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CONVERGENCE ANALYSIS OF AN ALTERNATED INERTIAL THREE-OPERATOR SPLITTING ALGORITHM WITH APPLICATIONS TO OPTIMAL CONTROL PROBLEMS

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Dedicated to the memory of Professor Kazimierz Goebel.

Abstract. In this work, we propose and analyse a three-operator splitting algorithm with alternated inertial step for finding a zero of monotone inclusion problems of the sum of three maximally monotone operators, where one is cocoercive. We show that some of the attractive properties of the proposed algorithm in general Hilbert spaces under mild conditions on the inertial parameters and relaxation parameters. Furthermore, we derive alternated inertial primal-dual splitting algorithms from the proposed algorithm. Finally, we conduct numerical experiments on optimal control problems. Numerical results demonstrate the advantage of the proposed alternated inertial three-operator splitting algorithms.

1. Introduction

We consider a problem of finding a zero of the sum of three maximally monotone operators, with one of them being cocoercive. This problem is called monotone inclusion problem. Due to the mathematical generality of maximally monotone operators, the monotone inclusion problem arises in many important modern applications, such as image recovery, signal processing, machine learning, sparse optimal control problems, etc; see, e.g., [3, 31, 27, 25, 29, 24]. The three-operator splitting algorithm, which was originally proposed by Davis and Yin (Set-valued Var. Anal. 25(4), 829-858, 2017) can be applied to solve the monotone inclusion problem and includes existing algorithms such as the forward-backward splitting algorithm [28, 19] and the Douglas-Rachford splitting algorithm [19, 15]. Moreover, a generalization of the three-operator splitting algorithm in the direction of larger stepsizes can be found in [11, 2].

For convex optimization, the three-operator splitting algorithm is known to exhibit a convergence rate of its objective function value of $o(1/\sqrt{k})$ under Lipschitz continuity of one of the proximal terms [12, Theorem 3.1]. This convergence rate is slow and hence various acceleration techniques have been investigated. Pedregosa and Gidel showed that linear convergence of an adaptive step-size variant of the

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three-operator splitting algorithm under additional assumptions of strong convexity of the smooth term and smoothness of one of the proximal terms [16, Theorem 3]. For monotone inclusion problems, Davis and Yin showed that the sequence generated by an accelerated variant of three-operator splitting algorithm converges to the solution with convergence rate $O(1/k^2)$ under the additional assumption that one of the involved operators is strongly monotone [12, Theorem 3.3]. Note that the existing accelerated algorithms require to know the strong monotonicity constant of involved operators. However, this constant is typically unknown, and so the question arises of whether it is possible to design an accelerated variant which does not require to assume the strong monotonicity of involved operators.

The inertial extrapolation technique has been widely used to improve the speed of convergence of algorithms without additional assumptions on involved operators. For instance, the inertial proximal point algorithm [1], the inertial forward-backward splitting algorithm [20], the inertial Douglas-Rachford splitting algorithm [6, 14], and the inertial forward-backward-forward splitting algorithm [5, 26]. Here, we are interested in variants inertial three-operator splitting algorithm. Cui, Tang and Yang [10] proposed an inertial three-operator splitting algorithm, which combines the inertial Kraselsnoskiĭ-Mann iteration [6] and the three-operator splitting algorithm [12]. Under certain conditions on the inertial parameters, they analyzed the converngece of their algorithm [23, Theorem 3.1]. However, the trajectories of sequences generated by inertial splitting algorithms exhibit undesirable oscillations. For example, there is an example that the inertial proximal point algorithm loses the Fejér monotonicity of the sequences with respect to the solution. This furthermore makes the sequence generated by the algorithm can oscillate around the set of solutions [23, Examples 1 and 2].

Our purpose in this paper is to study the convergence of an alternated inertial three-operator splitting algorithm for finding a zero of the sum of three maximally monotone operators where one of them is cocoercive. The idea is to employ the alternated inertial technique [23, 17] to the three-operator splitting algorithm with large stepsizes [11, 2]. We show that the sequence generated by the proposed algorithm converges weakly to a solution under mild conditions on the inertial parameters and the relaxation parameters, which are different from [10]. As a consequence, it is natural to generalize the convergence analysis of the existing algorithms in [23, 17, 30] to the proposed alternated inertial three-operator splitting algorithm. We emphasize that the proposed algorithm gives the Fejér monotonicity when the iteration counter is even. Furthermore, by introducing suitable product spaces, structured monotone inclusion problems involving parallel sums and compositions of maximally monotone operators with linear continuous ones can be considered as the special instance of the monotone inclusion problem (see [8, 5, 32]). As a consequence, the alternated inertial primal-dual splitting algorithm for solving structured monotone inclusion problems can be derived from the proposed algorithm. Finally, we conduct numerical experiments on optimal control problems [24]. Numerical results demonstrate the advantage of the proposed alternated inertial three-operator splitting algorithms.

The paper is organized as follows. Section 2 recalls some basic definitions. Section 3 presents and analyzes our alternated three-operator splitting algorithm. Section 4 derives the alternated inertial primal-dual splitting algorithms from our proposed algorithm. Section 5 reports numerical experiments results. Section 6 gives conclusions.

2. Preliminaries

This section reviews basic definitions, facts, and notation that will be used throughout the paper.

$\mathcal{H}$ denotes a real Hilbert space with endowed the *inner product* $\langle\cdot,\cdot\rangle$ and the *norm* $\|\cdot\|$. $\mathbb{R}$ and $\mathbb{N}$ denote the *set of real numbers* and the *set of positive integers*, respectively. We denote by $\mathbb{R}_{++}$ the *set of strictly positive real numbers.* For any $\{x_k\} \subset \mathcal{H}$ and $x \in \mathcal{H}$, $x_k \to x$ and $x_k \rightharpoonup x$ denote the *strong and weak convergences* of $\{x_k\}$ to x, respectively. Let $x, y \in \mathcal{H}$ and let $\alpha \in \mathbb{R}$. Then, the following identity will be used in the paper:

$$\|(1-\alpha)x + \alpha y\|^2 = (1-\alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

([3, Corollary 2.15]).

For an arbitrary set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$, $\mathrm{dom}(A)$ denotes the *domain of* A, i.e., $\mathrm{dom}(A) = \{x \in \mathcal{H} : A(x) \neq \emptyset\}$. $\mathrm{ran}(A)$ denotes the *range of* A, i.e., $\mathrm{ran}(A) = \bigcup\{A(x) : x \in \mathrm{dom}(A)\}$, $\mathrm{gr}(A)$ denotes the *graph of* A, i.e., $\mathrm{gr}(A) = \{(x, x^*) : x^* \in A(x)\}$. The *set of zero points of* A is denoted by $A^{-1}(0)$, i.e., $A^{-1}(0) = \{z \in \mathrm{dom}(A) : 0 \in A(z)\}$.

A set-valued operator $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be

(i) *monotone* if, for all $(x, x^*), (y, y^*) \in \mathrm{gr}(A)$,
$$\langle x - y, x^* - y^*\rangle \geq 0;$$
(ii) *maximally monotone* if A is monotone and $A = B$ whenever $B\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is a monotone mapping such that $\mathrm{gr}(A) \subset \mathrm{gr}(B)$;
(iii) *uniformly monotone* with modulus $\phi\colon [0,\infty) \to [0,\infty]$ if ϕ increasing, vanishes only at 0, and for all $(x, x^*), (y, y^*) \in \mathrm{gr}(A)$,
$$\langle x - y, x^* - y^*\rangle \geq \phi(\|x - y\|).$$

For operator $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$, and for $\gamma \in \mathbb{R}_{++}$, the *resolvent* $J_{\gamma A}\colon \mathcal{H} \rightrightarrows \mathcal{H}$ of A is defined by $J_{\gamma A} = (I + \gamma A)^{-1}$, where I is the identity mapping on $\mathcal{H}$. Moreover, if A is maximally monotone, then $J_{\gamma A}$ is single-valued and $\mathrm{dom}(J_{\gamma A}) = \mathcal{H}$.

For a function $f\colon \mathcal{H} \to \overline{\mathbb{R}}$, $\mathrm{dom}(f)$ denotes the domain of f, i.e., $\mathrm{dom}(f) = \{x \in \mathcal{H} : f(x) \in \mathbb{R}\}$. $\Gamma(\mathcal{H})$ denotes the family of proper, convex and lower semicontinuous extended real-valued functions. For a function $f \in \Gamma(\mathcal{H})$, the *subdifferential* $\partial f\colon \mathcal{H} \rightrightarrows \mathcal{H}$ of f at $x \in \mathcal{H}$ is defined by

$$\partial f(x) = \{x^* \in \mathcal{H} : f(y) \geq f(x) + \langle y - x, x^*\rangle \ (\forall y \in \mathcal{H})\}. \tag{2.2}$$

We know that the subdifferential ∂f is maximally monotone ([31, Theorem 4.6.6], [3, Theorem 20.40]) and its resolvent is given by $J_{\gamma\partial f} = \mathrm{prox}_{\gamma f}$, where

$$\mathrm{prox}_{\gamma f}(x) = \operatorname*{argmin}_{y\in\mathcal{H}}\{f(y) + \frac{1}{2\gamma}\|y - x\|^2\}$$

denotes the *proximal mapping* of f. We also know that the subdifferential of a uniformly convex function is uniformly monotone [3, Example 22.4 (iii)]. The *conjugate* of f is

$$f^*\colon \mathcal{H} \to f^*(p) = \sup\{\langle p, x\rangle - f(x) : x \in \mathcal{H}\}$$

for all $p \in \mathcal{H}$. Moreover, if $f \in \Gamma(\mathcal{H})$, then $f^* \in \Gamma(\mathcal{H})$, as well, and $(\partial f)^{-1} = \partial f^*$. For $g \in \Gamma(\mathcal{H})$, the *infimal convolution* $f\Box g\colon \mathcal{H} \to \overline{R}$ of f and g is defined by

$$f\Box g(x) = \inf_{y\in\mathcal{H}}\{f(y) + g(x - y)\}$$

for all $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces and let $L\colon \mathcal{H} \to \mathcal{G}$ be a nonzero bounded linear operator with induced norm $\|L\| = \sup\{\|Lx\| : x \in \mathcal{H}$ with $\|x\| \le 1\}$. The *adjoint operator* $L^*\colon \mathcal{G} \to \mathcal{H}$ *of* L is defined by $\langle Lx, y\rangle_{\mathcal{G}} = \langle x, L^*y\rangle_{\mathcal{H}}$ for all $x \in \mathcal{H}$ and all $y \in \mathcal{G}$.

