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ON A FRACTIONAL PARABOLIC PROBLEMS WITH VARIABLE EXPONENT

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Abstract. In this paper, we discuss the existence and uniqueness of weak solution to the following parabolic $p(.)$-Laplacian,

$$\begin{cases} u_t + (-\Delta)^s_{p(.)} u + |u|^{q(.)-2}u &= \lambda \frac{\partial F}{\partial u} \quad \text{in } Q_T := \Omega \times (0,T), \\ u &= 0 \quad \text{in } \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T), \\ u(.,0) &= u_0(.) \quad \text{in } \mathbb{R}^N, \end{cases}$$

where $\Omega \subset \mathbb{R}^N, N > 2$ is a bounded smooth domain, $F \in \mathcal{C}^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and locally Lipschitz uniformly in t while λ is a positive parameter and q is a continuous function on $\overline{\Omega}$.

The functional setting involves Lebesgue and Sobolev spaces with variable exponent. We use the semi-group approach and some a-priori estimates to obtain our results.

1. Introduction

We study the parabolic problem involving fractional $p(.)$-Laplacian,

$$(P_T)\begin{cases} u_t + (-\Delta)^s_{p(.)} u + |u|^{q(.)-2}u &= \lambda \frac{\partial F}{\partial u} \quad \text{in } Q_T := \Omega \times (0,T), \\ u &= 0 \quad \text{in } \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T), \\ u(.,0) &= u_0(.) \quad \text{in } \mathbb{R}^N, \end{cases}$$

where $s \in (0,1)$, p and q are two continuous functions $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow (0,\infty)$ and $q : \overline{\Omega} \longrightarrow (0,\infty)$. We assume that p is symmetric i.e $p(x,y) = p(y,x)$,

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le p(x,y) \le p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < \infty$$

and

$$1 < q^- = \min_{x\in\overline{\Omega}} q(x) \le q(x) \le q^+ = \max_{x\in\overline{\Omega}} q(x) < \infty.$$

Fractional parabolic problems with variable exponent are recent topic in partial differential equations, first introduced by Kaufmann *et al.* in [13]. The authors extended the Sobolev spaces with variable exponent to the fractional ones via compact embedding theorem. For more details about fractional Sobolev spaces with variable exponents, refer to [7].

The operator $(-\Delta)^s_{p(.)} u$ is defined by

$$(-\Delta)^s_{p(.)} u(x) = P.V \int_\Omega \frac{|u(x)-u(y)|^{p(.)-2}(u(x)-u(y))}{|x-y|^{N+s.p(.)}} dy, \; x \in \Omega,$$

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where P.V. is a commonly used abbreviation for the principal value. It is a fractional version of the $p(x)$-Laplacian operator given by $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, associated with the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$.

Problems with variable exponents have gained considerable attention as they arise in various scientific fields, including physics, finance, biology, and geophysics (see [1], [6], and [11] for more details).

Recently, M. Hsini *et al.* proved in [12], the existence and uniqueness of the following stationary problem using Ekeland's variational principle:

$$\begin{cases} (-\Delta)^s_{p(.)}u(x) + |u(x)|^{q(.)-2}u(x) & = \lambda\frac{\partial F}{\partial u}(x,u) \quad \text{in } \Omega, \\ u & = 0 \quad \text{in } \partial\Omega. \end{cases} \tag{1.1}$$

In [4], T. Boudjeriou used sub-differential approach to prove existence of a local solution to the following evolution problem involving fractional $p(x)$-laplacian:

$$\begin{cases} u_t + (-\Delta)^s_{p(.)}u & = |u|^{q(.)-2}u \quad \text{in } \Omega, t>0, \\ u & = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, t>0, \\ u(.,0) & = u_0(.) \quad \text{in } \Omega. \end{cases}$$

In [16] J. Giacomoni *et al.* have studied the following quasilinear parabolic problem with $p(x)$-Laplacian:

$$\begin{cases} u_t - \Delta_{p(x)}u = f(x,u) & \text{in } Q_T = (0,T)\times\Omega, \\ u = 0 & \text{on } \Sigma_T = (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where they have proven the existence and uniqueness of the weak solution, and discussed the global behaviour of solutions. Our aim is to extend the works made in [12, 16, 4]. Indeed, we take a source term $\lambda\frac{\partial F}{\partial u}$, which is not the case in [4]. We also study the parabolic version instead of elliptic one, which has been studied in [12]. Moreover, our operator (fractional $p(x)$-laplacian) is more general that the $p(x)$-Laplacian, which was used in [16]. We study problem (P_T) under the following assumptions:

(H₁) $F : \overline{\Omega}\times\mathbb{R} \longrightarrow \mathbb{R}$ is homogeneous of degree r, that is,

$$F(x,tu) = t^{r(x)}F(x,u) \text{ for all } t>0, x\in\overline{\Omega}, u\in\mathbb{R}.$$

(H₂) $\left|\frac{\partial F}{\partial t}(x,t)\right| \le CV(x)|t|^{r(x)-2}t$, for all $(x,t)\in\overline{\Omega}\times\mathbb{R}$, where C is a positive constant, $V\in L^{l(x)}(\Omega)$, $l,r\in C(\overline{\Omega})$ are such that for all $x\in\overline{\Omega}$, we have

$$1 < r(x) < p(x,x) < \frac{N}{s} < l(x) \text{ and } p(x,x) \le q(x) < p^*(x) := \frac{Np(x,x)}{N - s.p(x,x)}.$$

(H₃) There exists an $\Omega_0 \subset\subset \Omega$ with $|\Omega_0| > 0$ such that $F(x,t) > 0$ for all $(x,t)\in\Omega_0\times\mathbb{R}^*$.

(H₄) There exists $\alpha\in\mathbb{R}$ such that $x\mapsto\frac{\partial F}{\partial t}(x,\alpha)\in L^{q(.)}(\Omega)$ $(1<q(.)<+\infty)$.

Note that assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$, and $(\mathbf{H_3})$ have been used in reference [12] to establish the existence of a weak solution to problem (1.1), which is the stationary version of our problem (P_T). We also need assumption $(\mathbf{H_4})$, inspired by assumption (f_2) from reference [16], for the proof of existence of solution to the approximate problem. To prove the existence of weak solutions, we follow a semi-group approach, involving a semi-discretization in time method, that provides the existence of mild solutions. To establish our results, we use some former contributions about the validity of a strong comparison principle (see [16]) and the regularity of solutions.

