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ON OPTIMALITY CONDITIONS FOR BEST APPROXIMATION PROBLEMS

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This paper is dedicated for the memory of Professor K. Goebel.

ABSTRACT. We consider best approximation problems involving integral functions defined on $L^2_n[0,1]$ and conic constraints, and obtain optimality conditions for the problems which hold without any constraint qualification and which are expressed by sequences.

1. Introduction and preliminaries

Jeyakumar et al. [3] obtained the Lagrange multiplier optimality theorems for convex optimization problems, which held without any constraint qualification and which were expressed by sequences. Such optimality theorems have been investigated for many kinds of convex optimization problems [5, 7–12].

In this paper, we consider best approximation problems involving integral functions defined on $L^2_n[0,1]$ and conic constraints, and obtain optimality conditions for the problems which hold without any constraint qualification and which are expressed by sequences. Moreover, we give an example illustrating how to use the optimality conditions for getting the optimal solution for a best approximation problem (see Example 3.2).

Now we give the definition of the convex function and the conjugate function, and then we state the well-known results about the epigraphs of the conjugate functions of the convex functions.

Let E be a Banach space with norm $x \mapsto \|x\|$ and let E^* the dual of E.

Definition 1.1. The conjugate function of a function $f\colon E \to \mathbb{R} \cup \{+\infty\}$ is the function $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(x^*) := \sup_{x \in E} \{\langle x^*, x\rangle - f(x)\} \ \ (x^* \in E^*).$$

A function $g\colon E \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $t \in [0,1]$,

$$g((1-t)x + ty) \leq (1-t)g(x) + tg(y)$$

for all $x, y \in E$. Let $g\colon E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. We denote the epigraph of g by $\operatorname{epi} g$, that is , $\operatorname{epi} g := \{(x,r) \in E \times \mathbb{R} : g(x) \leq r\}$.

The following two propositions are well known (See [2] and [4]);

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Proposition 1.2. *Let E be a Banach space. Consider a family of proper lower semicontinuous convex functions $\phi_i \colon E \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, where I is an arbitrary index set. Suppose that $\sup_{i\in I} \phi_i$ is not identically $+\infty$. Then*

$$\operatorname{epi}(\sup_{i\in I} \phi_i)^* = \operatorname{clco} \bigcup_{\mathrm{i}\in \mathrm{I}} \operatorname{epi} \phi_{\mathrm{i}}^*.$$

Proposition 1.3. *Let E be a Banach space. Let $\phi_1 \colon E \to \mathbb{R}$, $\phi_2 \colon E \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous convex functions. Then*

$$\operatorname{epi}(\phi_1 + \phi_2)^* = \operatorname{cl}(\operatorname{epi} \phi_1^* + \operatorname{epi} \phi_2^*).$$

In addition, if one of the proper, lower semicontinuous convex functions is continuous, then

$$\operatorname{epi}(\phi_1 + \phi_2)^* = \operatorname{epi} \phi_1^* + \operatorname{epi} \phi_2^*.$$

2. Formulation

Now we formulate our best approximation problem defined on $L^2_n[0,1]$: Consider

$$\begin{array}{lll} \text{(P)} & \text{Minimize}_{x\in L^2_n[0,1]} & \dfrac{1}{2}\displaystyle\int_0^1 \|x(t)\|^2 dt \\ & \quad\text{subject to} & x \in K, \\ & & a_i(t)^T x(t) - b_i(t) = 0 \text{ a.e. } t \in [0,1],\ i = 1,\dots,m \end{array}$$

where K is a closed convex cone in $L^2_n[0,1]$ and $a_i \in L^\infty_n[0,1]$ and $b_i \in L^2[0,1]$, $i = 1,\dots,m$ are given, where $L^2_n[0,1] = \{x \mid x : [0,1] \to \mathbb{R}^n$: measurable and $\int_0^1 \|x(t)\|^2 dt < \infty\}$ and $L^\infty_n[0,1] = \{x \mid x : [0,1] \to \mathbb{R}^n$: measurable and there exists M such that $\|x(t)\| \leqq M$ a.e on $[0,1]\}\}$. We define the inner product $\langle\cdot,\cdot\rangle$ as $\langle f, g\rangle = \int_0^1 f(t)^T g(t) dt$ for any $f, g \in L^2_n[0,1]$. Then $L^2_n[0,1]$ is a Hilbert space with the inner product (refer [1] for the definitions and basic properties of the spaces $L^2_n[0,1]$ and $L^\infty_n[0,1]$).

Let $\triangle = \{x \in K \mid a_i(t)^T x(t) - b_i(t) = 0 \text{ a.e. } t \in [0,1]\}$. Assume that $\triangle \neq \emptyset$. We can prove that $\triangle$ is a closed convex subset of $L^2_n[0,1]$. So the problem (P) has the unique optimal solution.