Let $C\colon \mathcal{H} \to \mathcal{H}$ be a single-valued operator. C is said to be

(i) *nonexpansive* if for all $x, y \in \mathcal{H}$

$$\|C(x) - C(y)\| \le \|x - y\|;$$

(ii) α*-averaged* with $\alpha \in (0, 1)$ if there exists a nonexpansive $R : \mathcal{H} \to \mathcal{H}$ such that $C = (1 - \alpha)I + \alpha R$;

(iii) β*-cocoercive* with $\beta \in (0, \infty)$ if for all $x, y \in \mathcal{H}$,

$$\langle C(x) - C(y), x - y\rangle \ge \beta\|C(x) - C(y)\|^2.$$

If a function $g\colon \mathcal{H} \to \mathbb{R}$ is convex and differentiable function with a β^{-1}-Lipschitzian gradient, then ∇g is β-cocoercive [3, Corollary 18.17]. When C is α-averaged, the following inequality holds [3, Proposition 4.35]:

$$\|C(x) - C(y)\|^2 \le \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - C)x - (I - C)y\|^2 \ (\forall x, y \in \mathcal{H}). \tag{2.3}$$

The set of fixed points of C is denoted by

$$\mathrm{Fix}(C) = \{x \in \mathcal{H} : C(x) = x\}.$$

Let Ω be a nonempty subset of $\mathcal{H}$ and let $\{x_k\}$ be a sequence in $\mathcal{H}$. Then $\{x_k\}$ is *Fejér monotone* with respect to Ω if for all $x \in \Omega$, $\|x_{k+1} - x\| \le \|x_k - x\|$ $(\forall k \in \mathbb{N})$.

The next results are crucial for the proof of our main results.

Proposition 2.1. [2, Lemma 3.2] *Let $A, B\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be two maximally monotone operators and $C\colon \mathcal{H} \to \mathcal{H}$. Let $x, \hat{x} \in \mathcal{H}$ and $\gamma > 0$, and set $u := J_{\gamma A}(x)$ (resp. $\hat{u} := J_{\gamma A}(\hat{x})$) and $v := J_{\gamma B}(2u - x - \gamma C(u))$ (resp. $\hat{v} := J_{\gamma B}(2\hat{u} - \hat{x} - \gamma C(\hat{u}))$). Then, it holds*

$$0 \le \langle x - \hat{x}, (u - v) - (\hat{u} - \hat{v})\rangle - \|(u - v) - (\hat{u} - \hat{v})\|^2 - \gamma\langle C(u) - C(\hat{u}), v - \hat{v}\rangle. \tag{2.4}$$

Further, if A (resp. B) is uniformly monotone with modulus ϕ, then (8) holds with 0 replaced by $\gamma\phi(\|u-\hat{u}\|$ (resp. $\gamma\phi(\|v-\hat{v}\|)$).

Proposition 2.2. [11, Proposition 2.1 and Corollary 4.2] *Let $A, B\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, let $C\colon \mathcal{H} \to \mathcal{H}$ be β-cocoercive, and let $\gamma, \lambda \in (0,\infty)$. Define*

$$T_{A,B,C} := I - \lambda J_{\gamma A} + \lambda J_{\gamma B} \circ (2J_{\gamma A} - I - \gamma C \circ J_{\gamma A}). \tag{2.5}$$

Then, the following hold:

(a) $\mathrm{Fix}(T_{A,B,C}) \neq \emptyset$ *if and only if* $(A+B+C)^{-1}(0) \neq \emptyset$;
(b) $J_{\gamma A}(\mathrm{Fix}(T_{A,B,C})) = (A+B+C)^{-1}(0)$;
(c) *if* $\gamma \in (0,4\beta)$ *and* $\lambda \in (0, 2-\gamma/(2\beta))$, *then* $T_{A,B,C}$ *is* $(2\lambda\beta)/(4\beta-\gamma)$*-averaged.*

3. Alternated inertial three operator splitting algorithm and convergence results

In this paper, we consider a monotone inclusion problem. This problem is formulated as follows.

Problem 3.1. Let $A, B\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone operators and $C\colon \mathcal{H} \to \mathcal{H}$ a β-cocoercive mapping for some $\beta > 0$. We consider the following inclusion problem

$$\text{find } u \in \mathcal{H} \text{ such that } 0 \in (A+B+C)(u). \tag{3.1}$$

We provide an algorithm together with convergence results. We consider the following iterative algorithm.

Algorithm 3.2.

$$\begin{cases} w_k = \begin{cases} z_k & \text{(if } k \text{ is even)} \\ z_k + \alpha(z_k - z_{k-1}) & \text{(if } k \text{ is odd)} \end{cases} \\ u_k = J_{\gamma A}(w_k) \\ v_k = J_{\gamma B}(2u_k - w_k - \gamma C(u_k)) \\ z_{k+1} = w_k + \lambda(v_k - u_k) \end{cases}$$

where $k \geq 1$, $z_0, z_1 \in \mathcal{H}$, $\alpha \in [0,\infty)$ and $\lambda, \gamma \in (0,\infty)$.

3.1. Convergence Analysis.

To establish weak convergence of the sequence generated by Algorithm 3.2, we need the following assumptions:

Assumption 3.3. Assume that γ, λ and α satisfy the following conditions:

(A1) $\gamma \in (0,4\beta)$;
(A2) $\lambda \in (0, 2-\gamma/(2\beta))$;
(A3) $\alpha \in [0, (2-\gamma/(2\beta)-\lambda)/\lambda)$.

Remark 3.4.

- Assume that $\alpha = 0$. Then, Algorithm 3.2 can be considered as the three-operator splitting algorithm [12] with constant relaxation parameters. The stepsize γ in [12] is assumed to be in $(0, 2\beta\epsilon)$ with $\epsilon \in (0,1)$. A generalization in the direction of large stepsizes can be found in [11, 2]

- Assume that $w_k = z_k + \alpha(z_k - z_{k-1})$ for every $k \in \mathbb{N}$. Then, Algorithm 3.2 becomes the inertial three-operator splitting algorithm [10] with constant inertial parameters and constant relaxation parameters. The convergence of the inertial three-operator splitting algorithm was discussed under the following conditions:
  - $\gamma \in (0, 2\beta\epsilon)$, where $\epsilon \in (0, 1)$;
  - $\alpha \in [0, 1)$;
  - Let $\lambda, \sigma, \delta > 0$ such that
$$\delta > \frac{\alpha^2(1+\alpha)+\alpha\sigma}{1-\alpha^2} \text{ and } 0 < \lambda \leq \frac{\delta - \alpha[\alpha(1+\alpha)+\alpha\delta+\sigma]}{\delta(1+\alpha(1+\alpha)+\alpha\delta+\sigma)}.$$

  These conditions are different from Assumption 3.3. In fact, compared to $(0, 2\beta\epsilon)$, the range of γ in (A1) is large. For given $0 \leq \alpha < 1$, we consider the upper bound of λ. As addressed in [10], set $\delta := \frac{\alpha^2(1+\alpha)+\alpha\sigma}{1-\alpha^2} + 1$. For simplicity, we focus on the cases of $\sigma = 0.01$ and $\sigma = 0.1$. Then we get the following:
$$\begin{array}{rlrl} (\sigma = 0.01) & \alpha = 0.1 \Rightarrow \lambda < 0.8010 & & \alpha = 0.3 \Rightarrow \lambda < 0.4622 \\ & \alpha = 0.6 \Rightarrow \lambda < 0.1076 & & \alpha = 0.9 \Rightarrow \lambda < 0.0019 \\ (\sigma = 0.1) & \alpha = 0.1 \Rightarrow \lambda < 0.7388 & & \alpha = 0.3 \Rightarrow \lambda < 0.4261 \\ & \alpha = 0.6 \Rightarrow \lambda < 0.0986 & & \alpha = 0.9 \Rightarrow \lambda < 0.0017 \end{array}$$

  There is trade-off between choosing α and choosing λ.

  On the other hand, we consider (A2) and (A3). For simplicity, we focus on the case $\gamma = \beta$. In this case, (A2) and (A3) become $\lambda \in (0, 1.5)$ and $\alpha \in [0, (1.5 - \lambda)/\lambda)$. Then we get the following:
$$\begin{array}{lll} \lambda = 0.1 \Rightarrow \alpha < 14 & \lambda = 0.3 \Rightarrow \alpha < 4 & \lambda = 0.6 \Rightarrow \alpha < 1.5 \\ \lambda = 0.9 \Rightarrow \alpha < 0.6667 & \lambda = 1.2 \Rightarrow \alpha < 0.25 & \lambda = 1.4 \Rightarrow \alpha < 0.0714 \end{array}$$

  Therefore, our conditions are mild and easily verifiable.
- Algorithm 3.2 is a generalization of the existing alternated inertial algorithms in [23, 17, 30]. For other type of alternated inertial algorithm we refer to [22].

Theorem 3.5. *Assume that $(A+B+C)^{-1}(0) \neq \emptyset$ and Assumption 3.3 holds for γ, λ and α. Let $\{z_k\}$ be the sequence generated by Algorithm 3.2. Then the following hold:*

(i) *$\{z_{2k}\}$ is Fejér monotone with respect to $\mathrm{Fix}(T_{A,B,C})$, where $T_{A,B,C}$ is defined in (2.5).*
(ii) *$\{z_k\}$ converges weakly to a point $\overline{z}$ in $\mathrm{Fix}(T_{A,B,C})$.*
(iii) *$\{C(u_k)\}$ converges strongly to $C(J_{\gamma A}(\overline{z}))$.*
(iv) *$\{u_k\}$ converges weakly to $J_{\gamma A}(\overline{z}) \in (A+B+C)^{-1}(0)$.*
(v) *$\{v_k\}$ converges weakly to $J_{\gamma B}(2J_{\gamma A}(\overline{z}) - \overline{z} - \gamma C(J_{\gamma A}(\overline{z}))) \in (A+B+C)^{-1}(0)$.*
(vi) *If either A or B is uniformly monotone, then $\{u_k\}$ and $\{v_k\}$ converge strongly to the unique point in $(A+B+C)^{-1}(0)$.*

Proof. (i) Let $s \in \operatorname{Fix}(T_{A,B,C})$. From the definition of $\{z_k\}$, we observe that

$$\begin{aligned} z_{k+1} &= w_k + \lambda(v_k - u_k) \\ &= w_k + \lambda(J_{\gamma B}(2u_k - w_k - \gamma C(u_k) - J_{\gamma A}(w_k)) \\ &= T_{A,B,C}(w_k). \end{aligned} \tag{3.2}$$

Using (2.3) and Proposition 2.2 (c), we obtain

$$\begin{aligned} \|z_{2k+2} - s\|^2 &= \|T_{A,B,C}(w_{2k+1}) - s\|^2 \\ &\le \|w_{2k+1} - s\|^2 - \frac{1 - \frac{2\lambda\beta}{4\beta-\gamma}}{\frac{2\lambda\beta}{4\beta-\gamma}} \|w_{2k+1} - T_{A,B,C}(w_{2k+1})\|^2 \\ &= \|w_{2k+1} - s\|^2 - \frac{1}{\lambda}\left(2 - \frac{\gamma}{2\beta} - \lambda\right) \|w_{2k+1} - T_{A,B,C}(w_{2k+1})\|^2. \end{aligned} \tag{3.3}$$

Using (2.1) and (3.2), we obtain

$$\begin{aligned} \|w_{2k+1} - s\|^2 =& \|z_{2k+1} + \alpha(z_{2k+1} - z_{2k}) - s\|^2 \\ =& (1+\alpha)\|z_{2k+1} - s\|^2 - \alpha\|z_{2k} - s\|^2 + \alpha(1+\alpha)\|z_{2k} - z_{2k+1}\|^2 \\ =& (1+\alpha)\|z_{2k+1} - s\|^2 - \alpha\|z_{2k} - s\|^2 \\ &+ \alpha(1+\alpha)\|w_{2k} - T_{A,B,C}(w_{2k})\|^2. \end{aligned} \tag{3.4}$$

Again, by using Proposition 2.2, we obtain

$$\begin{aligned} \|z_{2k+1} - s\|^2 &= \|T_{A,B,C}(w_{2k}) - s\|^2 \\ &\le \|w_{2k} - s\|^2 - \frac{1}{\lambda}\left(2 - \frac{\gamma}{2\beta} - \lambda\right) \|w_{2k} - T_{A,B,C}(w_{2k})\|^2, \end{aligned}$$

and hence the above estimate together with (3.3) and (3.4) implies

$$\begin{aligned} \|z_{2k+2} - s\|^2 \le& \|z_{2k} - s\|^2 - \frac{1+\alpha}{\lambda}\left(2 - \frac{\gamma}{2\beta} - \lambda - \lambda\alpha\right) \|w_{2k} - T_{A,B,C}(w_{2k})\|^2 \\ &- \frac{1}{\lambda}\left(2 - \frac{\gamma}{2\beta} - \lambda\right) \|w_{2k+1} - T_{A,B,C}(w_{2k+1})\|^2. \end{aligned} \tag{3.5}$$

By using Assumption 3.3, we obtain that $\{z_{2k}\}$ is Fejér monotone with respect to $\operatorname{Fix}(T_{A,B,C})$.

