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APPROXIMATING COMMON FIXED POINTS USING MARTINEZ-YANES AND XU PROJECTION METHOD AND MEAN-VALUED ITERATION IN HILBERT SPACES

ATSUMASA KONDO

Dedicated to the late Professor Kazimierz Goebel

Abstract. In this paper, we prove a Nakajo and Takahashi type strong convergence theorem of for finding a common fixed point of nonlinear mappings. Incorporating the Martinez-Yanes and Xu's projection method, a result proved by Hojo, Kondo, and Takahashi in 2018 is further developed, exploiting a mean-valued iteration. The mappings are of a general type that includes nonexpansive mappings and other classes of well-known mappings. The shrinking projection method initiated in Takahashi, Takeuchi, and Kubota's paper is also studied.

1. Introduction

In this paper, we denote by H a real Hilbert space with an inner product $\langle \cdot,\ \cdot \rangle$. The norm is defined by $\|x\| = \sqrt{\langle x,\ x \rangle}$, where $x \in H$. A set of all fixed points of a mapping $T : C \to H$ is represented as follows:

$$F(T) = \{x \in C : Tx = x\},$$

where C is a nonempty subset of H. For a broader discussion on the fixed point theory, see Goebel and Kirk [5] and Goebel [6]. A *nonexpansive mapping* $T : C \to H$ is characterized by the condition

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C. \tag{1.1}$$

For two commutative nonexpansive mappings $S, T : C \to C$, Atsushiba and Takahashi [3] introduced the following iteration:

$$x_{n+1} = a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \tag{1.2}$$

for all $n \in \mathbb{N}$, where $a_n \in [0, 1]$ is a coefficient of a convex combination and satisfies certain conditions. They proved a weak convergence theorem for finding a common fixed point of S and T in the setting of a Banach space. Such a mean-valued iteration originates from Baillon [4] and Shimizu and Takahashi [29]. For more recent studies, see [1, 7, 9, 10, 15, 18, 20, 21, 23, 24].

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In 2006, basing on the methods of Ishikawa [12] and Nakajo and Takahashi [28], Martinez-Yanes and Xu [26] proved the following strong convergence theorem:

Theorem 1.1 ([26]). *Let C be a nonempty, closed, and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{a_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n \to 1$ and $0 \leq a_n \leq \overline{a} < 1$ for some $\overline{a} \in [0,1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &\in C \text{ is given,} \\ z_n &= \lambda_n x_n + (1-\lambda_n) T x_n, \\ y_n &= a_n x_n + (1-a_n) T z_n, \\ C_n &= \{h \in C : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\ &\quad - (1-a_n)\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right)\}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{1.3}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(T)$, where $\widehat{x} = P_{F(T)}x$.

In Theorem 1.1, $P_{C_n \cap Q_n}$ and $P_{F(T)}$ are the metric projections from H onto $C_n \cap Q_n$ and $F(T)$, respectively. Setting $\lambda_n = 1$ for all $n \in \mathbb{N}$, we have $z_n = x_n$ and then, Nakajo and Takahashi's strong convergence theorem is obtained.

General classes of mappings have been exploited to prove fixed point theorems and convergence theorems for finding fixed points. Among others, a mapping $T : C \to C$ is called a *normally 2-generalized hybrid mapping* [22] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_2 \left\|T^2 x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ &+\beta_2 \left\|T^2 x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0 \end{aligned} \tag{1.4}$$

for all $x, y \in C$, where the parameters satisfy $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$ and $\alpha_2 + \alpha_1 + \alpha_0 > 0$. If $\alpha_1 = 1$, $\beta_0 = -1$, and the other parameters are 0, then the condition (1.4) coincides with that of nonexpansive mappings (1.1). Other than nonexpansive mappings, the class of normally 2-generalized hybrid mappings contains *generalized hybrid mappings*; see Kocourek et al. [14]. *Nonspreading mappings* [16], *hybrid mappings* [32], and *λ-hybrid mappings* [2] are also included in the class of normally 2-generalized hybrid mappings because these classes of mappings are contained in the class of generalized hybrid mappings. Furthermore, *normally generalized hybrid mappings* [35] and 2-*generalized hybrid mappings* [27] are contained in the class of normally 2-generalized hybrid mappings; see Takahashi and Kondo [22]. For examples of these types of nonlinear mappings, see, e.g., [10, 11, 15, 17, 20].

In 2018, Hojo et al. [8] proved the following strong convergence theorem for commutative normally 2-generalized hybrid mappings using the mean-valued iteration (1.2):

Theorem 1.2 ([8]). *Let C be a nonempty, closed, and convex subset of H. Let S and T be normally 2-generalized hybrid mappings from C into itself that satisfy $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$ be a sequence of real numbers such that $0 \leq a_n \leq \overline{a} < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &\in C \text{ is given}, \\ y_n &= a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n, \\ C_n &= \{h \in C : \|y_n - h\| \leq \|x_n - h\|\}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{1.5}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S)\cap F(T)} x$.

They also proved various types of weak and strong convergence theorems, including a strong convergence theorem based on the shrinking projection method by Takahashi et al. [33].

