0$, then $t_0 > 0$. Since f is O-evenly quasiconvex, $tz_0 \in \mathrm{lev}(f, <, \alpha)$ for each $t > 0$. However, for sufficiently large t, $\langle v_0, tz_0\rangle > t_0$. This is a contradiction. If $\langle v_0, z_0\rangle = 0$, there exists $\bar{z} \in \mathrm{lev}(f, <, \alpha)$ such that $\langle v_0, \bar{z}\rangle > 0$ since $\mathrm{lev}(f, <, \alpha)$ is open. By the similar way, we can prove that $\langle v_0, t\bar{z}\rangle > t_0$ for sufficiently large t. Hence, for each $z \in \mathrm{lev}(f, <, \alpha)$, $\langle v_0, z\rangle < 0$.

This shows that $(v_0, 0) \in -\mathrm{epi}\delta_A^*$ and

$$-f^Q(v_0, 0) = \inf\{f(y) : \langle v_0, z\rangle \geq 0\} \geq \alpha > -\mathrm{val}(D)_0 \geq -f^Q(v_0, 0).$$

This is a contradiction. Hence $\mathrm{val}(P) = \mathrm{val}(D)_0$. □

Finally, we show a strong duality theorem for R-evenly quasiconvex objective functions.

Theorem 3.3. *Let f be an usc R-evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then*

$$\mathrm{val}(P) = -\min\{\mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}.$$

Proof. By Corollary 2.4,

$$\mathrm{val}(P) = -\min\{\mathrm{val}(D)_1, \mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\} \geq -\min\{\mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}.$$

Assume that $\mathrm{val}(P) > -\min\{\mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}$. Then, there exists $\alpha \in \mathbb{R}$ such that $\mathrm{val}(P) > \alpha > -\min\{\mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\}$. Since $(0, 0) \in -\mathrm{epi}\delta_A^*$,

$$\alpha > -\min\{\mathrm{val}(D)_0, \mathrm{val}(D)_{-1}\} \geq -f^Q(0, 0) = \inf_{x\in\mathbb{R}^n} f(x).$$

This shows that $\mathrm{lev}(f, <, \alpha)$ is nonempty. Furthermore, $\mathrm{lev}(f, <, \alpha)$ is open convex since f is usc R-evenly convex. By the separation theorem between A and $\mathrm{lev}(f, <, \alpha)$, there exists $(v_0, t_0) \in \mathbb{R}^{n+1}$ such that for each $x \in A$ and $z \in \mathrm{lev}(f, <, \alpha)$,

$$\langle v_0, x\rangle \geq t_0 > \langle v_0, z\rangle$$

If $t_0 = 0$, we can check that $\mathrm{val}(P) = -\mathrm{val}(D)_0$. Additionally, if $t_0 < 0$, then $\mathrm{val}(P) = -\mathrm{val}(D)_{-1}$.

Assume that $t_0 > 0$ and there exists $z_0 \in L(f, <, \alpha)$ such that $\langle v_0, z_0\rangle \geq 0$. If $\langle v_0, z_0\rangle = 0$, then there exists $\bar{z} \in L(f, <, \alpha)$ such that $\langle v_0, \bar{z}\rangle > 0$ since f is usc. Without loss of generality, we can assume that $\langle v_0, z_0\rangle > 0$. Since f is R-evenly quasiconvex, $tz_0 \in L(f, <, \alpha)$ for each $t \geq 1$ by Proposition 2.1 (v). However, for sufficiently large t, $\langle tv_0, z_0\rangle > t_0$. This is a contradiction. Hence, for each $x \in A$ and $z \in \mathrm{lev}(f, <, \alpha)$,

$$\langle v_0, x\rangle \geq t_0 > 0 > \langle v_0, z\rangle .$$

This shows that $(v_0, 0) \in -\mathrm{epi}\delta_A^*$ and

$$-f^Q(v_0, 0) = \inf\{f(y) : \langle v_0, z\rangle \geq 0\} \geq \alpha > -\mathrm{val}(D)_0 \geq -f^Q(v_0, 0).$$

This is a contradiction. This completes the proof. □

4. Discussion

In this section, we discuss about our results. In particular, we compare our results with duality theorems in [22, 23].