(ii) From (3.5), we obtain $\lim_{k\to\infty} \|w_{2k} - T_{A,B,C}(w_{2k})\| = \lim_{k\to\infty} \|w_{2k+1} - T_{A,B,C}(w_{2k+1})\| = 0$ and hence

$$\lim_{k\to\infty} \|w_k - T_{A,B,C}(w_k)\| = 0. \tag{3.6}$$

Since $T_{A,B,C}$ is $(2\lambda\beta)/(4\beta - \gamma)$-averaged, $T_{A,B,C}$ is nonexpansive. By (3.6) and [3, Theorem 5.14], $\{w_k\}$ converges weakly to a point in $\operatorname{Fix}(T_{A,B,C})$. Therefore, by (3.2) and (3.6), we have that $\{z_k\}$ converges weakly to a point $\overline{z} \in \operatorname{Fix}(T_{A,B,C})$.

(iii) Using the definition of $\{z_k\}$ we have $\lambda(v_k - u_k) = T_{A,B,C}(w_k) - w_k$. By (3.6), we obtain

$$\lim_{k\to\infty}\|v_k-u_k\|=0. \tag{3.7}$$

Since $\overline{z}\in \mathrm{Fix}(T_{A,B,C})$, we have $J_{\gamma A}(\overline{z}) = J_{\gamma B}(2J_{\gamma A}(\overline{z})-\overline{z}-\gamma C(J_{\gamma A}(\overline{z})))$. By applying Proposition 2.1 with $x:=\overline{z}$ and $\hat{x}:=w_k$, we obtain the inequality

$$0\le\langle\overline{z}-w_k,v_k-u_k\rangle-\|u_k-v_k\|^2-\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k)),J_{\gamma A}(\overline{z})-v_k\rangle. \tag{3.8}$$

Using the β-cocoercivity of C, we obtain

$$\begin{aligned}&-\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k),J_{\gamma A}(\overline{z})-v_k\rangle\\&=-\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k),J_{\gamma A}(\overline{z})-u_k\rangle+\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k),v_k-u_k\rangle\\&\le-\gamma\beta\|C(J_{\gamma A}(\overline{z}))-C(u_k)\|^2+\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k),v_k-u_k\rangle,\end{aligned}$$

which together with (3.8) yields that

$$\begin{aligned}\gamma\beta\|C(J_{\gamma A}(\overline{z}))-C(u_k)\|^2\le&\langle\overline{z}-w_k,v_k-u_k\rangle-\|u_k-v_k\|^2\\&+\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k),v_k-u_k\rangle.\end{aligned} \tag{3.9}$$

Since $\{w_k\}$ converges weakly, $\{w_k\}$ is bounded [3, Lemma 2.46] and hence $\{u_k\}$ and $\{v_k\}$ are bounded. From (3.7) and (3.9), we obtain

$$C(u_k)\to C(J_{\gamma A}(\overline{z}))\ (k\to\infty). \tag{3.10}$$

(iv) and (v) Since $\{u_k\}$ is bounded, there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$, which converges weakly to $\overline{u}$. It follows from (3.7) that

$$\begin{aligned}&u_{k_j}\rightharpoonup\overline{u},\quad v_{k_j}\rightharpoonup\overline{u},\quad C(u_{k_j})\to C(J_{\gamma A}(\overline{z})),\quad \frac{1}{\gamma}(w_{k_j}-u_{k_j})\rightharpoonup\frac{1}{\gamma}(\overline{z}-\overline{u}),\\&\text{and}\quad \frac{1}{\gamma}(2u_{k_j}-w_{k_j}-\gamma C(u_{k_j})-v_{k_j})\rightharpoonup\frac{1}{\gamma}(\overline{u}-\overline{z}-\gamma C(J_{\gamma A}(\overline{z}))).\end{aligned}$$

By [3, Corollary 26.8], we obtain $(\overline{u},\frac{1}{\gamma}(\overline{z}-\overline{u}))\in \mathrm{gr}(A)$ and hence $\overline{u}=J_{\gamma A}(\overline{z})\in(A+B+C)^{-1}(0)$. Therefore, $J_{\gamma A}(\overline{z})$ is the unique cluster point of $\{u_k\}$ and $\{v_k\}$.

(vi) Assume that A is uniformly monotone. Using uniform monotonicity in Lemma 2.2 with $x:=\overline{z}$ and $\hat{x}:=w_k$, we obtain the inequality

$$\begin{aligned}\phi(\|J_{\gamma A}(\overline{z})-u_k\|)\le&\langle\overline{z}-w_k,v_k-u_k\rangle-\|u_k-v_k\|^2\\&-\gamma\langle C(J_{\gamma A}(\overline{z}))-C(u_k),J_{\gamma A}(\overline{z})-v_k\rangle,\end{aligned} \tag{3.11}$$

where $\phi\colon[0,\infty)\to[0,\infty]$ an increasing function that vanishes only at 0 We therefore deduce from (3.7) and (3.10) that $\phi(\|J_{\gamma A}(\overline{z})-u_k\|)\to0$, which implies that $u_k\to\overline{z}$. If B is uniformly monotone, then the right-hand side of (3.11) is $\phi(\|J_{\gamma A}(\overline{z})-v_k\|)$. Thus, we conclude using the similar argument as in the case when A is uniformly monotone. □

Remark 3.6. Theorem 3.5 leads us to Fejér monotonicity of $\{z_{2k}\}$ and convergence of $\{z_k\}$, $\{C(u_k)\}$, $\{u_k\}$ and $\{v_k\}$. The inertial parameters and the relaxation parameters are different from [10].

4. Primal-dual splitting algorithms and convergence results

We will formulate in this section weakly convergent primal-dual splitting algorithms by using the general algorithm in Section 3. To this end, we first consider the following structured monotone inclusion problem [8, 5, 32].

Problem 4.1. Let m be a strictly positive integer and let $I := \{1, 2, \ldots, m\}$. We consider the following primal inclusion problem

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in A(x) + \sum_{i=1}^{m} L_i^* \left(\left(B_i \Box D_i \right) \left(L_i(x) \right) \right) + C(x), \tag{4.1}$$

and its dual inclusion problem

$$\begin{aligned} &\text{find } v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m \text{ such that} \\ &(\exists x \in \mathcal{H}) \begin{cases} -\sum_{i=1}^{m} L_i^*(v_i) \in A(x) + C(x) \\ v_i \in (B_i \Box D_i)(L_i(x)) \ (\forall i \in I) \end{cases} \end{aligned} \tag{4.2}$$

where

- $\mathcal{H}, \mathcal{G}_1, \ldots, \mathcal{G}_m$ are real Hilbert spaces.
- $A \colon \mathcal{H} \rightrightarrows \mathcal{H}$ and $B_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i$ $(i \in I)$ are maximally monotone operators.
- $C : \mathcal{H} \to \mathcal{H}$ is monotone and β-Lipschitzian operator for some $\beta > 0$.
- $D_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i$ $(i \in I)$ is maximal monotone.
- $L_i : \mathcal{H} \to \mathcal{G}$ $(i \in I)$ is a nonzero bounded linear operator with adjoint L_i^*.

We say that $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a primal-dual solution to Problem 4.1 if

$$-\sum_{j=1}^{m} L_j^* \overline{v}_j \in A(\overline{x}) + C(\overline{x}) \quad \text{and} \quad \overline{v}_i \in (B_i \Box D_i)(L_i(\overline{x})) \ i = 1, 2, \ldots, m. \tag{4.3}$$

If $\overline{x}$ is a solution to (4.1), then there exists $(\overline{v}_1, \ldots, \overline{v}_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ such that $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ is a primal-dual solution to Problem 4.1, and if $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution to (4.2), then there exists $\overline{x} \in \mathcal{H}$ such that $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ is a primal-dual solution to Problem 4.1. If $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ is a primal-dual solution to Problem 4.1, then $\overline{x}$ is a solution to (4.1) and $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution to (4.2). Various particular instances of (4.1) and (4.2) when applied to convex optimization problems can be found in [8, 32].

We consider the Hilbert space $\boldsymbol{\mathcal{G}} := \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ endowed with the inner product and associated norm defined for $\boldsymbol{u} = (u_1, \ldots, u_m), \boldsymbol{v} = (v_1, \ldots, v_m) \in \boldsymbol{\mathcal{G}}$ as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{G}}} := \sum_{i=1}^{m} \langle u_i, v_i \rangle_{\mathcal{G}_i} \text{ and } \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{G}}} := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{\boldsymbol{\mathcal{G}}}},$$

respectively. Furthermore, we let $\boldsymbol{\mathcal{K}} = \mathcal{H} \times \boldsymbol{\mathcal{G}}$ be the Hilbert space endowed with inner product and associated norm defined for every $(x, \boldsymbol{u}), (y, \boldsymbol{v}) \in \boldsymbol{\mathcal{K}}$ as

$$\langle (x,\boldsymbol{u}), (y,\boldsymbol{v}) \rangle_{\boldsymbol{\mathcal{K}}} := \langle x, y \rangle_{\mathcal{H}} + \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{G}}} \text{ and } \|(x,\boldsymbol{u})\|_{\boldsymbol{\mathcal{K}}} := \sqrt{\langle (x,\boldsymbol{u}), (x,\boldsymbol{u}) \rangle_{\boldsymbol{\mathcal{K}}}}, \tag{4.4}$$

respectively.

4.1. **A primal-dual algorithm of forward-backward-type.** This subsection presents an algorithm for solving Problem 4.1 under an additional assumption that

$$D_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i \ (i \in I) \text{ is } \nu_i\text{-strongly for some } \nu_i \in (0,\infty). \tag{4.5}$$

Consider the sequences generated by the following algorithm:

Algorithm 4.2.