The rest of the paper is organized as follows: In section 2, we recall some basic proprieties of Lebesgue and Sobolev spaces with variable exponent and in section 3, we state and prove our main results on the existence, uniqueness, regularity of solutions to (PT) and on the global behaviour of solutions.

2. Preliminary

To begin, we define the norm space and its open convex subset as follows:

$$C_{d(\Omega)} = \left\{u \in C_0(\overline{\Omega}) : \exists c \geq 0 \text{ such that } |u(x)| \leq cd(x), \forall x \in \Omega\right\}.$$

where $d(x) := dist(x, \partial\Omega)$, and

$$C^+_{d^s(\Omega)} = \left\{u \in C_{d(\Omega)} : \inf_{x\in\Omega} \frac{u(x)}{d^s(x)} > 0\right\}.$$

We consider the function space

$$L^{q(.)}(\Omega) = \left\{u \text{ (measurable) } : \Omega \longrightarrow \mathbb{R} : \exists\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{q(.)} dx < \infty\right\}.$$

$L^{q(.)}(\Omega)$ is separable, uniformly convex Banach space with variable exponent endowed with the norm:

$$|u|_{L^{q(.)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{q(.)} dx < 1\right\}.$$

$\left(L^{q(.)}(\Omega), |\cdot|_{L^{q(.)}}\right)$ is generalized Lebesgue space.

Hölder type inequality : if $q(.), q'(.) \in (1, +\infty)$ are such that $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$ and if $u \in L^{q(.)}(\Omega)$ and $v \in L^{q'(.)}(\Omega)$, then

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \left(\frac{1}{q-} + \frac{1}{q'-}\right)|u|_{q(.)}|v|_{q'(.)}.$$

Lemma 2.1 (see [12]). *If (u_n), $u \in L^{q(.)}(\Omega)$ and $q^+ < \infty$, then we have the following relations:*

(i) $$|u|_{q(.)} > 1 \Rightarrow |u|^{q^-}_{q(.)} \leq \int_\Omega |u|^{q(.)}dx \leq |u|^{q^+}_{q(.)}.$$

(ii)] $$|u|_{q(.)} < 1 \Rightarrow |u|^{q^+}_{q(.)} \leq \int_\Omega |u|^{q(.)}dx \leq |u|^{q^-}_{q(.)}.$$

(iii) $$|u_n - u|_{q(.)} \longrightarrow 0 \textit{ if and only if } \int_\Omega |u_n - u|^{q(.)}dx \longrightarrow 0.$$

Proposition 2.2 (see [9]). *Let γ and q be measurable functions such that $\gamma \in L^{\infty}(\mathbb{R}^N)$ and $1 \leq \gamma(.)q(.) \leq \infty$ for any $x \in \mathbb{R}^N$. Let $u \in L^{q(.)}(\mathbb{R}^N), u \neq 0$. Then*

$$\min\left(|u|^{\gamma^-}_{\gamma(.)q(.)}, |u|^{\gamma^+}_{\gamma(.)q(.)}\right) \leq ||u|^{\gamma(.)}|_{q(.)} \leq \max\left(|u|^{\gamma^-}_{\gamma(.)q(.)}, |u|^{\gamma^+}_{\gamma(.)q(.)}\right).$$

If k is a positive integer, we define the variable exponent Sobolev space as follows:

$$W^{k,q(.)}(\Omega) = \left\{u \in L^{q(.)}(\Omega) : D^{\alpha}u \in L^{q(.)}(\Omega), \text{ for all } |\alpha| \leq k\right\},$$

where $\alpha = (\alpha_1, ..., \alpha_N)$ is a multi-index, $D^{\alpha}u = \dfrac{\partial^{|\alpha|}u}{\partial^{|\alpha_1|}_{x_1}...\partial^{|\alpha_N|}_{x_N}}$, endowed with the norm

$$||u||_{k,q(.)} = \sum_{\|\alpha\| \leq k} |D^{\alpha}u|_{q(.)}.$$

We denote by $W^{s,q(.)}_0(\Omega)$ the closure of $C^{\infty}_0(\Omega)$ in $W^{s,q(.)}(\Omega)$.

As in [12], for $0 < s < 1$, we consider the variable exponent Sobolev fractional space as follows:

$$\begin{aligned} W &= W^{s,q(.),p(.)}(\Omega) \\ &= \left\{u : \Omega \longrightarrow \mathbb{R} : f \in L^{q(.)}(\Omega) : \int_{\Omega}\int_{\Omega} \frac{|u(x) - u(y)|^{p(.)}}{\lambda^{p(.)}|x - y|^{n+sp(.)}} dxdy < \infty, \forall \lambda > 0\right\}. \end{aligned}$$

Let

$$[u]_{s,p(.)} = \inf\left\{\lambda > 0, \int_{\Omega}\int_{\Omega} \frac{|u(x) - u(y)|^{p(.)}}{\lambda^{p(.)}|x - y|^{n+sp(.)}} dxdy < 1\right\}$$

be the variable exponent Gagliardo seminorm. W is a Banach space with the norm

$$\|u\|_W = [u]_{s,p(.)} + |u|_{q(.)}.$$

Lemma 2.3 (see Lemma 1.2 in [12]).

(i) *If $1 \leq [u]_{s,p(.)} < \infty$, then*

$$([u]_{s,p(.)})^{p-} \leq \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(.)}}{|x - y|^{n+sp(.)}} dxdy \leq ([u]_{s,p(.)})^{p+}. \tag{2.1}$$

(ii) *If $[u]_{s,p(.)} \leq 1$, then*

$$([u]_{s,p(.)})^{p+} \leq \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(.)}}{|x - y|^{n+sp(.)}} dxdy \leq ([u]_{s,p(.)})^{p-}. \tag{2.2}$$

We denote by $W_0 = W^{s,q(.),p(.)}_0(\Omega)$ the closure of $C^{\infty}_0(\Omega)$ in W, then W_0 is a Banach space with the norm $\|u\|_{W_0} = [u]_{s,p(.)}$.