In this paper, we develop the theorems in Hojo et al. [8], incorporating Martinez-Yanes and Xu's method (1.3). A general type of iteration than (1.5) is employed for finding a common fixed point of commutative nonlinear mappings. The mappings are of a general type that includes normally 2-generalized hybrid mappings and are not assumed to be continuous. In Section 2, we introduce background knowledge. In Section 3, we prove a main theorem that generalizes Theorem 1.2. As a recent paper Kondo [20] also used Martinez-Yanes and Xu's projection method with a mean-valued iteration, a comparative consideration is also provided in this section. In Section 4, the shrinking projection method that is initiated in Takahashi, Takeuchi, and Kubota's paper [33] is addressed.

2. Preliminaries

In this section, we present basic information and results. For more details, see Takahashi [30, 31]. Let x, y, z be elements of a real Hilbert space H, let $d \in \mathbb{R}$, and let C be a nonempty, closed, and convex subset of H. It can be ascertained that a set

$$D = \left\{h \in C : \|y - h\|^2 \leq \|x - h\|^2 + \langle z,\ h\rangle + d\right\} \tag{2.1}$$

is closed and convex; see Lemma 1.3 in Martinez-Yanes and Xu [26]. Following convention, we use P_C to denote a *metric projection* from H onto C; that is, $\|x - P_C x\| \leq \|x - h\|$ for any $x \in H$. In other words, the metric projection P_C maps $x \in H$ to the element $P_C x$ of C that is closest to x. Metric projections are nonexpansive. The closest point $\overline{x}$ of C from x is characterized as follows:

Lemma 2.1. *Let C be a nonempty and convex subset of H. Let $x \in H$ and $\overline{x} \in C$. Then, the following three conditions are equivalent:*

(a) $\|x - \overline{x}\| \leq \|x - h\|$ *for all* $h \in C$,
(b) $\langle x - \overline{x},\ \overline{x} - h\rangle \geq 0$ *for all* $h \in C$,
(c) $\|x - \overline{x}\|^2 + \|\overline{x} - h\|^2 \leq \|x - h\|^2$ *for all* $h \in C$.

Weak and strong convergence of a sequence $\{x_n\}$ in H to a point $x\,(\in H)$ are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Strong convergence $x_n \to x$ is characterized by the following condition: for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to x$. A closed and convex set C in H is weakly closed; that is, $\{x_n\} \subset C$ and $x_n \rightharpoonup x$ imply $x \in C$.

A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - q\| \leq \|x - q\|$$

for all $x \in C$ and $q \in F(T)$, where C is a nonempty subset of H. Itoh and Takahashi [13] demonstrated that the set of fixed points of a quasi-nonexpansive mapping is closed and convex. Kondo and Takahashi [22] showed that a normally 2-generalized hybrid mapping (1.4) with a fixed point is quasi-nonexpansive.

Lemma 2.2 ([22]). *Let* $T : C \to C$ *be a normally* 2*-generalized hybrid mapping with* $F(T) \neq \emptyset$*, where* C *is a nonempty subset of* H*. Then,* T *is quasi-nonexpansive.*

Hojo et al. [8] proved the following:

Lemma 2.3 ([8]). *Let* $S, T : C \to C$ *be normally* 2*-generalized hybrid mappings with* $ST = TS$ *and* $F(S) \cap F(T) \neq \emptyset$*, where* C *is a nonempty, closed, and convex subset of* H*. For a bounded sequence* $\{z_n\}$ *in* C*, define*

$$Z_n \equiv \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l z_n\,(\in C) \tag{2.2}$$

for each $n \in \mathbb{N}$*. Suppose that* $Z_{n_j} \rightharpoonup v \in H$*, where* $\{Z_{n_j}\}$ *is a subsequence of* $\{Z_n\}$*. Then,* $v \in F(S) \cap F(T)$*.*

In the next section, we use commutative quasi-nonexpansive mappings S and T that satisfy

$$Z_{n_j} \rightharpoonup v \Longrightarrow v \in F(S) \cap F(T), \tag{2.3}$$

where Z_n is defined by (2.2) and the assumptions in Lemma 2.3 are maintained.

Letting $S = I$ in Lemma 2.3, we have

$$Z_{n_j} \rightharpoonup v \Longrightarrow v \in F(T), \tag{2.4}$$

where $Z_n \equiv \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n\,(\in C)$ and I is the identity mapping. In Kondo [18, 20, 21, 23], the condition (2.4) was employed.

From Lemmas 2.2 and 2.3, two commutative normally 2-generalized hybrid mappings with a common fixed point are quasi-nonexpansive and satisfy the condition (2.3). The mappings addressed in this paper are of this type. As explained in Introduction, nonexpansive mappings, generalized hybrid mappings, normally generalized hybrid mappings, and 2-generalized hybrid mappings are special cases of normally 2-generalized hybrid mappings. Therefore, these classes of nonlinear mappings are targeted in this paper.

In the main theorems of this paper, we assume that commutative quasi-nonexpansive mappings that satisfy (2.3) have a common fixed point. The next theorem presents a set of sufficient conditions for that assumption to be fulfilled:

Theorem 2.4 ([7]; see also [9])**.** *Let C be a nonempty, closed, and convex subset of H and let $S,T : C \to C$ be normally 2-generalized hybrid mappings such that $ST = TS$. Assume that there exists an element $z \in C$ such that $\{S^kT^lz : k,l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $F(S) \cap F(T)$ is nonempty.*

Note that if C is bounded, then so is $\{S^kT^lx : k,l \in \mathbb{N} \cup \{0\}\}$ for all $x \in C$.