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & \text{(if } k \text{ is even)} \\ x_k + \alpha(x_k - x_{k-1}) & \text{(if } k \text{ is odd)} \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & \text{(if } k \text{ is even)} \\ v_{i,k} + \alpha(v_{i,k} - v_{i,k-1}) & \text{(if } k \text{ is odd)} \end{cases} \quad (\forall i \in I) \\ x_{k+1} = \overline{x}_k + \lambda \left(J_{\tau A} \left(\overline{x}_k - \tau \left(\sum_{i=1}^m L_i^* \overline{v}_{i,k} + C\overline{x}_k \right) \right) - \overline{x}_k \right) \\ v_{i,k+1} = \overline{v}_{i,k} + \lambda \left(J_{\sigma_i B_i^{-1}} \left(\overline{v}_{i,k} + \sigma_i \left(L_i(2x_{k+1} - \overline{x}_k) - D_i^{-1}\overline{v}_{i,k} \right) \right) - \overline{v}_{i,k} \right) \ (\forall i \in I) \end{cases}$$

where $(x_0, v_{1,0}, \ldots, v_{m,0}), (x_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\alpha \in [0,\infty]$, $\gamma, \lambda, \tau, \sigma_1, \ldots, \sigma_m \in (0,\infty)$.

We derive convergence of Algorithm 4.2.

Theorem 4.3. *In Problem 4.1, suppose that (4.5) and*

$$0 \in \operatorname{ran} \left(A + \sum_{i=1}^m L_i^* \left(B_i \square D_i \right) L_i + C \right) \tag{4.6}$$

hold. Define

$$\overline{\beta} := \min\{\beta, \nu_1, \ldots, \nu_m\},$$

$$\rho := \min\left\{\tau^{-1}, \sigma_1^{-1}, \ldots, \sigma_m^{-1}\right\} \left(1 - \sqrt{\tau \sum_{i=1}^m \sigma_i \|L_i\|^2} \right),$$

and suppose that $2\overline{\beta}\rho > 1$ holds. Let $\{(x_k, v_{1,k}, \ldots, v_{m,k})\} \subset \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ be a sequence generated by Algorithm 4.2 such that $\lambda \in (0, 2 - 1/(2\overline{\beta}\rho))$ and $\alpha \in [0, 1 - 1/(2\overline{\beta}\rho))$. Then the following hold:

(i) *$\{(x_{2k}, v_{1,2k}, \ldots, v_{m,2k})\}$ is Fejér monotone with respect to the set of primal-dual solutions of Problem 4.1.*
(ii) *There exists a primal-dual solution $\overline{\boldsymbol{v}} = (\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ to Problem 4.1 such that $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ converges weakly to $\overline{\boldsymbol{v}}$.*
(iii) *Suppose that C is uniformly monotone. Then $\{x_k\}$ converges strongly to $\overline{x}$.*

(iv) *Suppose that D_i^{-1} is uniformly monotone for some $i \in I$. Then $\{v_{i,k}\}$ converges strongly to $\overline{v}_i$.*

Proof. We consider the following operators $\boldsymbol{M}\colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}}$, $\boldsymbol{S}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ and $\boldsymbol{V}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ as follows:

$$\boldsymbol{M}\colon (x, \boldsymbol{u}) \mapsto (A(x) + \sum_{i=1}^{m} L_i^*(u_i), B_1^{-1}(u_1) - L_1(x), \dots, B_m^{-1}(u_m) - L_m(x)), \tag{4.7}$$

$$\boldsymbol{S}\colon (x, \boldsymbol{u}) \mapsto \left(C(x), D_1^{-1}(u_1), \dots, D_m^{-1}(u_m)\right). \tag{4.8}$$

$$\boldsymbol{V} : (x, \boldsymbol{u}) \mapsto \left(\frac{1}{\tau}x - \sum_{i=1}^{m} L_i^* u_i, -L_1 x + \frac{1}{\sigma_1}u_1, \dots, -L_m x + \frac{1}{\sigma_m}u_m\right). \tag{4.9}$$

Furthermore, consider the Hilbert space $\boldsymbol{\mathcal{K}_V}$ endowed with inner product and norm defined for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{K}}$ as

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{K}_V}} := \langle \boldsymbol{x}, \boldsymbol{V}\boldsymbol{y}\rangle_{\boldsymbol{K}} \text{ and } \|\boldsymbol{x}\|_{\boldsymbol{\mathcal{K}_V}} := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{K}_V}}} \tag{4.10}$$

respectively. Then the following hold:

- $\boldsymbol{M}$ and $\boldsymbol{S}$ are maximal monotone on $\boldsymbol{\mathcal{K}}$;
- $\boldsymbol{V}^{-1}\boldsymbol{M}$ and $\boldsymbol{V}^{-1}\boldsymbol{S}$ are maximal monotone on $\boldsymbol{\mathcal{K}_V}$;
- $\boldsymbol{V}^{-1}\boldsymbol{S}$ is $\beta\rho$-cocoercive on $\boldsymbol{\mathcal{K}_V}$;
-

$$\begin{aligned} &(\hat{x}, \hat{v}_1, \dots, \hat{v}_m) \in (\boldsymbol{V}^{-1}\boldsymbol{M} + \boldsymbol{V}^{-1}\boldsymbol{S})^{-1}(\boldsymbol{0}) \\ \Leftrightarrow\ &(\hat{x}, \hat{v}_1, \dots, \hat{v}_m) \in (\boldsymbol{M} + \boldsymbol{S})^{-1}(\boldsymbol{0}) \\ \Leftrightarrow\ &(\hat{x}, \hat{v}_1, \dots, \hat{v}_m) \text{ is a primal-dual solution to Problem 4.1;} \end{aligned} \tag{4.11}$$

- weak and strong convergence in $\boldsymbol{\mathcal{K}_V}$ are equivalent with weak and strong convergence in $\boldsymbol{\mathcal{K}}$;

([8, 32]). Let $\boldsymbol{A}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$, $\boldsymbol{B}\colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}}$, and $\boldsymbol{C}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ be defined by $\boldsymbol{A}(\boldsymbol{x}) := \boldsymbol{0}$, $\boldsymbol{B}(\boldsymbol{x}) := \boldsymbol{V}^{-1}\boldsymbol{M}(\boldsymbol{x})$, and $\boldsymbol{C}(\boldsymbol{x}) := \boldsymbol{V}^{-1}\boldsymbol{S}(\boldsymbol{x})$. For every $k \in \mathbb{N}$, define

$$\begin{aligned} \boldsymbol{z}_k &:= (x_k, v_{1,k}, \dots, v_{m,k}), \\ \boldsymbol{w}_k &:= (\overline{x}_k, \overline{v}_{1,k}, \dots, \overline{v}_{m,k}). \end{aligned} \tag{4.12}$$

It follows from (4.7), (4.8), (4.9) and (4.12) that Algorithm 4.2 can be equivalently written as

$$\boldsymbol{z}_{k+1} = \boldsymbol{w}_k + \lambda(J_{\boldsymbol{B}}(\boldsymbol{w}_k - \boldsymbol{C}\boldsymbol{w}_k) - \boldsymbol{w}_k). \tag{4.13}$$

Consequently, it has the structure of Algorithm 3.2 in the case when $A \equiv 0$. Hence, it is sufficient to check the assumptions in Theorem 3.5 to show our results.

(i) and (ii) Set $T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}} := J_{\boldsymbol{B}} \circ (\boldsymbol{I} - \boldsymbol{C})$. According to Theorem 3.5, the sequence $\{\boldsymbol{z}_{2k}\}$ is Fejér monotone with respect to $\mathrm{Fix}(T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}})$. Moreover, $\{\boldsymbol{z}_k\}$ converges weakly to $\overline{\boldsymbol{z}}$ in $\boldsymbol{\mathcal{K}_V}$ and consequently, in $\boldsymbol{\mathcal{K}}$ to $\overline{\boldsymbol{z}} \in (\boldsymbol{B} + \boldsymbol{C})^{-1}(\boldsymbol{0})$.

(iii) and (iv) We prove the statement in case C is uniformly monotone, the situation when D_i^{-1} ($i \in I$) fulfills this condition being similar. By taking into consideration (iii) of Theorem 3.5, we have

$$\boldsymbol{C}(\boldsymbol{v}_k) - \boldsymbol{C}(\overline{\boldsymbol{v}}) \to 0 \ (k \to \infty). \tag{4.14}$$

On the other hand, (4.14) and the strongly positivity of $\boldsymbol{V}$ [32, page 673] yield $\boldsymbol{V}^{-1}(\boldsymbol{S}(\boldsymbol{v}_k) - \boldsymbol{S}(\overline{\boldsymbol{v}})) = \boldsymbol{C}(\boldsymbol{v}_k) - \boldsymbol{C}(\overline{\boldsymbol{v}}) \to 0$ $(k \to \infty)$, which implies that $\boldsymbol{S}(\boldsymbol{v}_k) - \boldsymbol{S}(\overline{\boldsymbol{v}}) \to 0$ $(k \to \infty)$. Hence,

$$C(x_k) \to C(\overline{x}) \text{ and } D_i^{-1}(v_{i,k}) \to D_i^{-1}(\overline{v}_i) \ (\forall i \in I). \tag{4.15}$$

There exists an increasing function $\phi_C \colon [0, \infty) \to [0, \infty]$ that vanishes only at 0 such that

$$\phi_C(\|x_k - \overline{x}\|) \le \langle x_k - \overline{x}, C(x_k) - C(\overline{x})\rangle \le \|x_k - \overline{x}\| \|C(x_k) - C(\overline{x})\|. \tag{4.16}$$

Because $\{x_k - \overline{x}\}$ is bounded, it follows from (4.16) that $x_k \to \overline{x}$ $(k \to \infty)$.

□

Remark 4.4. In the case when $C \equiv 0$ and, for every $i \in \mathbb{N}$

$$D_i(v) = \begin{cases} \mathcal{G}_i & (v = 0), \\ \emptyset & (v \neq 0), \end{cases} \tag{4.17}$$

the conclusion of Theorem 4.3 remains valid with condition $2\overline{\beta}\rho > 1$ replaced by

$$\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 1 \tag{4.18}$$

[32, Remark 3.3].

The following convex minimization problem is strongly related to Problem 4.1.

Problem 4.5. Let $f \in \Gamma(\mathcal{H})$ and $h \colon \mathcal{H} \to \mathbb{R}$ a convex and differentiable function with a β-Lipschitzian gradient for $\beta > 0$. For every $i \in I$, let $\mathcal{G}_i$ be a real Hilbert space, let $g_i, l_i \in \Gamma(\mathcal{G}_i)$ such that l_i is $1/\nu_i$-strongly convex for $\nu_i > 0$. Let $L_i : \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator. Consider the primal problem

$$\inf_{x \in \mathcal{H}} \left\{ f(x) + \sum_{i=1}^{m} g_i \Box l_i(L_i(x)) + h(x) \right\} \tag{4.19}$$

and the dual problem

$$\sup_{v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m} \left\{ -f^* \Box h^* \left(-\sum_{i=1}^{m} L_i^*(v_i) \right) - \sum_{i=1}^{m} (g_i^*(v_i) + l^*(v_i)) \right\}. \tag{4.20}$$

Define

$$A := \partial f, B_i := \partial g_i, \ D_i := \partial l_i \ (i \in I) \text{ and } C := \nabla h.$$

The following result is a direct consequence of Theorem 4.3.