Lemma 2.4 (see Lemma 1.3 of [12]). *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain and $s \in (0, 1)$. Let q, p be continuous variable exponents with $s.p(x, y) < N$ for $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ and $q(x) \geq p(x, x)$ for $x \in \overline{\Omega}$. Assume that $\gamma : \overline{\Omega} \longrightarrow (1, \infty)$ is a continuous function such that*

$$p^*(x) = \frac{Np(x, x)}{N - s.p(x, x)} > \gamma(x) \geq \gamma^- = \inf_{x\in\Omega} \gamma(x), \text{ for } x \in \overline{\Omega}.$$

Then, there exists a constant C=C(N,s,p,q,Ω) such that for every $u \in W$, it holds that

$$|u|_{\gamma(.)} \leq C\|u\|_W.$$

That is, the space $W(\Omega)$ is continuously embedded in $L^{\gamma(.)}$. Moreover, this embedding is compact. In addition, if $u \in W_0$, the following inequality holds

$$|u|_{\gamma(.)} \leq C\|u\|_{W_0}.$$

Proposition 2.5. *For any $u, v \in W$ we have:*

$$\langle(-\Delta)^s_{p(.)}u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(.)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+s.p(.)}}dxdy.$$

Proposition 2.6 (see Proposition 3.7 of [12]). *If w_n converges weakly to w in W_0, then*

(i) $$\lim_{n\longrightarrow\infty}\int_\Omega \frac{\partial F}{\partial u}(x, w_n)(w_n - w)dx = 0.$$

(ii) $$\lim_{n\longrightarrow\infty}\int_\Omega |w_n|^{q(.)-2}w_n(w_n - w)dx = 0.$$

Lemma 2.7 (see[3]). *For all u, $v \in W_0$, we define $I : W_0 \longrightarrow W_0^*$ such that*

$$\langle I(u), v\rangle = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(.)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(.)}}dxdy. \tag{2.3}$$

Then

(i) *I is a bounded and strictly monotone operator.*
(ii) *I satisfies (S_+) condition, that is, if $u_n \rightharpoonup u$ in W_0 and $\limsup_{n\longrightarrow 0} I(u_n)(u_n - u) \leq 0$, then $u_n \longrightarrow u$ in W_0.*
(iii) *I is a homeomorphism.*

Lemma 2.8. *Let $t^+ = \max(t, 0)$. If $u \in W_0$, then*

$$|u(x)-u(y)|^{p-2}(u^+(x)-u^+(y))(u(x)-u(y)) \geq |u^+(x)-u^+(y)|.$$

Proposition 2.9. *Let $\phi : X \to \mathbb{R}$ be a continuous differentiable function and convex, then $\phi' : X \to X'$ is monotone.*

Lemma 2.10 (see [10]). *for all $u, v \in \mathbb{R}^N$,*

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle \geq \begin{cases} c|u-v|^p & if \quad p \geq 2, \\ c\frac{|u-v|^2}{(|u|+|v|)^{2-p}} & if \quad p \leq 2, \end{cases} \tag{2.4}$$

where c is a positive constant.

To end this section, we define the space $V(Q_T)$ by

$$V(Q_T) = \{u; u_t \in L^2(Q_T), u \in L^\infty(0, T, W_0^{s,q(.),p(.)}(\Omega))\}.$$

3. Main results

Let us now present the main results of this paper.

Theorem 3.1. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *hold, and* $u_0 \in C^+_{d^s(\Omega)}$. *Then, problem* (P_T) *admits a unique weak solution* u, *in the sense that* $u \in V(Q_T)$ *such that for every* $v \in C_0^\infty(Q_T)$,

$$\int_{Q_T} u_t v dx + \int_0^T \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(.)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(.)}} dxdydt$$

$$= \lambda \int_{Q_T} \frac{\partial F}{\partial u}(x,u)v(x)dx. \tag{3.1}$$

Proof. We make the proof of Theorem 3.1 in two subsections. Firstly, we deal with the existence of weak solutions to the auxilliary problem (S_T) and secondly, we deduce the existence result for problem (P_T).

3.1. Existence results for the auxilliary problem (S_T) corresponding to (P_T).

We consider the following evolution problem (S_T).

$$(S_T)\begin{cases} u_t + (-\Delta)^s_{p(.)}u &= g(x,t) \quad \text{in } Q_T := \Omega\times(0,T), \\ u &= 0 \quad \text{in } \left(\mathbb{R}^N\setminus\Omega\right)\times(0,T), \\ u(.,0) &= u_0(.) \quad \text{in } \mathbb{R}^N, \end{cases}$$

where $g \in L^{q(.)}(Q_T)$. Considering the initial data $u_0 \in W_0 \cap L^\infty(\Omega)$, the weak solution to problem (S_T) is defined as follows:

Definition 3.2. A function $u \in V(Q_T)$ is said to be solution of problem (S_T) if for every $v \in C_0^\infty(Q_T)$, we have:

$$\int_{Q_T} u_t v dx + \int_0^T \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(.)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(.)}} dxdydt$$

$$= \int_{Q_T} g(x,t)dxdt$$

and $u(x,0) = u_0(x)$ for a.e. $x \in \Omega$.

We give below our result of existence and uniqueness of weak solution of the problem (S_T).

Theorem 3.3. *Let* $T > 0$, $g \in L^{q(.)}(Q_T)$ $(1 < q(.) < +\infty)$ *and* $u_0 \in W_0 \cap L^\infty(\Omega)$. *Then, there exists a unique solution* u *to the problem* (S_T). *Moreover,* $u \in C([0,T], W_0)$.

Proof. To prove Theorem 3.3, we use the method of semi-groups. We begin by dealing with the following elliptic problem corresponding to (S_T):

$$(S)\begin{cases} u + \mu(-\Delta)^s_{p(.)}u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

where $\mu > 0$ and g$\in L^{q(\cdot)}(\Omega)$. The following proposition provides a result on the existence and uniqueness of a weak solution to the elliptic problem (S). It allows for a generalization of Lemma 4.7 of [10] to the case of a variable exponent.

Proposition 3.4. *For any $\mu > 0$, problem (S) admits a unique weak solution u in the sense that $u \in W_0$ satisfies*

$$\int_\Omega u(x)v(x)dx+\mu\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(\cdot)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(\cdot)}}dxdy$$

$$= \int_\Omega g(x)v(x)dx, \tag{3.2}$$

for all $v \in C_0^\infty(\Omega)$. Moreover, $u \in C_0(\overline{\Omega})$.