4.1. **H-quasiconjugate duality.** In [22], Thach studies duality theorems in terms of H-quasiconjugate functions. We consider the following maximization problem:

$$\begin{cases} \text{maximize } f(x), \\ \text{subject to } x \in A, \end{cases}$$

where f is an usc quasiconvex function and A is a compact convex subset of $\mathbb{R}^n$. Thach introduces the following dual problem:

$$\begin{cases} \text{minimize } f^H(v), \\ \text{subject to } v \in \mathbb{R}^n \setminus \mathrm{int}A^0, \end{cases}$$

where $A^0 = \{v \in \mathbb{R}^n : \forall x \in A, \langle v, x \rangle \leq 1\}$, and $\mathrm{int}A^0$ is the interior of A^0. Thach investigates a maximization problem of a quasiconvex function and shows duality theorems in terms of H-quasiconjugate. On the other hand, in this paper, we study a minimization problem of a quasiconvex function. In Theorem 3.1, we show strong duality theorem in terms of H-quasiconjugate.

4.2. **R-quasiconjugate duality.** In [23], Thach introduces the following dual problem:

$$(D)_R \begin{cases} \text{minimize } f^R(v), \\ \text{subject to } v \in -A^0. \end{cases}$$

Clearly, $(D)_R = (D)_{-1}$. Thach shows the following result, see Theorem 3.2 in [23].

Theorem 4.1 ([23]). $\mathrm{val}(P) = -\mathrm{val}(D)_R$ *and* $v^* \in \mathbb{R}^n$ *is a global minimizer of* $(D)_R$ *if and only if for each* $x \in A$ *and* $y \in \mathrm{lev}(f, <, \mathrm{val}(P))$,

$$\langle v^*, x \rangle \geq -1 > \langle v^*, y \rangle . \tag{4.1}$$

Thach shows a characterization of a global minimizer of $(D)_R$. However, even if f is R-evenly quasiconvex, the separation inequality (4.1) does not necessarily hold by Theorem 3.3. We show the following example.

Example 4.2. Let f be the following function on $\mathbb{R}^2$:

$$f(x) = \begin{cases} 0, & x \in S = \{(y_1, y_2) : y_1 > 0, y_1 y_2 > 1\}, \\ 1, & x \notin S, x_1 + x_2 > \frac{1}{2}, \\ 2, & \textit{otherwise}. \end{cases}$$

We can check easily that f is usc R-evenly quasiconvex.

Consider a minimization problem of f on $A = \{(x_1, x_2) : x_2 \leq 0\}$. Then $\mathrm{val}(P) = 1$, $((0,-1),0) \in -\mathrm{epi}\delta_A^*$, and

$$f^O((0,-1)) = -1,$$

that is, $\mathrm{val}(P) = -\mathrm{val}(D)_0$. However, $\mathrm{val}(P) \neq -\mathrm{val}(D)_{-1}$. Actually, let $v = (v_1, v_2) \in \mathbb{R}^n$ such that $((v_1, v_2), -1) \in -\mathrm{epi}\delta_A^*$, then $v_1 = 0$ and $v_2 \leq 0$. This shows that

$$f^R(v) = -\inf\{f(x) : \langle v, x\rangle \geq -1\} = 0.$$

Hence,

$$\mathrm{val}(P) = 1 > 0 = -\mathrm{val}(D)_{-1}.$$

In Theorem 3.1 and Theorem 3.2, we show strong duality theorems for H-evenly quasiconvex and O-evenly quasiconvex functions. On the other hand, in Theorem 3.3, we need not only R-quasiconjugate but also O-quasiconjugate for R-evenly quasiconvex functions. In the following theorem, we show a sufficient condition for $\mathrm{val}(P) = -\mathrm{val}(D)_{-1}$.

Theorem 4.3. *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Assume that $0 \in \mathrm{int}A$ and $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$. Then*

$$\mathrm{val}(P) = -\mathrm{val}(D)_{-1}.$$

Proof. Let $\alpha = \mathrm{val}(P)$. By the assumption, $\mathrm{lev}(f,<,\alpha)$ is nonempty. By the separation theorem for A and $\mathrm{lev}(f,<,\alpha)$, there exists $(v_0, t_0) \in \mathbb{R}^{n+1}$ such that for each $x \in A$ and $z \in \mathrm{lev}(f,<,\alpha)$,

$$\langle v_0, x\rangle \geq t_0 > \langle v_0, z\rangle .$$

Since $0 \in \mathrm{int}A$, $t_0 < 0$. Let $\bar{v} = \frac{v_0}{-t_0}$, then for each $x \in A$ and $z \in \mathrm{lev}(f,<,\alpha)$,

$$\langle \bar{v}, x\rangle \geq -1 > \langle \bar{v}, z\rangle .$$

This shows that $f^Q(\bar{v},-1) \leq -\alpha$ and $(\bar{v},-1) \in -\mathrm{epi}\delta_A^*$. By Theorem 2.3,

$$\mathrm{val}(P) = -\mathrm{val}(D) \geq -\mathrm{val}(D)_{-1} \geq -f^Q(\bar{v},-1) \geq \alpha.$$

This completes the proof. □

In Theorem 4.3, we do not assume that f is R-evenly quasiconvex. For R-evenly quasiconvex objective functions, we may need additional assumptions for $\mathrm{val}(P) = -\mathrm{val}(D)_{-1}$.

