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A NOTE ON COLLECTIVELY FIXED POINT RESULTS AND MINIMAX TYPE INEQUALITIES

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Dedicated to Professor K. Goebel with much admiration

Abstract. Using recent collectively fixed point results of the author for multivalued maps we will establish a new analytic alternative. This analytic alternative will then generate a minimax type inequality and as an application we will consider the existence of an equilbrium point for a generalized game.

1. Introduction

In this paper we begin in Section 1 by presenting a variety of new collectively fixed point results of the author [6] for multivalued maps in both the compact, condensing and coercive cases; we refer the reader to [2, 4, 5] for some results in the literature. These collectively fixed point results enable us in Section 2 to obtain an analytic alternative where the appropriate maps are either compact, condensing or coercive. This analytic alternative with an appropriate choice will then generate a minimax type inequality motivated from the inequalities of Ky Fan [3]. As an application we present equilibrium results (the existence of an equilibrium point) for a generalized game (or abstract economy).

Now we describe the class of maps considered. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and let F a multifunction. We say $F \in \Phi^\star(Z, W)$ [1] if W is convex and there exists a map $S : Z \to W$ with $S(x) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ and has convex values for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$.

First we recall a result for compact maps [6]. In the below I is an index set.

Theorem 1.1. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and in addition there exists a map $T_i : X \to X_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $x \in X$ there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Finally assume for each $i \in I$ that there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ (here x_i is the projection of x on X_i).*

Next we recall a result for condensing type maps [6].

Theorem 1.2. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ and in addition*

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there exists a map $T_i : X \to X_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $x \in X$ there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Finally assume there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i \in I} F_i(x)$ for $x \in X$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.

Finally we recall a result for coercive type maps [6].

Theorem 1.3. *Let $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \equiv \prod_{i \in I} X_i \to X_i$ and in addition there exists a map $T_i : X \to X_i$ with $T_i(x) \subseteq F_i(x)$ for $x \in X$, $T_i(x)$ has convex values for $x \in X$ and $T_i^{-1}(w)$ is open (in X) for each $w \in X_i$. Also suppose for each $x \in X$ there exists a $j \in I$ with $T_j(x) \neq \emptyset$. Finally assume there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in I$ with $T_j(x) \cap Y_j \neq \emptyset$. Then there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$.*

2. Applications

We begin by presenting an analytic alternative which will then generate our minimax type inequalities.

Theorem 2.1. *Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i and let $X = \prod_{i \in I} X_i$. For $i \in I$ let $f_i, g_i : X \times X_i \to \mathbf{R}$ with $g_i(x, y) \leq f_i(x, y)$ for all $(x, y) \in X \times X_i$, let $\lambda_i \in \mathbf{R}$ and let*

$$F_i(x) = \{z_i \in X_i : \ f_i(x, z_i) > \lambda_i\} \quad \text{and}$$
$$S_i(x) = \{z_i \in X_i : \ g_i(x, z_i) > \lambda_i\} \ \text{ for } \ x \in X.$$

Assume for each $i \in I$ that $S_i(x)$ is convex valued for each $x \in X$ and $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$. In addition suppose either

(1) *for each $i \in I$ there exists a convex compact set K_i with $F_i(X) \subseteq K_i \subseteq X_i$,*

or

(2) *there exists a convex compact set K of X with $F(K) \subseteq K$ where $F(x) = \prod_{i \in I} F_i(x)$ for $x \in X$,*

or

(3) *there is a compact subset K of X and for each $i \in I$ a convex compact subset Y_i of X_i such that for each $x \in X \backslash K$ there exists a $j \in I$ with $S_j(x) \cap Y_j \neq \emptyset$,*

hold. Then either

(A1) *there exists a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ (i.e. $f_i(x, x_i) > \lambda_i$),*

or

(A2) *there exists a $x \in X$ with $\sup_{z_i \in X_i} g_i(x, z_i) \leq \lambda_i$ for all $i \in I$*

occurs.

Proof. Note either (a). there exists a $x \in X$ with $S_i(x) = \emptyset$ for all $i \in I$ or (b). for each $x \in X$ there exists a $i \in I$ with $S_i(x) \neq \emptyset$.

Suppose (a) holds. Then for this x we have $S_i(x) = \emptyset$ for all $i \in I$ so for all $i \in I$ we have $g_i(x, z_i) \le \lambda_i$ for $z_i \in X_i$ (so $\sup_{z_i \in X_i} g_i(x, z_i) \le \lambda_i$).