Proof. Consider the energy functional $J_\mu : W_0 \longrightarrow \mathbb{R}$ corresponding to (S), given by

$$J_\mu(u) = \frac{1}{2}\int_\Omega (u(x))^2dx + \mu\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(\cdot)}}{p(.)|x-y|^{N+sp(\cdot)}}dxdy - \int_\Omega gudx. \tag{3.3}$$

We claim that J_μ is coercive in W_0. Indeed, based on Sobolev embedding theorem, we have.

$$\begin{aligned} J_\mu(u) &= \frac{1}{2}\int_\Omega (u(x))^2dx+\mu\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(\cdot)}}{p(.)|x-y|^{N+sp(\cdot)}}dxdy - \int_\Omega gudx \\ &\geq \frac{\mu}{p^+}\|u\|_{W_0}^{p^-} - C\|u\|_{W_0}(\text{for}\|u\|_{W_0} \geq 1). \end{aligned}$$

Hence, we conclude that J_μ is coercive. Additionally, J_μ is bounded below and strictly convex in W_0 (this is because the function $\xi \mapsto \frac{1}{p(x)}|\xi|^{p(x)}$ is C^1 and strictly convex). Therefore, J_μ possesses a global minimizer $u \in W_0$, which is the unique solution of problem (S). According to Theorem 4.4 of [10], we deduce that $u \in C_0(\overline{\Omega})$. ☐

Now, we prove that problem (S_T) admits a unique mild solution, which is also a weak solution. Let $L \in \mathbb{N}^*$, $T \geq 0$ and $\Delta t = \frac{T}{L}$. As in [10], we define $t_n = n\Delta t$, $u^n = u(t_n, .)$. We proceed in four steps to obtain the desired results.
Step 1 Approximation of g.

Let define g_{Δ_t} as

$$g_{\Delta t}(x,t) = g^n(x) := \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} g(x,z)dz.$$

Using the embedding $L^{p^+}(Q_T) \hookrightarrow L^{p(\cdot)}(Q_T)$, we deduce that

$$|g_{\Delta t}|_{L^{q(\cdot)}(Q_T)} \leq C\|g_{\Delta t}\|_{L^{q^+}(Q_T)}.$$

Now, by Jensen's inequality:

$$\|g_{\Delta t}\|_{L^{q^+}(Q_T)}^{q^+} = \Delta t\sum_{n=1}^{L}\|g^n\|_{L^{q^+}(\Omega)}^{q^+}$$

$$
\begin{aligned}
&= \Delta t \sum_{n=1}^{L} \| \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} g(x,z)dz \|_{L^{q^+}(\Omega)}^{q^+} \\
&= \sum_{n=1}^{L} \int_{t_{n-1}}^{t_n} \|g(x,z)\|_{L^{q^+}(\Omega)}^{q^+} dz \\
&\leq \|g\|_{L^{q^+}(Q_T)}^{q^+}.
\end{aligned}
$$

Therefore

$$|g_{\Delta t}|_{L^{q(\cdot)}(Q_T)} \leq C\|g\|_{L^{q^+}(Q_T)}^{q^+}.$$

Thus, $g_{\Delta t} \in L^{q(\cdot)}(Q_T)$. We also note that $g_{\Delta t}$ is bounded. Since $\big(L^{q(\cdot)}(\Omega), |\cdot|_{L^{q(\cdot)}}\big)$ is reflexive, we can extract a subsequence denoted again as $g_{\Delta t}$, such that

$$g_{\Delta t} \longrightarrow g \quad \text{in} \quad L^{q(\cdot)}(Q_T). \tag{3.4}$$

Step 2 Time-discretization of (S_T).

For $1 \leq n \leq L$, we consider the following iterative scheme $u^0 = u_0$ and for $n \geq 1$,

$$u_n \text{ is solution of } \begin{cases} \frac{u^n - u^{n-1}}{\Delta t} + (-\Delta)^s_{p(.)} u^n &= g^n \quad \text{in } \Omega, \\ u^n &= 0 \quad \text{in } R^N \backslash \Omega. \end{cases} \tag{3.5}$$

Note that the sequence $\big(u^n\big)_{1 \leq n \leq L}$ is well-defined. Indeed, existence and uniqueness of $u^1 \in W_0 \cap L^\infty(\Omega)$ follows from Proposition 3.4 with $g = g^1 \Delta t + u^0 \in L^{q(\cdot)}(\Omega)$. Hence by induction we obtain in the same way the existence of (u^n), for any $n = 2, \cdots, L$.

Now, we define $u_{\Delta t}$ and $\tilde{u}_{\Delta t}$ for $n = 1, \cdots, L$ and $t \in [t_{n-1}, t_n]$ as follows:

$$u_{\Delta t}(t) = u^n \quad \text{and} \quad \tilde{u}_{\Delta t}(t) = \frac{t - t_{n-1}}{\Delta t}(u^n - u^{n-1}) + u^{n-1}.$$

So, we obtain

$$\frac{\partial \tilde{u}_{\Delta t}}{\partial t} + (-\Delta)^s_{p(.)} u_{\Delta t} = g_{\Delta_t} \text{ in } Q_T. \tag{3.6}$$

Step 3 A priori estimates for $u_{\Delta t}$ and $\tilde{u}_{\Delta t}$.

Multiplying (3.5) by $u^n - u^{n-1}$ and umming up form $n = 1$ to L, we get

$$
\begin{aligned}
\sum_{1}^{L} \Delta t \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta t} \right)^2 dx + \sum_{1}^{L} \langle (-\Delta)^s_{p(.)} u^n, u^n - u^{n-1} \rangle \\
= \sum_{1}^{L} \int_\Omega g^n (u^n - u^{n-1}) dx.
\end{aligned}
$$

By Young inequality we obtain:

$$\sum_{1}^{L} \Delta t \int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta t} \right)^2 dx \quad + \quad \sum_{1}^{L} \langle (-\Delta)^s_{p(.)} u^n, u^n - u^{n-1} \rangle$$

$$\leq \frac{\Delta t}{2}\sum_{1}^{L}\left[\int_{\Omega}(g^n)^2+\left(\frac{u^n-u^{n-1}}{\Delta t}\right)^2\right]dx,$$

which gives

$$\frac{\Delta t}{2}\sum_{1}^{L}\int_{\Omega}\left(\frac{u^n-u^{n-1}}{\Delta t}\right)^2dx+\sum_{1}^{L}\langle(-\Delta)^s_{p(.)}u^n,u^n-u^{n-1}\rangle \leq \tfrac{\Delta t}{2}\textstyle\sum_{1}^{L}\int_{\Omega}(g^n)^2.$$
$$\leq C\|g\|^2_{L^2(Q_T)}.$$