3. Nakajo and Takahashi's method

In this section, we prove a strong convergence theorem for finding a common fixed point of commutative quasi-nonexpansive mappings that jointly satisfy the condition (2.3). We use Martinez-Yanes and Xu's projection method, which is based on Nakajo-Takahashi's projection method. The fundamentals of the proof were developed by many researchers; see, e.g., [1, 8, 10, 18, 19, 34]. The following lemma is useful to prove the main theorem.

Lemma 3.1. *Let $S,T : C \to C$ be quasi-nonexpansive mappings such that $F(S) \cap F(T) \neq \emptyset$, where C is a nonempty subset of a real Hilbert space H. Then, the following holds:*

$$\left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^kT^lx - q \right\| \leq \|x - q\|$$

for all $x \in C$ and $q \in F(S) \cap F(T)$, where $n \in \mathbb{N}$.

Proof. Let $x \in C$, $q \in F(S) \cap F(T)$, and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive, it follows that $\left\|S^kT^lx - q\right\| \leq \|x - q\|$ for all $k,l \in \mathbb{N} \cup \{0\}$. Using this, we obtain

$$\begin{aligned}
\left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^kT^lx - q \right\| &= \frac{1}{n^2} \left\| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^kT^lx - n^2q \right\| \\
&= \frac{1}{n^2} \left\| \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left(S^kT^lx - q\right) \right\| \\
&\leq \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left\|S^kT^lx - q\right\| \\
&\leq \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \|x - q\| \\
&= \|x - q\|.
\end{aligned}$$

This ends the proof. ☐

Setting $S = I$ in Lemma 3.1, we have

$$\left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x - q \right\| \leq \|x - q\| \tag{3.1}$$

for all $x \in C$ and $q \in F(T)$, where I is the identity mapping defined on C.

Theorem 3.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $ST = TS$, $F(S) \cap F(T) \neq \emptyset$, and the condition (2.3). Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $\lambda_n + \mu_n + \nu_n + \zeta_n + \eta_n + \theta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{a_n\}$ be a sequence of real numbers such that*

$$0 \leq a_n \leq \overline{a} < 1 \tag{3.2}$$

for all $n \in \mathbb{N}$, where $\overline{a} \in [0, 1)$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned}
x_1 &\in C \text{ is given,} \\
z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n \\
&\quad + \zeta_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + \eta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n + \theta_n \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n, \\
y_n &= a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l z_n, \\
C_n &= \{h \in C : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\
&\quad - (1 - a_n) \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ h \rangle \right) \}, \\
Q_n &= \{h \in C : \langle x - x_n,\ x_n - h \rangle \geq 0\}, \text{ and} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S) \cap F(T)} x$.

Proof. Let us assume that $x_n \in C$ is given for a moment. As S and T are quasi-nonexpansive mappings, it follows from (3.1) and Lemma 3.1 that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n - q \right\| \leq \|x_n - q\|, \quad \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \leq \|x_n - q\|, \text{ and} \tag{3.3}$$
$$\left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n - q \right\| \leq \|x_n - q\|$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. We show that

$$\|z_n - q\| \leq \|x_n - q\| \tag{3.4}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. Indeed, as S and T are quasi-nonexpansive,

$$\begin{aligned}
\|z_n - q\| &= \left\| \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \zeta_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + \eta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right. \\
&\quad \left. + \theta_n \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n - q \right\| \\
&= \left\| \lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q) + \zeta_n \left(\frac{1}{n} \sum_{k=0}^{n-1} S^k x_n - q \right) \right. \\
&\quad \left. + \eta_n \left(\frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right) + \theta_n \left(\frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n - q \right) \right\| \\
&\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| + \zeta_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n - q \right\| \\
&\quad + \eta_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| + \theta_n \left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n - q \right\| \\
&\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| + \zeta_n \|x_n - q\| + \eta_n \|x_n - q\| \\
&\quad + \theta_n \|x_n - q\| \\
&= \|x_n - q\|,
\end{aligned}$$

which indicates that (3.4) holds true.

Define

$$Z_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l z_n.$$

As C is convex, $\{Z_n\}$ is in C. Using this notation, we can write

$$y_n = a_n x_n + (1 - a_n) Z_n \in C.$$

From Lemma 3.1, we have

$$\|Z_n - q\| \leq \|z_n - q\| \tag{3.5}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$.

We verify that the sequence $\{x_n\}$ is properly defined. The set Q_n is closed and convex in C for all $n \in \mathbb{N}$. Also, C_n is closed and convex in C for all $n \in \mathbb{N}$ once x_n, y_n, z_n, and $w_n \in C$ are given. Indeed, it holds that

$$\begin{aligned}
\|y_n - h\|^2 \leq \|x_n - h\|^2 - (1 - a_n) \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ h \rangle \right) \\
\iff \|y_n - h\|^2 \leq \|x_n - h\|^2 \\
+ 2 (1 - a_n) \langle x_n - z_n,\ h \rangle - (1 - a_n) \left(\|x_n\|^2 - \|z_n\|^2 \right).
\end{aligned}$$

Hence, from (2.1), C_n is closed and convex.