Corollary 4.6. *In Problem 4.5, suppose that*

$$0 \in \operatorname{ran}\left(\partial f + \sum_{i=1}^{m} L_i^* \left(\partial g_i \Box \partial l_i\right) L_i + \nabla h\right). \tag{4.21}$$

Define

$$\overline{\beta} := \min\{\beta, \nu_1, \dots, \nu_m\},$$

$$\rho := \min\left\{\tau^{-1}, \sigma_1^{-1}, \dots, \sigma_m^{-1}\right\}\left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2}\right)$$

and suppose that $2\overline{\beta}\rho > 1$ holds. Let $\{(x_k, v_{1,k}, \dots, v_{m,k})\}$ be a sequence generated by

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \alpha(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & (\text{if } k \text{ is even}) \\ v_{i,k} + \alpha(v_{i,k} - v_{i,k-1}) & (\text{if } k \text{ is odd}) \end{cases} \quad (\forall i \in I) \\ x_{k+1} = \overline{x}_k + \lambda\left(\operatorname{prox}_{\tau f}\left(\overline{x}_k - \tau\left(\sum_{i=1}^{m} L_i^* \overline{v}_{i,k} + \nabla h(\overline{x}_k)\right)\right) - \overline{x}_k\right) \\ v_{i,k+1} = \overline{v}_{i,k} + \lambda\left(\operatorname{prox}_{\sigma_i g_i^*}\left(\overline{v}_{i,k} + \sigma_i\left(L_i(2x_{k+1} - \overline{x}_k)\right)\right) - \overline{v}_{i,k}\right) \quad (\forall i \in I), \end{cases}$$

where $(x_0, v_{1,0}, \dots, v_{m,0}), (x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\lambda \in (0, 2 - 1/(2\overline{\beta}\rho))$ and $\alpha \in [0, 1 - 1/(2\overline{\beta}\rho))$. Then the following hold:

- (i) *$\{(x_{2k}, v_{1,2k}, \dots, v_{m,2k})\}$ is Fejér monotone with respect to the set of primal-dual solutions of Problem 4.5.*
- (ii) *There exists $\overline{\boldsymbol{v}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \in \boldsymbol{\mathcal{K}}$ such that $\{(x_k, v_{1,k}, \dots, v_{m,k})\}$ converges weakly to $\overline{\boldsymbol{v}}$ and $\overline{x}$ is an optimal solution of the problem (4.19) and $(\overline{v}_1, \dots, \overline{v}_m)$ is an optimal solution of (4.20).*
- (iii) *Suppose that h is uniformly convex. Then $\{x_k\}$ converges strongly $\overline{x}$.*
- (iv) *Suppose that l_i^* is uniformly convex for some $i \in I$. Then $\{v_{i,k}\}$ converges strongly to $\overline{v}_i$.*

4.2. **A primal-dual algorithm of Douglas-Rachford-type.** This subsection presents an algorithm for solving Problem 4.1 under additional assumption that

$$C \equiv 0. \tag{4.22}$$

Consider the sequences generated by the following algorithm:

Algorithm 4.7.

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \alpha(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & (\text{if } k \text{ is even}) \\ v_{i,k} + \alpha(v_{i,k} - v_{i,k-1}) & (\text{if } k \text{ is odd}) \end{cases} \quad (\forall i \in I) \\ p_{1,k} = J_{\tau A}\left(\overline{x}_k - \frac{\tau}{2}\sum_{i=1}^{m} L_i^* \overline{v}_{i,k}\right) \\ w_{1,k} = 2p_{1,k} - \overline{x}_k \\ \begin{cases} p_{2,i,k} = J_{\tau B_i^{-1}}(\overline{v}_{i,k} + \frac{\sigma_i}{2} L_i w_{1,k}) \\ w_{2,i,k} = 2p_{2,i,k} - \overline{v}_{i,k} \end{cases} \quad (\forall i \in I) \\ z_{1,k} = w_{1,k} - \frac{\tau}{2}\sum_{i=1}^{m} L_i^* w_{2,i,k} \\ x_{k+1} = x_k + \lambda(z_{1,k} - p_{1,k}) \\ \begin{cases} z_{2,i,k} = J_{\sigma_i D_i^{-1}}\left(w_{2,i,k} + \frac{\sigma_i}{2} L_i(2z_{1,k} - w_{1,k})\right) \\ v_{i,k+1} = \overline{v}_{i,k} + \lambda(z_{2,i,k} - p_{2,i,k}) \end{cases} \quad (\forall i \in I) \end{cases}$$

where $(x_0, v_{1,0}, \dots, v_{m,0}), (x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\lambda \in (0,2)$, $\alpha \in [0, (2-\lambda)/\lambda)$, and $\tau, \sigma_1, \dots, \sigma_m \in (0, \infty)$ such that

$$\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 4. \tag{4.23}$$

Theorem 4.8. *In Problem 4.1, suppose that (4.22) and*

$$0 \in \operatorname{ran}\left(A + \sum_{i=1}^{m} L_i^* \left(B_i \Box D_i\right) L_i\right) \tag{4.24}$$

hold. Suppose that (4.18) holds. Let $\{(x_k, v_{1,k}, \dots, v_{m,k})\}$ be a sequence generated by Algorithm 4.7 such that $\lambda \in (0,2)$ and $\alpha \in [0, (2-\lambda)/\lambda)$. Then the following hold:

(i) $\{(x_{2k}, v_{1,2k}, \dots, v_{m,2k})\}$ *is Fejér monotone with respect to* $\operatorname{Fix}(T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}})$, *where* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are defined in (4.30).*

(ii) $\{(x_k, v_{1,k}, \dots, v_{m,k})\}$ *converges weakly to a point* $\overline{\boldsymbol{x}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m)$ *in* $\operatorname{Fix}(T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}})$.

(iii) $\{(p_{1,k}, p_{2,1,k}, \dots, p_{2,m,k})\}$ *converges weakly to* $\overline{\boldsymbol{p}} = (\overline{p}_1, \overline{p}_{2,1}, \dots, \overline{p}_{2,m})$, *where*

$$\overline{p}_1 = J_{\tau A}\left(\overline{x} - \frac{\tau}{2}\sum_{i=1}^{m} L_i^* \overline{v}_i\right),$$
$$\overline{p}_{2,i} = J_{\sigma_i B_i^{-1}}\left(\overline{v}_i + \frac{\sigma_i}{2} L_i(2\overline{p}_1 - \overline{x})\right) \quad (\forall i \in I),$$

and $\overline{\boldsymbol{p}}$ *is a primal-dual solution to Problem 4.1.*

(iv) $\{(z_{1,k}, z_{2,1,k}, \dots, z_{2,m,k})\}$ *converges weakly to* $\overline{\boldsymbol{p}}$.

(v) *if A and B_i^{-1} ($i \in I$) are uniformly monotone, then $\{(p_{1,k}, p_{2,1,k}, \dots, p_{2,m,k})\}$ and $\{(z_{1,k}, z_{2,1,k}, \dots, z_{2,m,k})\}$ converge strongly to the unique primal-dual solution to Problem 4.1.*

Proof. We consider the following operators $\overline{\boldsymbol{M}}\colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}}$, $\overline{\boldsymbol{S}}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ and $\overline{\boldsymbol{V}}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ as follows:

$$\overline{\boldsymbol{M}}\colon (x, \boldsymbol{u}) \mapsto (A(x) + \frac{1}{2}\sum_{i=1}^{m} L_i^*(u_i), B_1^{-1}(u_1) - \frac{1}{2}L_1(x), \dots, B_m^{-1}(u_m) - \frac{1}{2}L_m(x)), \tag{4.25}$$

$$\overline{\boldsymbol{S}}\colon (x, \boldsymbol{u}) \mapsto \left(\frac{1}{2}\sum_{i=1}^{m} L_i^*(u_i), D_1^{-1}(u_1) - \frac{1}{2}L_1(x), \dots, D_m^{-1}(u_m) - \frac{1}{2}L_m(x)\right), \tag{4.26}$$

$$\overline{\boldsymbol{V}}\colon (x, \boldsymbol{u}) \mapsto \left(\frac{x}{\tau} - \frac{1}{2}\sum_{i=1}^{m} L_i^*(u_i), \frac{1}{\sigma_1}u_1 - \frac{1}{2}L_1(x), \dots, \frac{1}{\sigma_m}u_m - \frac{1}{2}L_m(x)\right). \tag{4.27}$$

Furthermore, consider the Hilbert space $\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}$ endowed with inner product and norm defined for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{K}}$ as

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}} := \langle \boldsymbol{x}, \overline{\boldsymbol{V}}\boldsymbol{y}\rangle_{\boldsymbol{\mathcal{K}}} \text{ and } \|\boldsymbol{x}\|_{\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}} := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle_{\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}}} \tag{4.28}$$

respectively. Then the following hold [5]:

- $\overline{\boldsymbol{M}}$ and $\overline{\boldsymbol{S}}$ are maximal monotone on $\boldsymbol{\mathcal{K}}$;
- $\overline{\boldsymbol{V}}^{-1}\overline{\boldsymbol{M}}$ and $\overline{\boldsymbol{V}}^{-1}\overline{\boldsymbol{S}}$ are maximal monotone on $\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}$;
-

$$\begin{aligned} &(\hat{x}, \hat{v}_1, \dots, \hat{v}_m) \in (\overline{\boldsymbol{V}}^{-1}\overline{\boldsymbol{M}} + \overline{\boldsymbol{V}}^{-1}\overline{\boldsymbol{S}})^{-1}(\boldsymbol{0}) \\ \Leftrightarrow\ &(\hat{x}, \hat{v}_1, \dots, \hat{v}_m) \in (\overline{\boldsymbol{M}} + \overline{\boldsymbol{S}})^{-1}(\boldsymbol{0}) \\ \Leftrightarrow\ &(\hat{x}, \hat{v}_1, \dots, \hat{v}_m) \text{ is a primal-dual solution to Problem 4.1;} \end{aligned} \tag{4.29}$$

- weak and strong convergence in $\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}$ are equivalent with weak and strong convergence in $\boldsymbol{\mathcal{K}}$;

Let $\boldsymbol{A}\colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}}$, $\boldsymbol{B}\colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}}$, and $\boldsymbol{C}\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ be defined by

$$\boldsymbol{A}(\boldsymbol{x}) := \overline{\boldsymbol{V}}^{-1}\overline{\boldsymbol{M}}(\boldsymbol{x}),\ \boldsymbol{B}(\boldsymbol{x}) := \overline{\boldsymbol{V}}^{-1}\overline{\boldsymbol{S}}(\boldsymbol{x}),\ \text{and } \boldsymbol{C}(\boldsymbol{x}) := \boldsymbol{0} \tag{4.30}$$

For every $k \in \mathbb{N}$, define

$$\begin{aligned} \boldsymbol{v}_k &:= (x_k, v_{1,k}, \dots, v_{m,k}), \\ \overline{\boldsymbol{v}}_k &:= (\overline{x}_k, \overline{v}_{1,k}, \dots, \overline{v}_{m,k}), \\ \boldsymbol{y}_k &:= (p_{1,k}, \dots, p_{m,k}), \\ \boldsymbol{z}_k &:= (z_{1,k}, \dots, z_{m,k}). \end{aligned} \tag{4.31}$$

It follows from (4.25), (4.26), (4.27) and (4.31) that Algorithm 4.7 can equivalently be written in the form

$$\begin{cases} \overline{\boldsymbol{V}}(\overline{\boldsymbol{v}}_k - \boldsymbol{y}_k) \in \overline{\boldsymbol{M}}(\boldsymbol{y}_k) \\ \overline{\boldsymbol{V}}(2\boldsymbol{y}_k - \overline{\boldsymbol{v}}_k - \boldsymbol{z}_k) \in \overline{\boldsymbol{S}}(\boldsymbol{z}_k) \\ \boldsymbol{v}_{k+1} = \overline{\boldsymbol{v}}_k + \lambda(\boldsymbol{z}_k - \boldsymbol{y}_k). \end{cases} \tag{4.32}$$

We can verify from (4.32) that

$$\begin{cases} \boldsymbol{y}_k = J_{\boldsymbol{B}}(\overline{\boldsymbol{v}}_k), \\ \boldsymbol{z}_k = J_{\boldsymbol{A}}(2\boldsymbol{y}_k - \overline{\boldsymbol{v}}_k), \\ \boldsymbol{v}_{k+1} = \overline{\boldsymbol{v}}_k + \lambda(\boldsymbol{z}_k - \boldsymbol{y}_k). \end{cases} \tag{4.33}$$

Consequently, (4.33) has the structure of Algorithm 3.2.