Using the convexity properties of $\xi\longmapsto\frac{1}{p}|\xi|^p$, Lemma 2.1 and Lemma 2.3 we obtain:

$$\begin{aligned}\frac{1}{p^+}\left(\|u^n\|^{p^-}_{W_0}-\|u^{n-1}\|^{p^+}_{W_0}\right)&\leq\int_{\Omega\times\Omega}\frac{|u^n(x)-u^n(y)|^{p(.)}}{p(.)|x-y|^{N+sp(.)}}dxdy\\&\quad-\int_{\Omega\times\Omega}\frac{|u^{n-1}(x)-u^{n-1}(y)|^{p(.)}}{p(.)|x-y|^{N+sp(.)}}dxdy\\&\leq\int_{\Omega\times\Omega}\frac{|u^n(x)-u^n(y)|^{p(.)-2}(u^n(x)-u^n(y))}{|x-y|^{N+sp(.)}}\\&\qquad\left((u^n-u^{n-1})(x)-(u^n-u^{n-1})(y)\right)dxdy.\end{aligned}$$

Now we have

$$\frac{\Delta t}{2}\sum_{1}^{L}\int_{\Omega}\left(\frac{\partial\tilde{u}_{\Delta t}}{\partial t}\right)^2dx+\sum_{1}^{L}\frac{1}{p^+}\left(\|u^n\|^{p^-}_{W_0}-\|u^{n-1}\|^{p^-}_{W_0}\right)\leq C\|g\|^2_{L^2(Q_T)}. \tag{3.7}$$

Hence,

(3.8) $\left(\frac{\partial\tilde{u}_{\Delta t}}{\partial t}\right)_{\Delta t}$ is bounded in $L^2(Q_T)$ uniformly in Δt,

(3.9) $(u_{\Delta t})$ and $(\tilde{u}_{\Delta t})$ are bounded in $L^\infty(0,T,W_0)\cap L^\infty(Q_T)$ uniformly in Δt.

Furthermore, using (3.8) we deduce that

$$\sup_{[0,T]}\|u_{\Delta t}-\tilde{u}_{\Delta t}\|_{L^2(\Omega)}\leq\max_{n=1,\cdots,L}\|u^n-u^{n-1}\|_{L^2(\Omega)}\leq C(\Delta t)^{1/2}. \tag{3.10}$$

Therefore, for $\Delta t\to 0^+$ there exists u and v in $L^\infty(0,T,W_0)\cap L^\infty(Q_T)$ such that (up to a subsequence)

$$\tilde{u}_{\Delta t}\rightharpoonup^* u \text{ in } L^\infty(0,T,W_0),\qquad u_{\Delta t}\rightharpoonup^* v \text{ in } L^\infty(0,T,W_0), \tag{3.11}$$

$$\tfrac{\partial\tilde{u}\Delta t}{\partial t}\rightharpoonup\tfrac{\partial u}{\partial t} \text{ in } L^2(QT). \tag{3.12}$$

Now, we use (3.10) to deduce that $u\equiv v$.

Step 4 We pass to the limit to prove that u satisfies (S_T).

In the following, $o_{\Delta t}$ is a function such that $o_{\Delta t}\longrightarrow 0$ as $\Delta t\longrightarrow 0$. Using the aforementioned uniform boundedness results and since the embedding $W_0\hookrightarrow L^2(\Omega)$ is compact, as mentioned in [10], we can apply the Aubin-Simon lemma to deduce that $\tilde{u}_{\Delta t}$ is compact in $C\left([0,T],L^2(\Omega)\right)$. Using interpolation and Ascoli-Arzela theorem, we obtain:

$$\tilde{u}_{\Delta t}\longrightarrow u\in C([0,T],L^{q(.)}(\Omega)), \tag{3.13}$$

and hence, from (3.11) we deduce that

$$u_{\Delta t} \longrightarrow u \in L^{\infty}([0,T], L^{q^+}(\Omega)). \tag{3.14}$$

From (3.4) and (3.14) we get

$$\int_{Q_T} g_{\Delta t}(u_{\Delta t} - u) = o_{\Delta t}. \tag{3.15}$$

Multiplying (3.6) by $(u_{\Delta t} - u)$ and using our above convergence results, we get:

$$\int_0^T \int_\Omega \left(\frac{\partial \tilde{u}_{\Delta t}}{\partial t} - \frac{\partial u}{\partial t} \right) (\tilde{u}_{\Delta t} - u) dx dt + \int_0^T \langle (-\Delta)^s_{p(.)} u_{\Delta t} - (-\Delta)^s_{p(.)} u, u_{\Delta t} - u \rangle dt = o_{\Delta t}, \tag{3.16}$$

which gives

$$\frac{1}{2} \int_\Omega \left[|\tilde{u}_{\Delta t} - u|^2 \right]_0^T dx + \int_0^T \langle (-\Delta)^s_{p(.)} u_{\Delta t} - (-\Delta)^s_{p(.)} u, u_{\Delta t} - u \rangle dt = o_{\Delta t}. \tag{3.17}$$

Therefore, by (3.13) we have

$$\int_0^T \langle (-\Delta)^s_{p(.)} u_{\Delta t} - (-\Delta)^s_{p(.)} u, u_{\Delta t} - u \rangle dt = o_{\Delta t}. \tag{3.18}$$

In particular,

$$\int_0^T \langle (-\Delta)^s_{p^+} u_{\Delta t} - (-\Delta)^s_{p^+} u, u_{\Delta t} - u \rangle dt = o_{\Delta t} \tag{3.19}$$

and

$$\int_0^T \langle (-\Delta)^s_{p^-} u_{\Delta t} - (-\Delta)^s_{p^-} u, u_{\Delta t} - u \rangle dt = o_{\Delta t}. \tag{3.20}$$