Observe that

$$F(S) \cap F(T) \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N}.$$

To check this, we use a mathematical induction. (i) As $Q_1 = C$, $F(S) \cap F(T) \subset Q_1$. Let $q \in F(S) \cap F(T)$. From (3.5), it follows that

$$\begin{aligned} \|y_1 - q\|^2 &= \|a_1 x_1 + (1 - a_1) Z_1 - q\|^2 \\ &= \|a_1 (x_1 - q) + (1 - a_1)(Z_1 - q)\|^2 \\ &\le a_1 \|x_1 - q\|^2 + (1 - a_1) \|Z_1 - q\|^2 \\ &\le a_1 \|x_1 - q\|^2 + (1 - a_1) \|z_1 - q\|^2 \\ &= \|x_1 - q\|^2 + (1 - a_1) \left(\|z_1 - q\|^2 - \|x_1 - q\|^2 \right) \\ &= \|x_1 - q\|^2 - (1 - a_1) \left(\|x_1\|^2 - \|z_1\|^2 - 2 \langle x_1 - z_1,\ q \rangle \right). \end{aligned} \tag{3.6}$$

Therefore, $q \in C_1$, and we can conclude that $F(S) \cap F(T) \subset C_1$. (ii) Assume that

$$F(S) \cap F(T) \subset C_k \cap Q_k,$$

where $k \in \mathbb{N}$. As $F(S) \cap F(T) \neq \emptyset$ is assumed, $C_k \cap Q_k \neq \emptyset$ follows. As $C_k \cap Q_k$ is a nonempty, closed, and convex subset of $C\,(\subset H)$, the metric projection $P_{C_k \cap Q_k}$ from H onto $C_k \cap Q_k$ is defined. Consequently, x_{k+1} is defined as $x_{k+1} = P_{C_k \cap Q_k} x$. Furthermore, z_{k+1}, Z_{k+1}, $y_{k+1}\,(\in C)$, C_{k+1}, and $Q_{k+1}\,(\subset C)$ are also defined properly. We show that

$$F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}.$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. In a same way as (3.6), $q \in C_{k+1}$ can be proved and therefore, we omit it here. As $x_{k+1} = P_{C_k \cap Q_k} x$ and $q \in F(S) \cap F(T) \subset C_k \cap Q_k$, from Lemma 2.1-(b), it holds that

$$\langle x - x_{k+1},\ x_{k+1} - q \rangle \ge 0.$$

This shows that $q \in Q_{k+1}$. Hence, $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$ as claimed. We have demonstrated that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From the hypothesis that $F(S) \cap F(T) \neq \emptyset$, $C_n \cap Q_n$ is nonempty for all $n \in \mathbb{N}$. Therefore, the sequence $\{x_n\}$ is defined inductively.

From the definition of Q_n, it follows that $x_n = P_{Q_n} x$ for all $n \in \mathbb{N}$. Consequently, we have

$$\|x - x_n\| \le \|x - q\| \tag{3.7}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, as $x_n = P_{Q_n} x$ and $q \in F(S) \cap F(T) \subset C_n \cap Q_n \subset Q_n$, the inequality (3.7) holds true. From (3.7), $\{x_n\}$ is bounded. From (3.4), $\{z_n\}$ is also bounded.

Let us show that

$$\|x - x_n\| \le \|x - x_{n+1}\| \tag{3.8}$$

for all $n \in \mathbb{N}$. Indeed, from $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, we obtain (3.8), which means that the sequence $\{\|x - x_n\|\}$ is monotone increasing. As $\{x_n\}$ is bounded, so is $\{\|x - x_n\|\}$. Thus, $\{\|x - x_n\|\}$ is convergent in $\mathbb{R}$.

Next, we prove that

$$x_n - x_{n+1} \to 0. \tag{3.9}$$

As $x_n = P_{Q_n}x$ and $x_{n+1} = P_{C_n \cap Q_n}x \in Q_n$, from Lemma 2.1-(c), it follows that

$$\|x - x_n\|^2 + \|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 .$$

As $\{\|x - x_n\|\}$ is convergent, we have (3.9) as claimed.

It holds that $\{Sx_n\}$ is bounded. Indeed, let $q \in F(S)$. As S is quasi-nonexpansive,

$$\begin{aligned} \|Sx_n\| &\leq \|Sx_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\| . \end{aligned}$$

As $\{x_n\}$ is bounded, so is $\{Sx_n\}$. Similarly, $\{Tx_n\}$ is also bounded.

As $\{x_n\}$ is bounded, we have from (3.3) that

$$\left\{ \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n \right\}, \ \left\{ \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right\}, \text{ and } \left\{ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \right\}$$

are also bounded. Using these boundaries, we can demonstrate that

$$z_n - x_n \to 0. \tag{3.10}$$

In fact, as $\lambda_n \to 1$, it follows that $\mu_n, \nu_n, \zeta_n, \eta_n, \theta_n \to 0$. Using this, we obtain

$$\begin{aligned} & \|z_n - x_n\| \\ = & \left\| \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \zeta_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + \eta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right. \\ & \left. + \theta_n \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n - x_n \right\| \\ \leq & (1 - \lambda_n) \|x_n\| + \mu_n \|Sx_n\| + \nu_n \|Tx_n\| \\ & + \zeta_n \left\| \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n \right\| + \eta_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right\| + \theta_n \left\| \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \right\| \\ \to & 0. \end{aligned}$$

As $\{x_n\}$ and $\{z_n\}$ are bounded, it holds from (3.10) that

$$\|x_n\|^2 - \|z_n\|^2 \to 0. \tag{3.11}$$

We can verify (3.11) as follows:

$$\begin{aligned} \left| \|x_n\|^2 - \|z_n\|^2 \right| &= (\|x_n\| + \|z_n\|) \left| \|x_n\| - \|z_n\| \right| \\ &\leq (\|x_n\| + \|z_n\|) \|x_n - z_n\| \to 0. \end{aligned}$$

Thus, (3.11) holds true as claimed.