Suppose (b) holds. Note S_i is a selection of F_i so Theorem 1.1 (if (1) occurs)) or Theorem 1.2 (if (2) occurs) or Theorem 1.3 (if (3) occurs) guarantees a $x \in X$ and a $i \in I$ with $x_i \in F_i(x)$ so $f_i(x, x_i) > \lambda_i$ (i.e. (A1) occurs). □

Remark 2.2. Fix $i \in I$. If

$$\text{for each } \ x \in X,\ y \mapsto g_i(x,y) \ \text{ is quasi–concave on } \ X_i,$$

then $S_i(x)$ is convex valued for each $x \in X$, whereas if

$$\text{for each } \ y \in X_i,\ x \mapsto g_i(x,y) \ \text{ is lower semicontinuous on } \ X,$$

then $S_i^{-1}(w)$ is open (in X) for each $w \in X_i$.

Next we show how our analytic alternative generates minimax type inequalities.

Theorem 2.3. *Let I be an index set and $\{X_i\}_{i \in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i and let $X = \prod_{i \in I} X_i$. For $i \in I$ let $f_i, g_i : X \times X_i \to \mathbf{R}$ with $g_i(x, y) \le f_i(x, y)$ for all $(x, y) \in X \times X_i$. For $i \in I$ let $\lambda_i = \sup_{x \in X} [f_i(x, x_i)]$ and let $J = \{i \in I : \ \lambda_i = \infty\}$ (note J could be empty). For $i \in I \backslash J$ let*

$$F_i(x) = \{z_i \in X_i : \ f_i(x, z_i) > \lambda_i\} \ \text{ and }$$
$$S_i(x) = \{z_i \in X_i : \ g_i(x, z_i) > \lambda_i\} \ \text{ for } \ x \in X^\star = \prod_{i \in I \backslash J} X_i.$$

Assume for each $i \in I \backslash J$ that $S_i(x)$ is convex valued for each $x \in X^\star$ and $S_i^{-1}(w)$ is open (in $X^\star$) for each $w \in X_i$. In addition suppose either

(1) *for each $i \in I \backslash J$ there exists a convex compact set K_i with $F_i(X^\star) \subseteq K_i \subseteq X_i$,*

or

(2) *there exists a convex compact set K of $X^\star$ with $F(K) \subseteq K$ where $F(x) = \prod_{i \in I \backslash J} F_i(x)$ for $x \in X^\star$,*

or

(3) *there is a compact subset K of $X^\star$ and for each $i \in I \backslash J$ a convex compact subset Y_i of X_i such that for each $x \in X^\star \backslash K$ there exists a $j \in I \backslash J$ with $S_j(x) \cap Y_j \neq \emptyset$,*

hold. Then there exists a $y \in X$ with

$$\sup_{z_i \in X_i} g_i(y, z_i) \le \sup_{x \in X} f_i(x, x_i) \ \text{ for all } \ i \in I. \tag{2.1}$$

Proof. If $i \in J$ then $\lambda_i = \sup_{x \in X} [f_i(x, x_i)] = \infty$ so trivially

$$\sup_{z_i \in X_i} g_i(y, z_i) \le \sup_{x \in X} f_i(x, x_i) \ \text{ for all } \ i \in J \ \text{ and for all } \ y \in X. \tag{2.2}$$

Next let $i \in I\backslash J$ so from Theorem 2.1 (with I replaced by $I\backslash J$ and X replaced by $X^\star$) either $(A1)$. there exists a $x \in X^\star$ and a $i \in I\backslash J$ with $f_i(x, x_i) > \lambda_i$, or $(A2)$. there exists a $x \in X^\star$ with $\sup_{z_i \in X_i} g_i(x, z_i) \leq \lambda_i$ for all $i \in I\backslash J$. Next we note $(A1)$ cannot occur since if there exists a $x \in X^\star$ and a $i \in I\backslash J$ with $f_i(x, x_i) > \lambda_i$, then this means $f_i(x, x_i) > \sup_{x \in X} [f_i(x, x_i)]$, which is a contradiction. Thus there exists a $y \in X^\star$ with $\sup_{z_i \in X_i} g_i(y, z_i) \leq \lambda_i$ for all $i \in I\backslash J$ i.e. there exists a $y \in X^\star$ with

$$(2.3) \qquad \sup_{z_i \in X_i} g_i(y, z_i) \leq \sup_{x \in X} f_i(x, x_i) \ \text{ for all } \ i \in I\backslash J$$

Now (2.2) and (2.3) guarantee (2.1). $\square$

A special case of Theorem 2.3 is if I is the singleton set (see [3]).

Corollary 2.4. *Let X be a convex set in a Hausdorff topological vector space E. Let $f, g : X \times X \to \mathbf{R}$ with $g(x, y) \leq f(x, y)$ for all $(x, y) \in X \times X$. Let $\lambda = \sup_{x \in X} [f(x, x)]$ (if $\lambda = \infty$ we have trivially* (2.4) *below) and assume $\lambda < \infty$. Let*

$$F(x) = \{z \in X : \ f(x, z) > \lambda\} \ \text{ and } \ S(x) = \{z \in X : \ g(x, z) > \lambda\} \ \text{ for } \ x \in X.$$

Assume $S(x)$ is convex valued for each $x \in X$ and $S^{-1}(w)$ is open (in X) for each $w \in X$. In addition suppose either