It follows that

$$\int_0^T \langle (-\Delta)^s_{p^+} u_{\Delta t} - (-\Delta)^s_{p^+} u, u_{\Delta t} - u \rangle dt \longrightarrow 0 \text{ as } o_{\Delta t} \longrightarrow 0 \tag{3.21}$$

and

$$\int_0^T \langle (-\Delta)^s_{p^-} u_{\Delta t} - (-\Delta)^s_{p^-} u, u_{\Delta t} - u \rangle dt \longrightarrow 0 \text{ as } o_{\Delta t} \longrightarrow 0. \tag{3.22}$$

Using inequality (2.4) and Lemma 2.3, we have:

Firstly, if $2 \le p(.) < \infty$, for all $x, y \in \Omega$ we deduce that

$$||u_{\Delta t} - u||^{P^+}_{W_0} = o_{\Delta t} \text{ and } ||u_{\Delta t} - u||^{P^-}_{W_0} = o_{\Delta t}.$$

As

$$\int_0^T \int_{\Omega \times \Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p(.)}}{|x-y|^{N+sp(.)}} \le \max \left(||u_{\Delta t} - u||^{P^+}_{W_0}, ||u_{\Delta t} - u||^{P^-}_{W_0} \right),$$

then, according to Lemma 2.3, we deduce that

$$\int_0^T \int_{\Omega \times \Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p(.)}}{|x-y|^{N+sp(.)}} = o_{\Delta t}. \tag{3.23}$$

Secondly, if $1 \leq p(.) \leq 2$, we deduce:
► For $p = p^+$,

$$\begin{aligned} 0 \leq & \int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p^+}}{|x-y|^{N+sp^+}} \\ \leq & \left(\int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^2}{(|u_{\Delta t}(x) - u_{\Delta t}(y)|^{p^+} + |u(x)-u(y)|^{p^+})^{\frac{2-p^+}{p^+}} |x-y|^{N+sp^+}} \right)^{\frac{p^+}{2}} \\ & \times \left(||u_{\Delta t}||_{W_0}^{p^+} + (||u||_{W_0}^{p^+} \right) = o_{\Delta t}. \end{aligned}$$

It follows that

$$\int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p^+}}{|x-y|^{N+sp^+}} = o_{\Delta t}. \tag{3.24}$$

► For $p = p^-$, we also have

$$\begin{aligned} 0 \leq & \int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p^-}}{|x-y|^{N+sp^-}} \\ \leq & \left(\int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^2}{(|u_{\Delta t}(x) - u_{\Delta t}(y)|^{p^-} + |u(x)-u(y)|^{p^-})^{\frac{2-p^-}{p^-}} |x-y|^{N+sp^-}} \right)^{\frac{p^-}{2}} \\ & \times \left(||u_{\Delta t}||_{W_0}^{p^-} + (||u||_{W_0}^{p^-} \right) = o_{\Delta t}. \end{aligned}$$

Then,

$$\int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p^-}}{|x-y|^{N+sp^-}} = o_{\Delta t}. \tag{3.25}$$

Hence, using (3.24) and (3.25) we deduce:

$$\int_0^T \int_{\Omega\times\Omega} \frac{|u_{\Delta t}(x) - u_{\Delta t}(y) - u(x) + u(y)|^{p(.)}}{|x-y|^{N+sp(.)}} dxdydt \longrightarrow 0.$$

Consequently, $u_{\Delta t}$ converges to u in $L^{p(.)}(0,T,W_0)$, and $(-\Delta)^s_{p(.)} u_{\Delta t} \longrightarrow (-\Delta)^s_{p(.)} u$. Hence, by passing to the limit in equation (3.6), we conclude that u is a weak solution of problem (S_T).
Now, we proceed to prove the uniqueness of the solution. Let us assume that there exists another solution of (S_T) denoted by v. By selecting an arbitrary $t_0 \in [0,T]$ and using the test function $u - v$, we obtain:

$$\begin{aligned} \int_0^{t_0} \int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)dxdt \quad + & \quad \int_0^{t_0} \langle (-\Delta)^s_{p(.)} u - (-\Delta)^s_{p(.)} v, u - v \rangle \\ + & \quad \int_0^{t_0} \int_\Omega (|u|^{q(.)-2}u - |v|^{q(.)-2}v)(u-v) = 0. \end{aligned}$$

Since $(-\Delta)^s_{p(.)}$ are monotone (according to lemma 2.7) this together with $u(0) = v(0)$, using Proposition 2.9 we get following inequalities:

$$\int_0^{t_0}\int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)dxdt \le 0.$$

In addition,

$$\begin{aligned}\int_0^{t_0}\int_\Omega \frac{\partial(u-v)}{\partial t}(u-v)dxdt &= \int_0^{t_0}\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}(u-v)^2dxdt \\ &= \frac{1}{2}\int_\Omega (u(.,t_0)-v(.,t_0))^2dx \le 0. \qquad (3.26)\end{aligned}$$

It follows that $u \equiv v$.

Using compact embedding $W_0 \hookrightarrow L^{q(\cdot)}$ and the convergence (3.13), we obtain that the solution to problem (S_T) is in $C([0,T], L^{q(\cdot)})$. □

3.2. **Existence of solution for** P_T**.** We proceed as in the proof of Theorem 3.3 splitting the proof in several steps.

Step 1 Semi-discretization in time of (P_T).

Let introduce the following iterative scheme (u^n) defined as

$$\begin{cases} u^0 = u_0 & \\ u^n + \Delta t((-\Delta)^s_{p(.)}u^n + |u^{n-1}|^{q(\cdot)-2}u^{n-1}) = u^{n-1} + \lambda\Delta t\frac{\partial F}{\partial u}(x,u^{n-1}) & \text{in } \Omega \\ u^n = 0 & \text{in } R^N\backslash\Omega. \end{cases}$$

Assume that **(H4)** holds, then $\frac{\partial F}{\partial u}(.,u^0) \in L^{q(\cdot)}(\Omega)$ and since $|u^0|^{q(\cdot)-2}u^0 \in L^{q(\cdot)}(\Omega)$, it follows that $\frac{\partial F}{\partial u}(.,u^0) - |u^0|^{q(\cdot)-2}u^0 \in L^{q(\cdot)}(\Omega)$. Thus using Proposition 3.4 with $g = u^0(id - |u^0|^{q(\cdot)-2}) + \lambda\Delta t\frac{\partial F}{\partial u}(.,u^0)$ we get existence of $u^1 \in W_0 \cap L^\infty(\Omega)$ and iteratively we get u^n.