We define the nonnegative dual cone of K as

$$K^* = \{z \in L^2_n[0,1] \mid \int_0^1 z(t)^T x(t) dt \geqq 0 \quad \forall x \in K\}.$$

We recall that $x = y$ in $L^2_n[0,1]$ if and only if $x(t) = y(t)$ a.e on $[0,1]$.

Example 2.1. Consider the following best approximation problem defined on $L^2_2[0,1]$:

$$\begin{array}{lll} (P)_1 & \text{Minimize}_{x\in L^2_2[0,1]} & \dfrac{1}{2}\displaystyle\int_0^1 \Big[x_1(t)^2 + x_2(t)^2\Big] dt \\ & \quad\text{subject to} & x_1(t) + x_2(t) = t \text{ a.e. } t \in [0,1]. \end{array}$$

Then $(\frac{t}{2}, \frac{t}{2})$ is the unique solution of (P_1) and its optimal value is $\frac{1}{12}$.

Consider another best approximation problem defined on $L_n^2[0,1]$:

$$(P)_2 \qquad \text{Minimize}_{x\in L_2^2[0,1]} \quad \frac{1}{2}\int_0^1 \Big[x_1(t)^2 + x_2(t)^2\Big]dt$$
$$\text{subject to} \qquad \int_0^1 \Big[x_1(t) + x_2(t)\Big]dt = \frac{1}{2}.$$

Then $(\frac{1}{4}, \frac{1}{4})$ is the unique solution of $(P)_2$ and its optimal value is $\frac{1}{16}$.

Jeyakumar et al. [6] studied the $(P)_2$ type best approximation problem, but in this paper, we consider the type $(P)_1$ best approximation problem. The example says that the two type best approximation problems are different.

3. Optimality conditions

Now we give the following optimality theorem for (P):

Theorem 3.1. *Let $\bar{x} \in \triangle$. Then the following are equivalent:*

(i) *$\bar{x}$ is an optimal solution of (P);*

(ii) *there exist $\bar{v} \in L_n^2[0,1]$, $\lambda_i^l \in L^2[0,1]$, $k_l^* \in K^*$, $r \geqq 0$ and $r_l \geqq 0$, $i = 1,\ldots,m$ such that*

$$\bar{v} + \lim_{l\to\infty}\Big[\sum_{i=1}^m \lambda_i^l a_i - k_l^*\Big] = 0$$
$$\textit{and} \quad -\frac{1}{2}\|\bar{x}\|^2 = \frac{1}{2}\|\bar{v}\|^2 + r + \lim_{l\to\infty}\Big[\sum_{i=1}^m \Big\langle \lambda_i^l, b_i\Big\rangle + r_l\Big];$$

(iii) *there exist $\lambda_i^l \in L^2[0,1]$ and $k_l^* \in K^*$ such that*

$$\bar{x} + \lim_{l\to\infty}\Big[\sum_{i=1}^m \lambda_i^l a_i - k_l^*\Big] = 0$$
$$\textit{and} \lim_{l\to\infty} \langle k_l^*, \bar{x}\rangle = 0,$$

where $\lambda_i^l a_i = (\lambda_i^l a_i^1, \ldots, \lambda_i^l a_i^l)$ $(a_i = (a_i^1, \ldots, a_i^l))$.

Proof. (i)$\Rightarrow$ (ii): Suppose that $\bar{x}$ is an optimal solution of (P). Let $f(x) = \frac{1}{2}\int_0^1 \|x(t)\|^2 dt$. Let $D = \{x \in L_n^2[0,1] \mid a_i(t)^T x(t) - b_i(t) = 0$ a.e. $t \in [0,1],\ i = 1,\ldots,m\}$ and $\triangle = \{x \in K \mid a_i(t)^T x(t) - b_i(t) = 0$ a.e. $t \in [0,1],\ i = 1,\ldots,m\}$. Then $\triangle = D \cap K$. Let $h_i(x) = a_i(\cdot)^T x(\cdot) - b_i(\cdot)$. Then $h_i : L_n^2[0,1] \to L_n^2[0,1], i = 1,\ldots,m$, is continuous and affine,

$$D = \{x \in L_n^2[0,1] \mid h_i(x) = 0,\ i = 1,\ldots,m\}.$$

By Proposition 1.3, we have,

$$\begin{aligned}(0, -f(\bar{x})) &\in \text{epi}(f + \delta_\triangle)^* \\ &= \text{epi}f^* + \text{epi}\delta_\triangle^* \\ &= \text{epi}f^* + \text{cl}(\text{epi}\delta_{\text{D}}^* + \text{epi}\delta_{\text{K}}^*).\end{aligned}$$