(i), (ii) and (iii) According to Theorem 3.5, the sequence $\{\boldsymbol{v}_{2k}\}$ is Fejér monotone with respect to $\mathrm{Fix}(T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}})$. Moreover, $\{\boldsymbol{z}_k\}$ converges weakly to $\overline{\boldsymbol{z}}$ in $\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}$ and consequently, in $\boldsymbol{\mathcal{K}}$ to $\overline{\boldsymbol{z}} \in (\boldsymbol{A}+\boldsymbol{B})^{-1}(\boldsymbol{0})$.

(v) The uniform monotonicity of A and B_i ($\forall i \in I$) implies uniform monotonicity of $\boldsymbol{A}$ on $\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}$ [5, Theorem 2.1 (ii)]. Therefore, the claim follows from Theorem 3.5 (vi).

□

We consider the following convex optimization problems.

Problem 4.9. Let $f \in \Gamma(\mathcal{H})$. For every $i \in I$, let $\mathcal{G}_i$ be a real Hilbert space, let $g_i, l_i \in \Gamma(\mathcal{G}_i)$ and $L_i : \mathcal{H} \to \mathcal{G}_i$ a nonzero bounded linear operator. Consider the primal problem

$$\inf_{x\in\mathcal{H}} \left\{ f(x) + \sum_{i=1}^{m} (g_i \Box l_i)(L_i(x)) \right\} \tag{4.34}$$

and the dual problem

$$\sup_{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m} \left\{ -f^*\left(-\sum_{i=1}^{m} L_i^* v_i\right) - \sum_{i=1}^{m} (g_i^*(v_i) + l_i^*(v_i)) \right\}. \tag{4.35}$$

Define

$$A := \partial f, B_i := \partial g_i \text{ and } D_i := \partial l_i \ (i \in I).$$

The following result is a direct consequence of Theorem 4.8.

Corollary 4.10. *In Problem 4.9, suppose that*

$$0 \in \mathrm{ran}\left(\partial f + \sum_{i=1}^{m} L_i^* \left(\partial g_i \Box \partial l_i\right) L_i\right). \tag{4.36}$$

Let $\{(x_k, v_{1,k}, \dots, v_{m,k})\}$ *be a sequence generated by*

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \alpha(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & (\text{if } k \text{ is even}) \\ v_{i,k} + \alpha(v_{i,k} - v_{i,k-1}) & (\text{if } k \text{ is odd}) \end{cases} \quad (\forall i \in I) \\ p_{1,k} = \operatorname{prox}_{\tau f}\left(\overline{x}_k - \frac{\tau}{2}\sum_{i=1}^m L_i^* \overline{v}_{i,k}\right) \\ w_{1,k} = 2p_{1,k} - \overline{x}_k \\ \begin{cases} p_{2,i,k} = \operatorname{prox}_{\tau g_i^*}(\overline{v}_{i,k} + \frac{\sigma_i}{2} L_i w_{1,k}) \\ w_{2,i,k} = 2p_{2,i,k} - \overline{v}_{i,k} \end{cases} \quad (\forall i \in I) \\ z_{1,k} = w_{1,k} - \frac{\tau}{2}\sum_{i=1}^m L_i^* w_{2,i,k} \\ x_{k+1} = x_k + \lambda(z_{1,k} - p_{1,k}) \\ \begin{cases} z_{2,i,k} = \operatorname{prox}_{\sigma_i l_i^*}\left(w_{2,i,k} + \frac{\sigma_i}{2} L_i(2z_{1,k} - w_{1,k})\right) \\ v_{i,k+1} = \overline{v}_{i,k} + \lambda(z_{2,i,k} - p_{2,i,k}) \end{cases} \quad (\forall i \in I) \end{cases}$$

such that $(x_0, v_{1,0}, \dots, v_{m,0}), (x_1, v_{1,1}, \dots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ $\lambda \in (0, 2)$ *and* $\alpha \in [0, (2-\lambda)/\lambda)$. *Suppose that (4.18) holds. The following hold:*

(i) $\{(x_{2k}, v_{1,2k}, \dots, v_{m,2k})\}$ *is Fejér monotone with respect to* $\operatorname{Fix}(T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}})$, *where* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are defined in (4.30) in case* $A := \partial f$, $B_i := \partial g_i$ *and* $D_i := \partial l_i$ $(i \in I)$.

(ii) $\{(x_k, v_{1,k}, \dots, v_{m,k})\}$ *converges weakly to a point* $\overline{\boldsymbol{x}} = (\overline{x}, \overline{v}_1, \dots, \overline{v}_m)$ *in* $\operatorname{Fix}(T_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}})$.

(iii) $\{(p_{1,k}, p_{2,1,k}, \dots, p_{2,m,k})\}$ *converges weakly to* $\overline{\boldsymbol{p}}$, *where* $\overline{\boldsymbol{p}} = (\overline{p}_1, \overline{p}_{2,1}, \dots, \overline{p}_{2,m})$,

$$\overline{p}_1 = \operatorname{prox}_{\tau f}\left(\overline{x} - \frac{\tau}{2}\sum_{i=1}^m L_i^* \overline{v}_i\right),$$

$$\overline{p}_{2,i} = \operatorname{prox}_{\sigma_i g_i^*}\left(\overline{v}_i + \frac{\sigma_i}{2} L_i(2\overline{p}_1 - \overline{x})\right) \ (\forall i \in I),$$

and $\overline{\boldsymbol{p}}$ *is a primal-dual solution to Problem 4.9.*

(iv) $\{(z_{1,k}, z_{2,1,k}, \dots, z_{2,m,k})\}$ *converges weakly to* $\overline{\boldsymbol{p}}$.

(v) *if* f *and* g_i^* $(i \in I)$ *are uniformly convex, then* $\{(p_{1,k}, p_{2,1,k}, \dots, p_{2,m,k})\}$ *and* $\{(z_{1,k}, z_{2,1,k}, \dots, z_{2,m,k})\}$ *converge strongly to the unique primal-dual solution to Problem 4.9.*

5. Numerical experiments

In this section, we provide numerical experiments and compare our proposed algorithms with some existing algorithms in the literature. All codes were written in MATLAB R2022a and performed on a PC Desktop Intel Core i7 4.0GHz, RAM 16.00 GB.

We consider a linear time-invariant system represented by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t),\ t \geq 0, \tag{5.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^d$, $u(t) \in \mathbb{R}$, $A \in \mathbb{R}^{d\times d}$, and $\boldsymbol{b} \in \mathbb{R}^d$. $\boldsymbol{x}(t)$ is called *state*, $u(t)$ is called *control*, and (5.1) is called *state equation.* The state equation can describe some problems arising in engineering, statistics, etc. Here, we are interested in the optimal control problem.

Let $T > 0$ be the final time of control. For the system (5.1), $u(t)$ is said to be *feasible* if $u(t)$ steers the state from a given initial state $\boldsymbol{x}(0) = \boldsymbol{\xi}$ to $\boldsymbol{x}(T) = \boldsymbol{0}$, and satisfies the magnitude constraints $\|u\|_\infty \leq 1$, where $\|u\|_\infty = \sup_{t\in[0,T]} |u(t)|$.

5.1. **L^1-optimal control problem.** Now we consider the following L^1-optimal control problem.

Problem 5.1. (L^1-optimal control problem) For a given initial state $\boldsymbol{\xi} \in \mathbb{R}^d$, find a feasible control u that minimizes $\|u\|_1$, where $\|u\|_1 = \int_0^T |u(t)|dt$.

The solution of Problem 5.1 is said to be L^1-*optimal control.* Here, we assume the existence of the L^1-optimal control.

Remark 5.2. Problem 5.1 is known as minimum fuel control, which was widely studied for rocket control [21, 24].

To solve Problem 5.1, we consider the discretization technique for the time interval $[0, T]$. To this end, we discretize $[0, T]$ into n subintervals $[0, T] = [0, h) \cup \cdots \cup [(n-1)h, nh]$, where h is the discretization step chosen such that $T = nh$. Here, we assume that the state $\boldsymbol{x}(t)$ and the control $u(t)$ are constant over each subinterval. Define $\boldsymbol{x}_d(l) := \boldsymbol{x}(lh)$, $u_d(l) := u(lh), l = 0, 1, \ldots, n-1$, $\boldsymbol{x}_d(n) := \boldsymbol{x}(T)$, $A_d := e^{Ah}$ and $\boldsymbol{b}_d := \int_0^h e^{At}\boldsymbol{b}dt$. Then, for $t = 0, h, \ldots, nh$, the state equation (5.1) is described as

$$\boldsymbol{x}_d(l+1) = A_d\boldsymbol{x}_d(l) + \boldsymbol{b}_d u_d(l), l = 0, 1, \ldots, n-1. \tag{5.2}$$

Let $\boldsymbol{\xi}$ be the initial state, that is, $\boldsymbol{x}(0) = \boldsymbol{\xi}$. Define $\boldsymbol{u} := (u_d(0), u_d(1), \ldots, u_d(n-1))^T \in \mathbb{R}^n$, $\Phi := (A_d^{n-1}\boldsymbol{b}_d, A_d^{n-2}\boldsymbol{b}_d, \ldots, \boldsymbol{b}_d)$ and $\boldsymbol{\zeta} := -A_d\boldsymbol{\xi}$. Then the final state $\boldsymbol{x}(T)$ can be described as

$$\boldsymbol{x}(T) = \boldsymbol{x}_d(n) = -\boldsymbol{\zeta} + \Phi\boldsymbol{u}. \tag{5.3}$$

From the discussion above, Problem 5.1 can be approximated by the following n-dimensional l^1 optimization problem:

Problem 5.3. (l^1 optimization problem)

$$\min_{\boldsymbol{u}\in\mathbb{R}^n} \left\{\|\boldsymbol{u}\|_{l^1} + i_{\mathcal{C}_1}(\boldsymbol{u}) + i_{\mathcal{C}_2}(\Phi\boldsymbol{u})\right\}, \tag{5.4}$$

where $\mathcal{C}_1 = \{\boldsymbol{v} \in \mathbb{R}^n : \|\boldsymbol{v}\|_{l^\infty} \leq 1\}$, $\mathcal{C}_2 = \{\boldsymbol{\zeta}\}$, $\|\boldsymbol{x}\|_{l^1}$ is the l^1 norm defined by $\|\boldsymbol{x}\|_{l^1} = \sum_{i=1}^n |x_i|$, $\|\boldsymbol{x}\|_{l^\infty}$ is the l^∞ norm defined by $\|\boldsymbol{x}\|_{l^\infty} = \max_{i\in\{1,2,\ldots,n\}} |x_i|$, and $i_{\mathcal{C}}$ is the indicator function of a nonempty closed convex set $\mathcal{C}$ defined by

$$i_{\mathcal{C}}(\boldsymbol{x}) = \begin{cases} 0 & (\boldsymbol{x} \in \mathcal{C}) \\ \infty & (\text{otherwise}) \end{cases}.$$

Here, we consider a simple example with a 2-dimensional (i.e. $d = 2$) linear control system (5.1) with $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ [24, Example 7.2]. We set the final time $T = 5$, and the initial and final states as $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{x}(5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Figure 1 shows an approximated L^1-optimal control obtained by a software package cvx[1] with MATLAB and an associated optimal state trajectory $\{\begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T : 0 \leq t \leq 5\}$.