Let sequence $u_{\Delta t}$ and $\tilde{u}_{\Delta t}$ as previously defined and such that $u_{\Delta t} = u_0$ for $t < 0$, then (3.6) is satisfied with

$$g_{\Delta t}(x,t) = \lambda\frac{\partial F}{\partial u}(x, u_{\Delta t}(t-\Delta t, x)) - |u_{\Delta t}(t-\Delta t,x)|^{q(\cdot)-2}u_{\Delta t}(t-\Delta t,x).$$

Step 2 We pass to the limit to prove that u satisfies (P_T).

As in the proof of Theorem 3.3 we obtain :

$\frac{\partial \tilde{u}_{\Delta t}}{\partial t}$ is bounded in $L^2(Q_T)$ uniformly in Δt,

$u_{\Delta t}$ and $\tilde{u}_{\Delta t}$ are bounded in $L^\infty(0,T,W_0)\cap L^\infty(Q_T)$ uniformly in Δt,

$$u_{\Delta t}\ ,\ \tilde{u}_{\Delta t} \rightharpoonup^* u \text{ in } L^\infty(0,T,W_0)\cap L^\infty(Q_T), \text{ and } \frac{\partial \tilde{u}_{\Delta t}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(Q_T). \qquad (3.27)$$

Also

$$\tilde{u}_{\Delta t} \longrightarrow u \in C([0,T], L^{q(\cdot)}(\Omega)) \quad \text{and} \quad u_{\Delta t} \longrightarrow u \in L^\infty([0,T], L^{q(\cdot)}(\Omega)). \qquad (3.28)$$

According to assumption on $F (F \in C^1)$ and since $\xi \longmapsto \frac{1}{p(x)}|\xi|^{p(x)} \in C^1$, using the above convergence results, we obtain:

$$|u_{\Delta t}|^{q(\cdot)-2}u_{\Delta t} \longrightarrow |u|^{q(\cdot)-2}u \quad \text{in} \quad L^\infty([0,T], L^{q(\cdot)}(\Omega)) \qquad (3.29)$$

and

$$\frac{\partial F}{\partial u}(x, u_{\Delta t}(t-\Delta t, x)) \to \frac{\partial F}{\partial u}(., u(., t)) \quad \text{in} \quad L^\infty([0,T], L^{q(.)}(\Omega)). \tag{3.30}$$

It follows that $g_{\Delta t}(., t) \to \frac{\partial F}{\partial u}(., u(., t)) - |u|^{q(.)-2}u$ in $L^\infty(0, T, W_0)$. By Theorem 3.3, it follows that problem (P_T) admits a weak solution.

Taking another weak solution denoted v we proof uniqueness of solution to (P_T). Indeed, for arbitrary $t_0 \in [0, T]$ we have:

$$\begin{aligned} \frac{1}{2}\int_\Omega (u(., t_0) - v(., t_0))^2 dx \quad &+ \quad \int_0^{t_0} \langle (-\Delta)^s_{p(.)} u - (-\Delta)^s_{p(.)} v, u - v\rangle dt \\ &+ \quad \int_0^{t_0}\int_\Omega \left(|u|^{q(.)-2}u - |v|^{q(.)-2}v\right)(u-v)dxdt \\ &= \quad \int_0^{t_0}\int_\Omega \left(\frac{\partial F}{\partial u}(., u) - \frac{\partial F}{\partial v}(., v)\right)(u-v)dxdt. \end{aligned} \tag{3.31}$$

Since $u \longmapsto (-\Delta)^s_{p(.)} u$ and $u \longmapsto |u|^{q(.)-2}u$ are monotones, we obtain the following inequality:

$$\frac{1}{2}\int_\Omega (u(., t_0) - v(., t_0))^2 dx \le \int_0^{t_0}\int_\Omega \left(\frac{\partial F}{\partial u}(., u) - \frac{\partial F}{\partial v}(., v)\right)(u-v)dxdt. \tag{3.32}$$

Lipschitz condition on $\frac{\partial F}{\partial u}$, Hölder inequality and (**H4**) give:

$$\frac{1}{2}\int_\Omega (u(., t_0) - v(., t_0))^2 dx \le C\int_0^{t_0} ||u-v||^2_{L^2(\Omega)} dt, \tag{3.33}$$

implies,

$$||u-v||^2_{L^2(\Omega)} dx \le C\int_0^{t_0} ||u-v||^2_{L^2(\Omega)} dt. \tag{3.34}$$

Now, Gronwall's lemma gives $u \equiv v$. The proof of Theorem 3.1 is then complete. □

We end this section by investigate the asymptotic behavior of global solution of (P_T), in particular the convergence to a stationary solution. For this, we study the following stationary problem (P) corresponding to (P_T).

$$(P)\begin{cases} (-\Delta)^s_{p(.)} u(x) + |u(x)|^{q(.)-2}u(x) &= \lambda \frac{\partial F}{\partial u}(x, u) \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega. \end{cases}$$

According to Theorem 3.4 in [12], for all $\lambda > 0$, problem (P) admits at least a weak solution $\tilde{u}$ in the sense that

$$\begin{aligned} &\int_{\Omega\times\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p(.)-2}(\tilde{u}(x) - \tilde{u}(y))(v(x) - v(y))}{|x-y|^{N+sp(.)}} dxdy \\ &\quad + \int_{\Omega\times\Omega} |\tilde{u}(x)|^{q(.)-2}\tilde{u}(x)v(x)dx \\ &= \lambda\int_{\Omega\times\Omega} \frac{\partial F}{\partial u}(x, \tilde{u})v(x)dx, \end{aligned}$$

for every $v \in C_0^\infty(Q_T)$.
Note that $\tilde{u}$ is unique. Indeed, the Euler Lagrange functional (corresponding to problem (P)) $\psi_\lambda : W_0 \to \mathbb{R}$ define by

$$\psi_\lambda(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(.)}}{p(.)|x - y|^{N+sp(.)}} dxdy + \int_\Omega \frac{|u|^{q(.)}}{q(.)} dx - \lambda \int_\Omega F(x, u)dx \tag{3.35}$$

is bounded below and strictly convex (this holds since for any x and y, the function $\xi \longmapsto \frac{1}{p(x,y)} \xi^{p(x,y)}$ is strictly convex).