4.3. **Conjugate functions in convex and quasiconvex programming.** We show the following table of conjugate functions in convex and quasiconvex programming:

convexity of function	conjugate	domain of conjugate
convex	f^*	$\mathbb{R}^n$
evenly quasiconvex	f^Q	$\mathbb{R}^{n+1}$
H-evenly quasiconvex	$f^H = f^Q(\cdot, 1)$	$\mathbb{R}^n$
O-evenly quasiconvex	$f^O = f^Q(\cdot, 0)$	$\mathbb{R}^n$
R-evenly quasiconvex	$f^R = f^Q(\cdot, -1)$	$\mathbb{R}^n$

In convex programming, Fenchel conjugate plays an essential role. In quasiconvex programming, we need extra parameters to obtain duality results, for example t in $f^Q(v, t)$. In this paper, we show duality results for quasiconvex programming without extra parameters by using f^H, f^O, and f^R. Similar to Fenchel conjugate, these three quasiconjugate functions are defined on $\mathbb{R}^n$. Additionally, these quasiconjugates are closely related to f^Q.

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References

[1] A. Daniilidis and J. E. Martínez-Legaz, *Characterizations of evenly convex sets and evenly quasiconvex functions*, J. Math. Anal. Appl. **273** (2002), 58–66.

[2] M. D. Fajardo, M. A. Goberna, M. M. L. Rodríguez and J. Vicente-Pérez, *Even Convexity and Optimization*, EURO Advanced Tutorials on Operational Research. Springer, 2020.

[3] W. Fenchel, *On conjugate convex functions*, Can. J. Math. **1** (1949), 73–77.

[4] W. Fenchel, *A remark on convex sets and polarity*, Comm. Sém. Math. Univ. Lund Tome Suppl. (1952), 82–89 .

[5] H. J. Greenberg and W. P. Pierskalla, *Quasi-conjugate functions and surrogate duality*, Cah. Cent. Étud. Rech. Opér. **15** (1973), 437–448.

[6] J. E. Martínez-Legaz, *Quasiconvex duality theory by generalized conjugation methods. Optimization*, **19** (1988), 603–652.

[7] J. E. Martínez-Legaz, *A generalized concept of conjugation*, Lecture Notes in Pure and Appl. Math. **86** (1983), 45–59.

[8] J. E. Martínez-Legaz, *A new approach to symmetric quasiconvex conjugacy*, Lecture Notes in Econom. and Math. Systems. **226** (1984), 42–48.

[9] J. E. Martínez-Legaz, *Characterization of R-evenly quasiconvex functions*, J. Optim. Theory Appl. **95** (1997), 717–722.

[10] J. J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. **49** (1970), 109–154.

[11] U. Passy and E. Z. Prisman, *Conjugacy in quasi-convex programming*, Math. Program. **30** (1984), 121–146

[12] U. Passy and E. Z. Prisman, *A convex-like duality scheme for quasi-convex programs*, Math. Program. **32** (1985), 278–300.

[13] J. P. Penot, *What is quasiconvex analysis?*, Optimization **47** (2000), 35–110.

[14] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[15] I. Singer, *The lower semi-continuous quasi-convex hull as a normalized second conjugate*, Nonlinear Anal. Theory Methods & Appl. **7** (1983), 1115–1121.

[16] I. Singer, *Conjugate functionals and level sets*, Nonlinear Anal. Theory Methods & Appl. **8** (1984), 313–320.

[17] S. Suzuki, *Set containment characterization with strict and weak quasiconvex inequalities*, J. Global Optim. **47** (2010), 273–285.

[18] S. Suzuki, *Duality theorems for quasiconvex programming with a reverse quasiconvex constraint*, Taiwanese J. Math. **21** (2017), 489–503.

[19] S. Suzuki, *Conjugate dual problem for quasiconvex programming*, J. Nonlinear Convex Anal. **23** (2022), 879–889.

[20] S. Suzuki, *Optimality conditions for quasiconvex programming in terms of quasiconjugate functions*, Minimax Theory Appl. to appear.

[21] S. Suzuki and D. Kuroiwa, *Set containment characterization for quasiconvex programming*, J. Global Optim. **45** (2009), 551–563.

[22] P. T. Thach, *Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications*, J. Math. Anal. Appl. **159** (1991), 299–322.

[23] P. T. Thach, *Diewert-Crouzeix conjugation for general quasiconvex duality and applications*, J. Optim. Theory Appl. **86** (1995), 719–743.

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