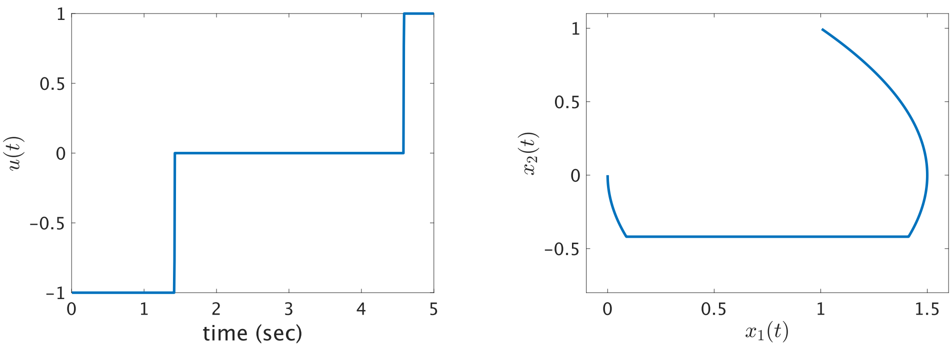

FIG. 1. The left figure shows an approximated L^1-optimal control. The right figure shows the state $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$ according to $u(t)$ in the left figure.

We now consider the case where the number of subintervals is $1,000$ (i.e. $n = 1,000$). By setting $f := \| \cdot \|_{l^1}$, $g_1 := i_{\mathcal{C}_1}$, $g_2 := i_{\mathcal{C}_2}$, $h(\boldsymbol{x}) := \mathbf{0}$, $L_1 := I_n$, $L_2 := \Phi$, $l_1 = l_2 := i_{\{\mathbf{0}\}}$, Problem 5.3 is reduced to the from of (4.19). Then the algorithm in Corollary 4.6 can be implemented. As addressed in [24], $\operatorname{prox}_{\tau f}$ is the soft thresholder on $\{0\}$ and $\operatorname{prox}_{\sigma_i g_i} = P_{\mathcal{C}_i}$ is given by

$$P_{\mathcal{C}_1}(\boldsymbol{u}) = \begin{pmatrix} \operatorname{sgn}(u_1)\min\{|u_1|, 1\} \\ \operatorname{sgn}(u_2)\min\{|u_2|, 1\} \\ \vdots \\ \operatorname{sgn}(u_n)\min\{|u_n|, 1\} \end{pmatrix} \text{ and } P_{\mathcal{C}_2}(\boldsymbol{v}) = \boldsymbol{\zeta},$$

and hence the computation in the algorithm is simple. Furthermore, by setting $f := \| \cdot \|_{l^1}$, $g_1 := i_{\mathcal{C}_1}$, $g_2 := i_{\mathcal{C}_2}$, $L_1 := I_n$, $L_2 := \Phi$, $l_1 = l_2 := i_{\{\mathbf{0}\}}$, Problem 5.3 is also reduced to the from of (4.34). Then the algorithm in Corollary 4.10 can be implemented. Now we are ready to apply the proposed algorithms to Problem 5.3.

We give numerical comparisons of the proposed algorithms (Proposed 1 and Proposed 2) in Corollaries 4.6 and 4.10, respectively and other existing algorithms, namely the algorithms of the primal-dual forward-backward-type (FB) from [7, 4,

[1] http://cvxr.com/cvx/

9, 32], the inertial primal-dual forward-backward-type (IFB) from [13], the primal-dual Douglas-Rachford-type (DR) from [5] and the inertial primal-dual Douglas-Rachford-type (IDR) from [6]. We also give numerical comparisons of the proposed algorithms with different choices of inertial parameter α. For the proposed algorithms, we used $\lambda = 1$ and set $\tau = 1$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.5$, which satisfy (4.18) and (4.23). We chose 10 random initial points $(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)}) = (x_1^{(i)}, v_{1,1}^{(i)}, v_{2,1}^{(i)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ $(i = 1, 2, \dots, 10)$ and every entry of the initial point is uniformly generated from $[-100, 100]$. The computational results are reported in Figures 2 and 3. Figure 2 demonstrates the following functions

$$
\begin{aligned}
D_k^{(i)} &:= \|(x_k^{(i)}, v_{1,k}^{(i)}, v_{2,k}^{(i)}) - (\overline{x}, \overline{v}_1, \overline{v}_2)\|_{\boldsymbol{\mathcal{K}_V}} \text{ and} \\
D_k &:= (1/10) \sum_{i=1}^{10} D_k^{(i)},
\end{aligned} \tag{5.5}
$$

where $(\overline{x}, \overline{v}_1, \overline{v}_2)$ denotes the primal-dual solution obtained by cvx, $(x_k^{(i)}, v_{1,k}^{(i)}, v_{2,k}^{(i)})$ is the sequence generated by $(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)})$ and each of Proposed 1, FB and IFB, and $\|\cdot\|_{\boldsymbol{\mathcal{K}_V}}$ is defined in (4.10). The computation results of Proposed 1, FB and IFB are presented in the left part of Figure 2. The inertial parameter for IFB was chosen as $\alpha \equiv 0.2$ according to the theoretic results provided in [13, 6, 14]. One can observe that Proposed 1 is faster than both FB and IFB. Furthermore, the right part of Figure 2 shows the computational results of Proposed 1 with different inertial parameters $(\alpha = 0.1, 0.5, 0.9)$. As we can see, the increase of α implies a faster approach of the solution. We see from Figure 2 that both the sequence generated by our algorithm is convergent.

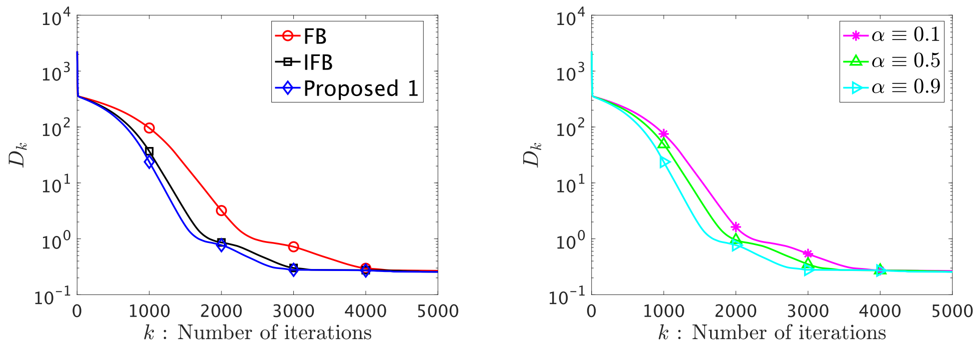

FIG. 2. The computation results of the relation between the distance to a solution and iteration number. The left figure shows the numerical comparison of (5.4) with Proposed 1, FB and IFB. The right figure shows the result of Proposed 1 for the case in which correspond to different choices of α $(\alpha = 0.1, 0.5, 0.9)$.

Figure 3 demonstrates the following functions

$$\begin{aligned} \overline{D}_k^{(i)} &:= \|(p_{1,k}^{(i)}, p_{2,1,k}^{(i)}, p_{2,2,k}^{(i)}) - (\overline{p}_1, \overline{p}_{2,1}, \overline{p}_{2,2})\|_{\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}} \text{ and} \\ \overline{D}_k &:= (1/10)\sum_{i=1}^{10} \overline{D}_k^{(i)}, \end{aligned} \tag{5.6}$$

where $(\overline{p}_1, \overline{p}_{2,1}, \overline{p}_{2,2})$ is the primal-dual solution obtained by cvx, $(p_{1,k}^{(i)}, p_{2,1,k}^{(i)}, p_{2,2,k}^{(i)})$ is the sequence generated by $(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)})$ and each of Proposed 2, DR and IDR, and $\|\cdot\|_{\boldsymbol{\mathcal{K}}_{\overline{\boldsymbol{V}}}}$ is defined in (4.28). The computation results of Proposed 2, DR and IDR are presented in the left part of Figure 3. The inertial parameter for IDR was chosen as $\alpha \equiv 0.2$ according to the theoretic results provided in [6]. One can observe that Proposed 2 is faster than both DR and IDR. Furthermore, the right part of Figure 3 shows the computational results of Proposed 2 with different inertial parameters ($\alpha = 0.1, 0.5, 0.9$). As we can see, the increase of α implies a faster approach of the solution. We see from Figure 3 that both the sequence generated by our algorithm is convergent.

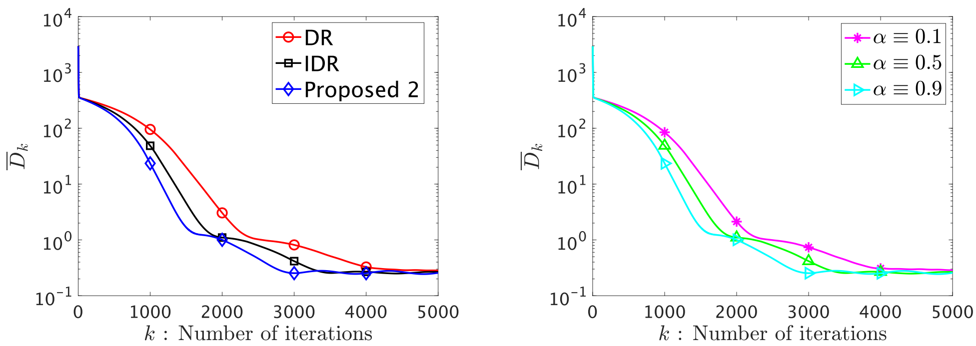

FIG. 3. The computation results of the relation between the distance to a solution and iteration number. The left figure shows the numerical comparison of (5.4) with Proposed 2, DR and IDR. The right figure shows the result of Proposed 2 for the case in which correspond to different choices of α ($\alpha = 0.1, 0.5, 0.9$).

5.2. L^1/L^2-optimal control problem.

Next we consider the following L^1/L^2-optimal control problem.

Problem 5.4. (L^1/L^2-optimal control problem) Let $\eta > 0$ be the weight parameter. For a given initial state $\boldsymbol{\xi} \in \mathbb{R}^d$, find a feasible control u that minimizes $\eta\|u\|_1 + \frac{1}{2}\|u\|_2^2$, where $\|u\|_2^2 = \int_0^T |u(t)|^2 dt$.