It is easy to see that ψ_λ is coercive and the uniqueness of solution to (P) follows.

The following proposition is inspired by Proposition 4.3 in reference [10]. We employ embedding $L^{p^+}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ in order to generalize to the case of a variable exponent.

Proposition 3.5. *Let F satisfying assumption $(\mathbf{H_2})$ and $u \in W_0$ be a weak solution of (P). Then $u \in L^\infty(\Omega)$.*

Proof. let adapt argument from [10] and use embedding $L^{p^+}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ to conclude.

First we note that due to the homogeneity of the problem (P), it suffices to prove that

$$||u^+||_{L^\infty(\Omega)} \leq 1 \text{ whenever } |u^+|_{L^{p(\cdot)}(\Omega)} \leq \delta \text{ for some } \delta > 0. \tag{3.36}$$

A similar assertion can be established for u^- where $u^+(x) = \max(u(x), 0)$ and $u^-(x) = \max(-u(x), 0)$. Therefore $u \in L^\infty$. For $k \geq 1$, set $w_k(x) = \big(u(x) - 1 - 2^{-k}\big)^+$.

We have the following properties about $w_k(x)$.

(i) $w_{k+1}(x) \leq w_k(x)$.
(ii) $u(x) \leq (2^{k+1} + 1)w_k(x)$ for $x \in \{w_k(x) > 0\}$.
(iii) $\{w_{k+1}(x) > 0\} \subset \{w_k(x) > 2^{-(k+1)}\}$.

Now let $U_k = ||w_k||^{p^+}_{L^{p^+}(\Omega)}$. Taking $v = u - (1 - 2^{-(k+1)})$.
Using lemma 2.8, (i), (ii) above, for $||w_k||_{W_0} > 1$ we get

$$\begin{aligned}
&||w_k||^{p^+}_{W_0^{s,p^+}(\Omega)} \\
&= \int_{\Omega\times\Omega} \frac{|w_{k+1}(x) - w_{k+1}(y)|^{p^+}}{|x - y|^{N+sp^+}} \\
&\leq \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p^+-2}(w_{k+1}(x) - w_{k+1}(y))(u(x) - u(y))}{|x - y|^{N+sp^+}} \\
&\leq \int_{\Omega\times\Omega} |\frac{\partial F(x, u)}{\partial u}| w_{k+1}(x)dx \\
&\leq \int_{\{w_{k+1}(x)>0\}} (CV(x)|u|^{p^+-1}) w_{k+1}dx
\end{aligned}$$

$$\leq C\left(\int_{\{w_{k+1}(x)>0\}}|V(x)|^{\frac{p^+}{p^+-1}}\right)^{\frac{p^+-1}{p^+}}\left(\int_{\{w_{k+1}(x)>0\}}|(2^{k+1}+1)^{p^+-1}w_k^{p^+}|^{p^+}\right)^{\frac{1}{p^+}}$$
$$\leq C(2^{k+1}+1)^{p^+-1}\left(\int_{\{w_{k+1}(x)>0\}}|V(x)|^{\frac{p^+}{p^+-1}}\right)^{\frac{p^+-1}{p^+}}\left(\int_{\{w_{k+1}(x)>0\}}|w_k^{p^+}|^{p^+}\right)^{\frac{1}{p^+}}$$
$$\leq C^k U_k.$$

From Hölder's inequality we have

$$\begin{aligned} U_{k+1} &= \int_{\{w_{k+1}(x)>0\}} w_{k+1}^{p^+}dx \\ &\leq \left(\int_{\{w_{k+1}(x)>0\}} w_k^{\frac{N}{N-sp^+}}\right)^{\frac{N-sp^+}{N}} \big|\{x\in\Omega : w_{k+1}(x)>0\}\big|^{\frac{sp^+}{N}} \\ &\leq C'\|w_{k+1}\|^{p^+}_{W^{s,p^+}}\big(2^{(k+1)p^+}U_k\big)^{\frac{sp^+}{N}}. \end{aligned}$$

Hence,

$$\begin{aligned} U_{k+1} &\leq C'C^kU_k\big(2^{(k+1)p^+}U_k\big)^{\frac{sp^+}{N}} \\ &\leq C'C^k\big(2^{k+1}+1\big)^{p^+\left(1+\frac{sp^+}{N}\right)}U_k^{1+\frac{sp^+}{N}} \\ &\leq C'^kU_k^{1+\alpha}, \end{aligned}$$

where $c>0$ and $\alpha=\frac{sp^+}{N}$.
It follows that $\lim\limits_{k\to\infty}U_k=0$, provides that $||u^+||^{p^+}_{L^{p^+}(\Omega)}=U_0\leq C^{\frac{-1}{\alpha^2}}\delta_1^{p^+}$. Then, $w_k(x)\to(u(x)-1)^+$.
Now, according to embedding $L^{p^+}(\Omega)\hookrightarrow L^{p(.)}(\Omega)$ we obtain

$$|u^+|_{L^{p(.)}(\Omega)}\leq C||u^+||_{L^{p^+}(\Omega)},$$

which impmlies

$$|u^+|_{L^{p(.)}(\Omega)}\leq\delta,\ \text{with}\ \delta=C\delta_1.$$

□

Let recall Theorem 4.4 of [10] that provides the $C^\alpha(\overline{\Omega})$ regularity of weak solution of (P).

Theorem 3.6. *Assume that $u\in W_0$ is solution of (P). There exists $\alpha=\alpha(N,\Omega,p,s)\in(0,s]$ and $C=C(N,p,s,\Omega,||u||_{L^\infty(\Omega)})$, such that $u\in C^\alpha(\overline{\Omega})$ and $||u||_{C^\alpha(\overline{\Omega})}\leq C$.*

Note that $A:=(-\Delta)^s_{p(.)}$ with Dirichlet conditions is m-accretif of domain

$$D(A)=\{\ u\in W_0\cap L^\infty(\Omega):Au\in L^\infty(\Omega)\}.$$

Using [10, Theorem 3.10], we show that $u(t)\to\tilde{u}$ as $t\to+\infty$, where $u(t)$ is the solution of problem (P_T) and $\tilde{u}$ the solution of problem (P).

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