We demonstrate that

$$y_n - x_{n+1} \to 0. \tag{3.12}$$

In fact, as $x_{n+1} = P_{C_n \cap Q_n} x \in C_n$, we have the following:

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 \\ - (1 - a_n) \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ x_{n+1} \rangle \right).$$

From (3.9)-(3.11), we obtain (3.12). From (3.9) and (3.12), it holds true that

$$x_n - y_n \to 0. \tag{3.13}$$

Next, observe that

$$x_n - Z_n \to 0. \tag{3.14}$$

It holds that

$$\|x_n - y_n\| = \|x_n - [a_n x_n + (1 - a_n) Z_n]\| = (1 - a_n) \|x_n - Z_n\|.$$

It follows from (3.2) and (3.13) that $x_n - Z_n \to 0$ as claimed.

Our aim is to show that $x_n \to \widehat{x} \left(= P_{F(S) \cap F(T)} x\right)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. We prove that there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x}$. As $\{x_{n_i}\}$ is bounded, there are a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ and an element v of H such that $x_{n_j} \rightharpoonup v$. From (3.14), $Z_{n_j} \rightharpoonup v$. As S and T satisfy the condition (2.3), we have $v \in F(S) \cap F(T)$.

We prove that $\left\{x_{n_j}\right\}$ converges strongly to v. As $v \in F(S) \cap F(T)$, from (3.7), we obtain

$$\begin{aligned} \left\|x_{n_j} - v\right\|^2 &= \left\|x_{n_j} - x\right\|^2 + 2 \left\langle x_{n_j} - x,\ x - v \right\rangle + \|x - v\|^2 \\ &\leq \|x - v\|^2 + 2 \left\langle x_{n_j} - x,\ x - v \right\rangle + \|x - v\|^2 \\ &= 2 \|x - v\|^2 + 2 \left\langle x_{n_j} - x,\ x - v \right\rangle. \end{aligned}$$

From $x_{n_j} \rightharpoonup v$, it holds that

$$\begin{aligned} \left\|x_{n_j} - v\right\|^2 &\leq 2 \|x - v\|^2 + 2 \left\langle x_{n_j} - x,\ x - v \right\rangle \\ &\to 2 \|x - v\|^2 + 2 \langle v - x,\ x - v \rangle = 0. \end{aligned}$$

Therefore, $x_{n_j} \to v$, as claimed.

As a final step, we show that $v \left(= \lim_{j\to\infty} x_{n_j}\right) = \widehat{x} \left(= P_{F(S) \cap F(T)} x\right)$. As $\widehat{x} = P_{F(S) \cap F(T)} x$ and $v \in F(S) \cap F(T)$, it suffices to demonstrate that $\|x - v\| \leq \|x - \widehat{x}\|$. As $\widehat{x} \in F(S) \cap F(T)$, from (3.7), we have

$$\left\|x - x_{n_j}\right\| \leq \|x - \widehat{x}\|$$

for all $j \in \mathbb{N}$. As $x_{n_j} \to v$, it is true that $\|x - v\| \leq \|x - \widehat{x}\|$. Thus, we obtain $v = \widehat{x}$. We have proved that for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x}\,(= v)$. Therefore, $x_n \to \widehat{x}$. This completes the proof. $\square$

We present the following corollary for an illustration because Theorem 3.2 seems to be a bit complicated.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $ST = TS$, $F(S) \cap F(T) \neq \emptyset$, and the condition* (2.3)*. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $\lambda_n + \mu_n + \nu_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{a_n\}$ be a sequence of real numbers such that $0 \leq a_n \leq \overline{a} < 1$ for all $n \in \mathbb{N}$, where $\overline{a} \in [0, 1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$
\begin{aligned}
x_1 &\in C \text{ is given,} \\
z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \\
y_n &= a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l z_n, \\
C_n &= \{h \in C : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\
&\qquad - (1 - a_n) \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ h \rangle \right) \}, \\
Q_n &= \{h \in C : \langle x - x_n,\ x_n - h \rangle \geq 0\}, \text{ and} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S) \cap F(T)} x$.

Some remarks concerning Theorem 3.2 are provided below. First, a condition required on the parameters are only $\lambda_n \to 1$ except for (3.2) as Martinez-Yanes and Xu [26]. Second, set $\lambda_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2 (or Corollary 3.3) and let S and T be normally 2-generalized hybrid mappings. This special case corresponds to Theorem 5.1 in Hojo et al. [18], which is reproduced in this paper as Theorem 1.2. Thus, Theorem 3.2 is a generalization of Theorem 1.2. Third, as nonexpansive mappings, generalized hybrid mappings, normally generalized hybrid mappings, and 2-generalized hybrid mappings are all special cases of normally 2-generalized hybrid mappings, Theorem 3.2 is effective for these classes of mappings.