Thus

$$(0,-f(\bar{x}))\in \mathrm{epi}f^{*}+\mathrm{cl}(\mathrm{epi}\delta^{*}_{\mathrm{D}}+\mathrm{epi}\delta^{*}_{\mathrm{K}}). \tag{3.1}$$

Since $f^{*}(v)=\frac{1}{2}\|v\|^{2}$, $\mathrm{epi}f^{*}=\left\{\left(v,\frac{1}{2}\|v\|^{2}\right)|v\in L^{2}_{n}[0,1]\right\}+\{0\}\times\mathbb{R}_{+}$. Since $\delta_{D}(x)=\sup\limits_{\lambda_i\in L^2[0,1]}\sum_{i=1}^{m}\langle\lambda_i,h_i(x)\rangle$, it follows from Proposition 1.2 that

$$\begin{aligned}\mathrm{epi}\delta^{*}_{\tilde{D}} &= \mathrm{cl}\bigcup_{\lambda_i\in L^2[0,1]}\sum_{i=1}^{m}\mathrm{epi}\Big(\langle\lambda_i,h_i(\cdot)\rangle\Big)^{*}\\ &= \mathrm{cl}\Big[\bigcup_{\lambda_i\in L^2[0,1]}\Big\{\sum_{i=1}^{m}(\lambda_i a_i,\int_0^1\lambda_i(t)b_i(t)dt)\Big\}+\{0\}\times\mathbb{R}_{+}\Big]\end{aligned}$$

and $\mathrm{epi}\delta^{*}_{K}=(-K^{*})\times\mathbb{R}_{+}$. Thus, from (3.1),

$$\begin{aligned}(0,-\frac{1}{2}\|\bar{x}\|^{2}) \in& \left\{\left(v,\frac{1}{2}\|v\|^{2}\right)|v\in L^{2}_{n}[0,1]\right\}+\{0\}\times\mathbb{R}_{+}\\ &+\mathrm{cl}\Big[\bigcup_{\lambda_i\in L^2[0,1]}\Big\{\sum_{i=1}^{m}(\lambda_i a_i,\int_0^1\lambda_i(t)b_i(t)dt)\Big\}+\{-K^{*}\}\times\mathbb{R}_{+}\Big].\end{aligned}$$

Hence there exist $\bar{v}\in L^{2}_{n}[0,1]$, $\lambda^{l}_{i}\in L^{2}[0,1]$, $k^{*}_{l}\in K^{*}$, $r\geqq 0$ and $r_l\geqq 0,\ i=1,\dots,m$ such that

$$\begin{aligned}\Big(0,-\frac{1}{2}\|\bar{x}\|^{2}\Big) =& \Big(\bar{v},\frac{1}{2}\|\bar{v}\|^{2}\Big)+(0,r)\\ &+\lim_{l\to\infty}\Big[\sum_{i=1}^{m}(\lambda^{l}_{i}a_i,\int_0^1\lambda^{l}_{i}(t)b_i(t)dt)+(-k^{*}_{l},r_l)\Big].\end{aligned}$$

Therefore there exist $\bar{v}\in L^{2}_{n}[0,1]$, $\lambda^{l}_{i}\in L^{2}[0,1]$, $k^{*}_{l}\in K^{*}$, $r\geqq 0$ and $r_l\geqq 0,\ i=1,\dots,m$ such that

$$\bar{v}+\lim_{l\to\infty}\Big[\sum_{i=1}^{m}\lambda^{l}_{i}(\cdot)a_i(\cdot)-k^{*}_{l}\Big]=0 \tag{3.2}$$

$$\text{and}\quad -\frac{1}{2}\|\bar{x}\|^{2}=\frac{1}{2}\|\bar{v}\|^{2}+r+\lim_{l\to\infty}\Big[\sum_{i=1}^{m}\int_0^1\lambda^{l}_{i}(t)b_i(t)dt+r_l\Big]. \tag{3.3}$$

Thus (ii) holds.

(ii) $\Rightarrow$ (iii): Suppose that (ii) holds. From (3.2) and (3.3), there exist $\lambda^{l}_{i}\in L^{2}[0,1]$, $k_l\in K^{*}$ and $r_l\in\mathbb{R}_{+}$ such that

$$0 = \langle\bar{v},\bar{x}\rangle+\left\langle\lim_{l\to\infty}\Big(\sum_{i=1}^{m}\lambda^{l}_{i}a_i-k^{*}_{l}\Big),\bar{x}\right\rangle \tag{3.4}$$

$$0 = \frac{1}{2}\langle\bar{x},\bar{x}\rangle+\frac{1}{2}\langle\bar{v},\bar{v}\rangle+r+\lim_{l\to\infty}\Big[\sum_{i=1}^{m}\int_0^1\lambda^{l}_{i}(t)b_i(t)dt+r_l\Big]. \tag{3.5}$$