(1) *there exists a convex compact set K with $F(X) \subseteq K \subseteq X$,*

or

(2) *there exists a convex compact set K of X with $F(K) \subseteq K$,*

or

(3) *there is a compact subset K of X and a convex compact subset Y of X such that for each $x \in X\backslash K$ we have $S(x) \cap Y \neq \emptyset$,*

hold. Then there exists a $y \in X$ with

$$(2.4). \qquad \sup_{z \in X} g(y, z) \leq \sup_{x \in X} f(x, x).$$

Now we consider an application to games. Let I be a (possible infinite) set of players and each player must select a strategy in a set determined by the strategies chosen by the other players. Here X_i denotes the set of strategies of the i^{th} player and each element of $X = \prod_{i \in I} X_i$ determines an outcome. The payoff to the i^{th} player is h_i (which is defined on X). Let x^i be given in X^i (the strategies of all the others). For $x \in X$, $i \in I$, $y_i \in X_i$ we write (x^i, y_i) as a point in X having the same components as x except the i^{th} component is replaced by y_i; note any $x \in X$ can be written as (x^i, x_i) for any $i \in I$ where x^i denotes the projection of x onto X^i. The i^{th} player chooses $y_i \in X_i$ so as to obtain $\sup_{y_i \in X_i} h_i(x^i, y_i)$. An equilibrium point is a strategy point $x \in X$ such that for all $i \in I$ we have

$$x_i \in X_i \ \text{ and } \ h_i(x) = \sup_{y_i \in X_i} h_i(x^i, y_i).$$

Theorem 2.5. *Let I be an index set and $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological vector space E_i, let h_i $(i \in I)$ be as described above and let $X = \prod_{i\in I} X_i$. For $i \in I$ let $f_i : X \times X_i \to \mathbf{R}$ be given by $f_i(x, y_i) = h_i(x^i, y_i) - h_i(x)$ and let $g_i(x, y_i) = f_i(x, y_i)$ for all $x \in X$ and $y_i \in X_i$. Also for $i \in I$ let $\lambda_i = \sup_{x\in X} [f_i(x, x_i)]$ and let $J = \{i \in I : \lambda_i = \infty\}$. For $i \in I\backslash J$ let*

$$F_i(x) = \{z_i \in X_i : \; f_i(x, z_i) > \lambda_i\} \;\; \textit{for} \;\; x \in X^\star = \prod_{i\in I\backslash J} X_i.$$

and let $S_i(x) = F_i(x)$ for $x \in X^\star$, Assume for each $i \in I\backslash J$ that $F_i(x)$ is convex valued for each $x \in X^\star$ and $F_i^{-1}(w)$ is open (in $X^\star$) for each $w \in X_i$. In addition suppose either

(1) *for each $i \in I\backslash J$ there exists a convex compact set K_i with $F_i(X^\star) \subseteq K_i \subseteq X_i$,*

or

(2) *there exists a convex compact set K of $X^\star$ with $F(K) \subseteq K$ where $F(x) = \prod_{i\in I\backslash J} F_i(x)$ for $x \in X^\star$,*

or

(3) *there is a compact subset K of $X^\star$ and for each $i \in I\backslash J$ a convex compact subset Y_i of X_i such that for each $x \in X^\star\backslash K$ there exists a $j \in I\backslash J$ with $S_j(x) \cap Y_j \neq \emptyset$,*

hold. Then there exists an equilibrium point i.e. there exists a $x \in X$ such that for all $i \in I$ we have

$$x_i \in X_i \;\; \textit{and} \;\; h_i(x) = \sup_{y_i\in X_i} h_i(x^i, y_i).$$

Proof. From Theorem 2.3 we deduce that there exists a $z \in X$ with

$$\sup_{y_i\in X_i} f_i(z, y_i) \leq \sup_{x\in X} f_i(x, x_i) \;\; \text{for all} \;\; i \in I$$

i.e.

$$\sup_{y_i\in X_i} [h_i(z^i, y_i) - h_i(z)] \leq \sup_{x\in X} [h_i(x^i, x_i) - h_i(x)] = 0 \;\; \text{for all} \;\; i \in I.$$

Thus $\sup_{y_i\in X_i} h_i(z^i, y_i) \leq h_i(z)$ for all $i \in I$. However on the other hand note for $i \in I$ that $h_i(z) = h_i(z^i, z_i) \leq \sup_{y_i\in X_i} h_i(z^i, y_i)$ so combining gives $h_i(z) = \sup_{y_i\in X_i} h_i(z^i, y_i)$. □

Conclusion

In this paper using the fixed point results in [6] we presented a new analytic alternative, a new minimax inequality and a new equilibrium result for a generalized game. In the future we hope to extend Theorem's 1.1–1.3 for more general classes of maps than the $\Phi^\star$ type maps considered here. As a result the theory in Section 2 would extend immediately for these new classes of maps.

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