Fourth, as you can see from the proof of Theorem 3.2, the condition $ST = TS$ is not used explicitly in the proof. The condition $ST = TS$ is only necessary for the condition (2.3) to hold, which is assumed. Therefore, as long as we assumed the condition (2.3), there was no need to actually assume the condition $ST = TS$. However, in view of the importance of the condition $ST = TS$ for this theorem to hold, the author decided to make an explicit assumption.

Fifth, the following theorem, which was proved by Alizadeh and Moradlou [1], is derived from Theorem 3.2:

Corollary 3.4 ([1])**.** *Let C be a nonempty, closed, and convex subset of H. Let T be a 2-generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{a_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $\lambda_n \to 1$ and $0 \leq a_n \leq \overline{a} < 1$ for all $n \in \mathbb{N}$, where $\overline{a} \in [0, 1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$
x_1 \in C \text{ is given,}
$$

$$
\begin{aligned}
z_n &= \lambda_n x_n + (1-\lambda_n) T x_n, \\
y_n &= a_n x_n + (1-a_n) \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\
C_n &= \{h \in C : \|y_n - h\|^2 \le \|x_n - h\|^2 \\
&\quad - (1-a_n) \Big(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ h \rangle \Big) \}, \\
Q_n &= \{h \in C : \langle x - x_n,\ x_n - h \rangle \ge 0\}, \text{ and} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(T)$, where $\widehat{x} = P_{F(T)} x$.

Proof. Set $S = I$ and $\mu_n = \zeta_n = \eta_n = \theta_n = 0$ in Theorem 3.2, where I is the identity mapping defined on C. Then, the desired result follows. □

Note that Alizadeh and Moradlou [1] dealt with an m-generalized hybrid mapping. As a final remark concerning Theorem 3.2, let us make a comparison to a recent result in Kondo [20]:

Theorem 3.5 ([20]). *Let C be a nonempty, closed, and convex subset of H. Let S and T be quasi-nonexpansive and mappings from C into itself such that $F(S) \cap F(T) \ne \emptyset$ with the condition (2.4). Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n + \theta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{\lambda'_n\}$, $\{\mu'_n\}$, $\{\nu'_n\}$, $\{\xi'_n\}$, and $\{\theta'_n\}$ be sequences of real numbers in $[0,1]$ such that $\lambda'_n + \mu'_n + \nu'_n + \xi'_n + \theta'_n = 1$ for all $n \in \mathbb{N}$ and $\lambda'_n \to 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$
\varliminf_{n\to\infty} a_n b_n > 0, \text{ and } \varliminf_{n\to\infty} a_n c_n > 0. \tag{3.15}
$$

Define a sequence $\{x_n\}$ in C as follows:

$$
\begin{aligned}
x_1 &\in C \text{ is given}, \\
z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \\
w_n &= \lambda'_n x_n + \mu'_n S x_n + \nu'_n T x_n + \xi'_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + \theta'_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \\
y_n &= a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n, \\
C_n &= \{h \in C : \|y_n - h\|^2 \le \|x_n - h\|^2 \\
&\quad - b_n \Big(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ h \rangle \Big) \\
&\quad - c_n \Big(\|x_n\|^2 - \|w_n\|^2 - 2 \langle x_n - w_n,\ h \rangle \Big) \},
\end{aligned}
$$

$$\begin{aligned} Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$, *where* $\widehat{x} = P_{F(S)\cap F(T)} x$.

One difference between Theorem 3.2 and 3.5 is in the conditions (3.2) and (3.15) on the coefficients a_n, b_n, c_n of a convex combination. The condition (3.2) allows the case where $a_n = 0$ for all n. Also, for Theorem 3.2, the assumption $ST = TS$ for mappings is required while it is dispensable for Theorem 3.5.

4. Shrinking projection method

In this section, we develop the shrinking projection method introduced by Takahashi, Takeuchi, and Kubota [33], incorporating the Martinez-Yanes and Xu method (1.3) and mean-valued iteration (1.2). The basic skeleton of the proof has been polished by many researchers; see, for instance, [8, 10, 18, 19, 25, 34].

The main theorem in this section can be proved under a more relaxed condition on mappings than in the previous section. Recall the setting of Lemma 2.3: Let C be a nonempty, closed, and convex subset of H, let $S, T : C \to C$ with $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$, and let $\{z_n\}$ be a bounded sequence in C. Define

$$Z_n = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l z_n \, (\in C).$$

Consider the following condition:

$$Z_{n_j} \to v \Longrightarrow v \in F(S) \cap F(T), \tag{4.1}$$

where $\{Z_{n_j}\}$ is a subsequence of $\{Z_n\}$. Mappings S and T with the condition (2.3) satisfy (4.1). Therefore, from Lemma 2.3, commutative two normally 2-generalized hybrid mappings S and T with a common fixed point satisfy this condition (4.1). In the main theorem of this section, we focus on commutative quasi-nonexpansive mappings with the condition (4.1).