From (3.4) and (3.5),

$$\begin{aligned} 0 &= -\frac{1}{2}\langle \bar{x}, \bar{x}\rangle + \langle \bar{v}, \bar{x}\rangle - \frac{1}{2}\langle \bar{v}, \bar{v}\rangle - r \\ &\quad + \lim_{l\to\infty}\Big[\Big\langle \sum_{i=1}^{m} \lambda_i^l a_i, \bar{x}\Big\rangle - \sum_{i=1}^{m}\int_0^1 \lambda_i^l(t) b_i(t)dt - \langle k_l^*, \bar{x}\rangle - r_l\Big] \\ &= -\frac{1}{2}\|\bar{x}-\bar{v}\|^2 - r - \lim_{l\to\infty}\Big(\langle k_l^*, \bar{x}\rangle + r_l\Big) \end{aligned}$$

Thus $\|\bar{x}-\bar{v}\|^2 = 0$, that is, $\bar{x} = \bar{v}$, $r = 0$, $\lim_{l\to\infty}\langle k_l^*, \bar{x}\rangle = 0$ and $\lim_{l\to\infty} r_l = 0$. Thus, from (3.2), there exist $\lambda_i^l \in L^2[0,1]$ and $k_l^* \in K^*$ such that

$$\begin{aligned} &\bar{x} + \lim_{l\to\infty}\Big[\sum_{i=1}^{m}\lambda_i^l a_i - k_l^*\Big] = 0 \\ &\text{and } \lim_{l\to\infty}\langle k_l^*, \bar{x}\rangle = 0. \end{aligned}$$

Hence (iii) holds.

(iii)$\Rightarrow$ (i): Suppose that (iii) holds. Then for any $x \in \triangle$, we have

$$\begin{aligned} 0 &= \langle \bar{x}, x-\bar{x}\rangle + \lim_{l\to\infty}\Big\langle \sum_{i=1}^{m}\lambda_i^l a_i - k_l^*, x-\bar{x}\Big\rangle \\ &= \langle \bar{x}, x\rangle - \|\bar{x}\|^2 + \lim_{l\to\infty}\Big[\sum_{i=1}^{m}\Big\langle \lambda_i^l, b_i\Big\rangle - \langle k_l^*, x\rangle - \sum_{i=1}^{m}\Big\langle \lambda_i^l, b_i\Big\rangle + \langle k_l^*, \bar{x}\rangle\Big] \\ &= \langle \bar{x}, x\rangle - \|\bar{x}\|^2 - \lim_{l\to\infty}\langle k_l^*, x\rangle . \end{aligned}$$

Thus for any $x \in \triangle$, $\langle \bar{x}, x\rangle - \|\bar{x}\|^2 \geqq 0$ and so $\|x-\bar{x}\|^2 \leqq \|x\|^2 - \|\bar{x}\|^2$. Thus $\bar{x}$ is an optimal solution of (P) and so (i) holds. □

Example 3.2. Consider the problem $(\mathrm{P})_1$ in Example 2.1. Then by Theorem 3.1, $(\bar{x}_1, \bar{x}_2)$ is the unique optimal solution of $(\mathrm{P})_1$ if and only if

$$\begin{aligned} &\bar{x}_1(t) + \bar{x}_2(t) = t \ \text{ a.e. } \ t \in [0,1] \ \text{ and} \\ &\text{there exists } \lambda^l \in L^2[0,1] \ \text{ such that} \\ &(\bar{x}_1, \bar{x}_2) = \lim_{l\to\infty}\lambda^l(1,1) = 0, \end{aligned}$$

equivalently, $\bar{x}_1(t) + \bar{x}_2(t) = t$ a.e. $t \in [0,1]$ and there exist $\lambda^l \in L^2[0,1]$ such that

$$(\bar{x}_1(t), \bar{x}_2(t)) = (-\lim_{l\to\infty}\lambda^l(t), -\lim_{l\to\infty}\lambda^l(t)) \ a.e. \ t \in [0,1].$$

Hence $\lambda^l(t) = -\frac{t}{2}$ $(a.e.\ t \in [0,1])$ and $(\bar{x}_1(t), \bar{x}_2(t)) = \left(\frac{t}{2}, \frac{t}{2}\right)$ $(a.e.\ t \in [0,1])$ is the unique optimal solution of $(\mathrm{P})_1$.

Remark 3.3. We do not know the example which shows that the limits in Theorem 3.1 can not be removed. We do not know whether the limits in Theorem 3.1 for best approximation problem can be removed or not. We had examples which show that such limits could not be removed in other convex or linear fractional optimization problems (see [3, 8–12]).

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