The solution of Problem 5.4 is said to be *L^1/L^2-optimal control*. Here, we assume the existence of the L^1/L^2-optimal control.

Remark 5.5. L^1/L^2-optimal control was first proposed in [25]. L^1/L^2-optimal control is desirable for some applications in which the actuators cannot move abruptly [24].

From the discussion in Subsection 5.1, Problem 5.4 can be approximated by the following n-dimensional l^1/l^2 optimization problem:

Problem 5.6. (l^1/l^2 optimization problem)

$$\min_{\boldsymbol{u}\in\mathbb{R}^n} \left\{ \eta\|\boldsymbol{u}\|_{l^1} + \frac{1}{2}\|\boldsymbol{u}\|_{l^2}^2 + i_{\mathcal{C}_1}(\boldsymbol{u}) + i_{\mathcal{C}_2}(\Phi\boldsymbol{u}) \right\}, \tag{5.7}$$

where $\|\boldsymbol{x}\|_{l^2}$ is the l^2 norm defined by $\|\boldsymbol{x}\|_{l^2} = \sqrt{\sum_{i=1}^n x_i^2}$.

We consider the same example in Subsection 5.1 (i.e., $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{x}(5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$) and set $\eta = 1$. Figure 4 shows an approximated L^1/L^2-optimal control obtained by cvx and an associated optimal state trajectory $\{\begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T : 0 \le t \le 5\}$.

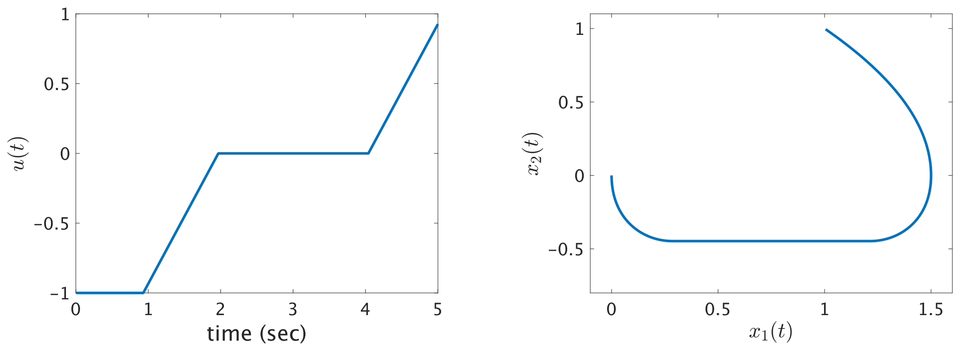

FIG. 4. The left figure shows an approximated L^1/L^2-optimal control. The right figure shows the state $\boldsymbol{x}(t) = \begin{pmatrix} x_1(t) & x_2(t) \end{pmatrix}^T$ according to $u(t)$ in the left figure.

By setting $f := \|\cdot\|_{l^1}$, $g_1 := \frac{1}{2}\|\cdot\|_{l^2}^2$, $g_2 := i_{\mathcal{C}_1}$, $g_3 := i_{\mathcal{C}_2}$, $L_1 := I_n$, $L_2 := I_n$, $L_3 := \Phi$, Problem 5.6 is reduced to the from of (4.19), and hence Proposed 1 can be implemented. Note that the proximal mapping of g_1^* is given by $\mathrm{prox}_{\sigma_1 g^*}(x) = I - \sigma_1 \mathrm{prox}_{(1/\sigma_1)g} \circ \sigma_1^{-1} I$ from the extended Moreau's decomposition formula. Furthermore, by setting $f := \|\cdot\|_{l^1}$, $g_1 := \frac{1}{2}\|\cdot\|_{l^2}^2$, $g_2 := i_{\mathcal{C}_1}$, $g_3 := i_{\mathcal{C}_2}$, $L_1 := I_n$, $L_2 := I_n$, $L_3 := \Phi$, $l_1 = l_2 = l_3 := i_{\{\mathbf{0}\}}$, Problem 5.6 is reduced to the from of (4.34), and hence Proposed 2 can be implemented. Now we are ready to apply the proposed algorithms to Problem 5.6.

For the proposed algorithms, we used $\lambda = 1$ and set $\tau = 1$, $\sigma_1 = 0.3$, $\sigma_2 = 0.3$ and $\sigma_3 = 0.5$ which satisfy (4.18) and (4.23). We chose 10 random initial points

$(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)}, v_{3,0}^{(i)}) = (x_1^{(i)}, v_{1,1}^{(i)}, v_{2,1}^{(i)}, v_{3,1}^{(i)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ $(i = 1, 2, \ldots, 10)$ and every entry of the initial point is uniformly generated from $[-100, 100]$. The computational results are reported in Figures 5 and 6. Figure 5 demonstrates the following functions

$$D_k^{(i)} := \|(x_k^{(i)}, v_{1,k}^{(i)}, v_{2,k}^{(i)}, v_{3,k}^{(i)}) - (\overline{x}, \overline{v}_1, \overline{v}_2, \overline{v}_3)\|_{\boldsymbol{\mathcal{K}_V}} \text{ and}$$
$$D_k := (1/10)\sum_{i=1}^{10} D_k^{(i)}, \tag{5.8}$$

where $(\overline{x}, \overline{v}_1, \overline{v}_2, \overline{v}_3)$ is the primal-dual solution obtained by cvx, $(x_k^{(i)}, v_{1,k}^{(i)}, v_{2,k}^{(i)}, v_{3,k}^{(i)})$ is the sequence generated by $(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)}, v_{3,0}^{(i)})$ and each of Proposed 1, FB and IFB, and $\|\cdot\|_{\boldsymbol{\mathcal{K}_V}}$ is defined in (4.10). The computation results of Proposed 1, FB and IFB are presented in the left part of Figure 5. The inertial parameter for IFB was chosen as $\alpha \equiv 0.2$ according to the theoretic results provided in [13, 6, 14]. One can observe that Proposed 1 is faster than both FB and IFB. Furthermore, the right part of Figure 5 shows the computational results of Proposed 1 with different inertial parameters ($\alpha = 0.1, 0.5, 0.9$). As we can see, the increase of α implies a faster approach of the solution. We see from Figure 5 that both the sequence generated by our algorithm is convergent

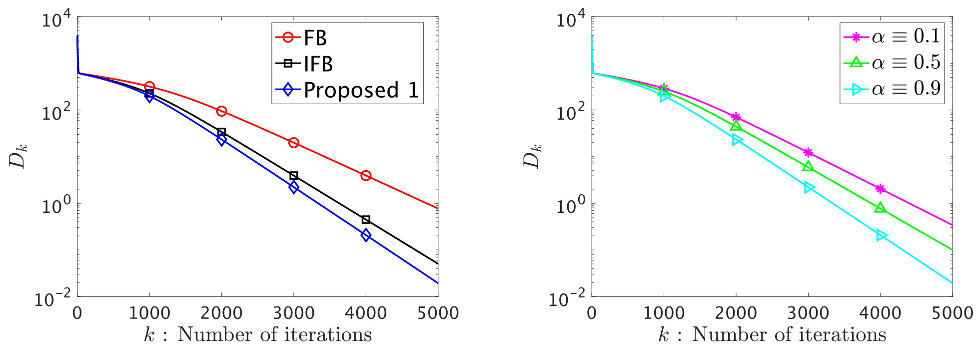

FIG. 5. The computation results of the relation between the distance to a solution and iteration number. The left figure shows the numerical comparison of (5.7) with Proposed 1, FB and IFB. The right figure shows the result of Proposed 1 for the case in which correspond to different choices of α ($\alpha = 0.1, 0.5, 0.9$).

Figure 6 demonstrates the following functions

$$\overline{D}_k^{(i)} := \|(p_{1,k}^{(i)}, p_{2,1,k}^{(i)}, p_{2,2,k}^{(i)}, p_{2,3,k}^{(i)}) - (\overline{p}_1, \overline{p}_{2,1}, \overline{p}_{2,2}, \overline{p}_{2,3})\|_{\boldsymbol{\mathcal{K}_{\overline{V}}}} \text{ and}$$
$$\overline{D}_k := (1/10)\sum_{i=1}^{10} \overline{D}_k^{(i)}, \tag{5.9}$$

where $(\overline{p}_1, \overline{p}_{2,1}, \overline{p}_{2,2}, \overline{p}_{2,3})$ is the primal-dual solution obtained by cvx, $(p_{1,k}^{(i)}, p_{2,1,k}^{(i)}, p_{2,2,k}^{(i)}, p_{2,3,k}^{(i)})$ is the sequence generated by $(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)}, v_{3,0}^{(i)})$ and each of

Proposed 2, DR and IDR, and $\|\cdot\|_{\mathcal{K}_{\overline{V}}}$ is defined in (4.28). The computation results of Proposed 2, DR and IDR are presented in the left part of Figure 6. The inertial parameter for IDR was chosen as $\alpha \equiv 0.2$ according to the theoretic results provided in [6]. One can observe that adding alternated inertial terms is no guarantee of improving the algorithm's efficiency during the early iterations. Furthermore, the right part of Figure 6 shows the computational results of Proposed 2 with different inertial parameters ($\alpha = 0.1, 0.5, 0.9$). As we can see, the increase of α implies a faster approach of the solution.

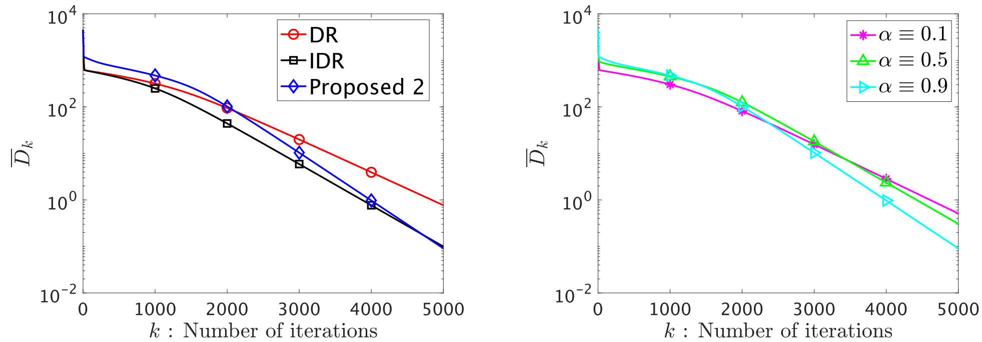

FIG. 6. The computation results of the relation between the distance to a solution and iteration number. The left figure shows the numerical comparison of (5.7) with Proposed 2, DR and IDR. The right figure shows the result of Proposed 1 for the case in which correspond to different choices of α ($\alpha = 0.1, 0.5, 0.9$).

6. Conclusions

In this paper, we have proposed an alternated inertial three-operator splitting algorithm and studied its convergence properties. The proposed algorithm can be applied for finding a zero point of the sum of three maximally monotone operators, where one is cocoercive. In contrast to the existing algorithms for this class of problems [10], our new algorithm is Fejér monotone and the assumptions on inertial parameters and relaxation parameters are mild. Moreover, by making use of the proposed algorithm and primal-dual techniques, we present the weakly convergent alternated inertial primal-dual splitting algorithm. Numerical experiments have been illustrated to show the effectiveness of the proposed algorithms.

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