Theorem 4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $ST = TS$, $F(S) \cap F(T) \neq \emptyset$, and the condition* (4.1). *Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval* $[0,1]$ *such that $\lambda_n + \mu_n + \nu_n + \zeta_n + \eta_n + \theta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{a_n\}$ be a sequence of real numbers such that $0 \leq a_n \leq \overline{a} < 1$ for all $n \in \mathbb{N}$, where $\overline{a} \in [0,1)$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u \, (\in H)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &\in C \text{ is given,} \\ C_1 &= C, \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n \\ &\quad + \zeta_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + \eta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n + \theta_n \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n, \end{aligned}$$

$$\begin{aligned} y_n &= a_n x_n + (1-a_n)\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l z_n, \\ C_{n+1} &= \{h \in C_n : \|y_n - h\|^2 \le \|x_n - h\|^2 \\ &\quad -(1-a_n)\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right)\},\ \textit{and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{u}$ *of* $F(S)\cap F(T)$, *where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. Define

$$Z_n = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l z_n \in C.$$

Then, we can write $y_n = a_n x_n + (1-a_n) Z_n \in C$. It holds that

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} S^k x_n - q\right\| \le \|x_n - q\|, \quad \left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l x_n - q\right\| \le \|x_n - q\|, \tag{4.2}$$

$$\left\|\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l x_n - q\right\| \le \|x_n - q\|, \quad \|Z_n - q\| \le \|z_n - q\|, \text{ and} \tag{4.3}$$

$$\|z_n - q\| \le \|x_n - q\| \tag{4.4}$$

for all $n \in \mathbb{N}$ and $q \in F(S)\cap F(T)$. The inequalities (4.2) and (4.3) follow from (3.1) and Lemma 3.1. The inequality (4.4) can be proved in the same way as (3.4).

We check that C_n is a closed and convex subset of C and

$$F(S)\cap F(T) \subset C_n$$

for all $n \in \mathbb{N}$ using a mathematical induction. (i) For $n = 1$, the results follow from $C_1 = C$. (ii) Assume that C_k is closed and convex and

$$F(S)\cap F(T) \subset C_k,$$

where $k \in \mathbb{N}$. As $F(S)\cap F(T) \neq \emptyset$ is assumed, the assumption $F(S)\cap F(T) \subset C_k$ implies that $C_k \neq \emptyset$. Hence, the metric projection P_{C_k} is defined. Furthermore, x_k, z_k, Z_k, y_k, and C_{k+1} are also defined properly. It holds that C_{k+1} is closed and convex from the induction assumption that C_k is closed and convex and (2.1). Observe that

$$F(S)\cap F(T) \subset C_{k+1}.$$

Let $q \in F(S)\cap F(T)$. Using (4.3) yields

$$\begin{aligned} \|y_k - q\|^2 &= \|a_k x_k + (1-a_k) Z_k - q\|^2 \\ &= \|a_k (x_k - q) + (1-a_k)(Z_k - q)\|^2 \\ &\le a_k \|x_k - q\|^2 + (1-a_k)\|Z_k - q\|^2 \\ &\le a_k \|x_k - q\|^2 + (1-a_k)\|z_k - q\|^2 \\ &= \|x_k - q\|^2 + (1-a_k)\left(\|z_k - q\|^2 - \|x_k - q\|^2\right) \end{aligned}$$

$$= \|x_k - q\|^2 - (1-a_k)\Big(\|x_k\|^2 - \|z_k\|^2 - 2\langle x_k - z_k,\ q\rangle\Big).$$

From this, we obtain $q \in C_{k+1}$. Thus, it follows that $F(S) \cap F(T) \subset C_{k+1}$ as claimed. We have demonstrated that C_n is a closed and convex subset of C and $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. From the hypothesis $F(S) \cap F(T) \neq \emptyset$, it holds true that $C_n \neq \emptyset$ for all $n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ is defined inductively.

Define $\overline{u}_n = P_{C_n}u$. As $\overline{u}_n \in C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence in C. We prove that

$$\|u - \overline{u}_n\| \leq \|u - q\| \tag{4.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This follows from $\overline{u}_n \equiv P_{C_n}u$ and $q \in F(S) \cap F(T) \subset C_n$. From (4.5), $\{\overline{u}_n\}$ is bounded.

Observe that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\| \tag{4.6}$$

for all $n \in \mathbb{N}$. As $\overline{u}_n = P_{C_n}u$ and $\overline{u}_{n+1} = P_{C_{n+1}}u \in C_{n+1} \subset C_n$, we have the inequality (4.6) as claimed. This implies that $\{\|u - \overline{u}_n\|\}$ is monotone increasing. As $\{\overline{u}_n\}$ is bounded, so is $\{\|u - \overline{u}_n\|\}$. Hence, $\{\|u - \overline{u}_n\|\}$ is convergent in $\mathbb{R}$.

We show that the sequence $\{\overline{u}_n\}$ is convergent in C; in other words, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{4.7}$$

For that aim, it is sufficient to demonstrate that $\{\overline{u}_n\}$ is a Cauchy sequence in C. Let $m, n \in \mathbb{N}$ with $m \geq n$. As $\overline{u}_n = P_{C_n}u$ and $\overline{u}_m = P_{C_m}u \in C_m \subset C_n$, from Lemma 2.1-(c), it follows

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2.$$

As $\{\|u - \overline{u}_n\|\}$ is convergent, we have $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$. This means that $\{\overline{u}_n\}$ is a Cauchy sequence in C. Consequently, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$.

Next, let us prove that $\{x_n\}$ has the same limit point with $\{\overline{u}_n\}$, namely,

$$x_n \to \overline{u}. \tag{4.8}$$

As the metric projection is nonexpansive, we have from the assumption $u_n \to u$ and (4.7) that

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0, \end{aligned}$$

which means that (4.8) holds true as claimed. Thus, $\{x_n\}$ is bounded. From (4.4), $\{z_n\}$ is also bounded. Furthermore, $\{Sx_n\}$ is also bounded. This can be verified as follows: Let $q \in F(S)$. As S is quasi-nonexpansive,

$$\begin{aligned} \|Sx_n\| &\leq \|Sx_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\|, \end{aligned}$$

which indicates that $\{Sx_n\}$ is bounded. Similarly, $\{Tx_n\}$ is also bounded. From (4.2) and (4.3),

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1}S^kx_n\right\},\ \left\{\frac{1}{n}\sum_{l=0}^{n-1}T^lx_n\right\},\ \text{and}\ \left\{\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}S^kT^lx_n\right\}$$

are also bounded.

Using these facts, we prove that

$$z_n - x_n \to 0. \tag{4.9}$$

From the assumption $\lambda_n \to 1$, it holds that $\mu_n, \nu_n, \zeta_n, \eta_n, \theta_n \to 0$. Hence, we can prove (4.9) as follows:

$$\begin{aligned}
& \|z_n - x_n\| \\
= & \left\|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \zeta_n \frac{1}{n}\sum_{k=0}^{n-1}S^kx_n + \eta_n\frac{1}{n}\sum_{l=0}^{n-1}T^lx_n\right. \\
& \left. +\theta_n\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}S^kT^lx_n - x_n\right\| \\
\leq & (1-\lambda_n)\|x_n\| + \mu_n\|Sx_n\| + \nu_n\|Tx_n\| \\
& +\zeta_n\left\|\frac{1}{n}\sum_{k=0}^{n-1}S^kx_n\right\| + \eta_n\left\|\frac{1}{n}\sum_{l=0}^{n-1}T^lx_n\right\| + \theta_n\left\|\frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1}S^kT^lx_n\right\| \\
\to & 0.
\end{aligned}$$

As $\{x_n\}$ and $\{z_n\}$ are bounded, using (4.9), we can verify that

$$\|x_n\|^2 - \|z_n\|^2 \to 0. \tag{4.10}$$

Indeed,

$$\begin{aligned}
\left|\|x_n\|^2 - \|z_n\|^2\right| &= (\|x_n\| + \|z_n\|)\,|\|x_n\| - \|z_n\|| \\
&\leq (\|x_n\| + \|z_n\|)\,\|x_n - z_n\| \to 0.
\end{aligned}$$

We demonstrate that

$$y_n - x_{n+1} \to 0. \tag{4.11}$$

Indeed, as $x_{n+1} = P_{C_{n+1}}u_{n+1} \in C_{n+1}$, we obtain

$$\begin{aligned}
\|y_n - x_{n+1}\|^2 \leq\ & \|x_n - x_{n+1}\|^2 \\
& - (1-a_n)\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ x_{n+1}\rangle\right).
\end{aligned}$$

From (4.8), $x_n - x_{n+1} \to 0$. From (4.9) and (4.10), (4.11) holds true as claimed. As $x_n - x_{n+1} \to 0$ and $y_n - x_{n+1} \to 0$, we obtain

$$x_n - y_n \to 0. \tag{4.12}$$

Next, observe that

$$x_n - Z_n \to 0. \tag{4.13}$$

It holds that

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - [a_n x_n + (1 - a_n) Z_n]\| \\ &= (1 - a_n) \|x_n - Z_n\| .\end{aligned}$$

As $0 \le a_n \le \overline{a} < 1$, we have from (4.12) that $x_n - Z_n \to 0$ as claimed. From (4.8) and (4.13), we have that $Z_n \to \overline{u}$. As S and T jointly satisfy (4.1), we obtain $\overline{u} \in F(S) \cap F(T)$.

From (4.8), it is sufficient to show that

$$\overline{u}\left(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n\right) = \widehat{u}\left(= P_{F(S)\cap F(T)} u\right).$$

As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, it is sufficient to show that

$$\|u - \overline{u}\| \le \|u - \widehat{u}\| .$$

Using (4.5) for $q = \widehat{u} \in F(S) \cap F(T)$, we have $\|u - \overline{u}_n\| \le \|u - \widehat{u}\|$ for all $n \in \mathbb{N}$. From (4.7), $\|u - \overline{u}\| \le \|u - \widehat{u}\|$. This implies that $\overline{u} = \widehat{u}$. From (4.8), we obtain $x_n \to \widehat{u}$. This completes the proof. □

Setting $\lambda_n = 1$ in Theorem 4.1, we obtain the following result, which was proved in Hojo et al. [8]:

Corollary 4.2 ([8]). *Let C be a nonempty, closed, and convex subset of H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $ST = TS$, $F(S) \cap F(T) \neq \emptyset$, and the condition (4.1). Let $\{a_n\}$ be a sequence of real numbers such that $0 \le a_n \le \overline{a} < 1$ for all $n \in \mathbb{N}$, where $\overline{a} \in [0, 1)$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C \text{ is given}, \\ &C_1 = C, \\ &y_n = a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n, \\ C_{n+1} &= \{h \in C_n : \|y_n - h\| \le \|x_n - h\|\}, \text{ and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Notice again remarks presented after Theorem 3.2 in the previous section. A corresponding result in Kondo [20] which closely relates with Theorem 4.1 is Theorem 4.1 in that paper.

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Atsumasa Kondo
Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan
E-mail address: a-kondo@biwako.shiga-u.ac.jp