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CONVERGENCE OF INEXACT ITERATES FOR MAPPINGS OF CONTRACTIVE TYPE IN GENERALIZED METRIC SPACES

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ABSTRACT. In the present paper we prove the existence a fixed point for certain operators of contractive type in generalized metric spaces. We also show the convergence of inexact iterates to a fixed point.

1. Introduction

For more than sixty years now, there has been a lot of research activity regarding the fixed point theory of contractive and of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 4, 5, 8, 9, 10, 13, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26] and references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed point problems and variational inequalities, which find important applications in mathematical analysis, optimization theory, and in engineering, medical and the natural sciences [3, 6, 7, 23, 24, 25, 26].

In [22] we have first introduced certain generalized metric spaces by extending the concept of a modular space studied in [11, 12, 14] and then established a fixed point theorem for certain Rakotch type contractive operators which map a closed subset into the space and have finite orbits of arbitrary lengths. In the present paper we extend this result for operators of contractive type considered in [18]. We also show the convergence of inexact iterates to a fixed point.

2. Modular space

Let X be a vector space. A functional $\rho : X \to [0,\infty]$ is called a *modular* [11, 12, 14] if the following three properties hold:

(1) $\rho(x) = 0$ if and only $x = 0$;

(2) $\rho(-x) = \rho(x)$ for all $x \in X$;

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for each $x, y \in X$ and each $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = 1$.

The vector space

$$X_\rho := \{x \in X : \ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

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is called a *modular space*.

Assume that ρ is a modular defined on a vector space X. We say that the modular ρ satisfies a Δ_2-type condition if there exists a number $M > 0$ such that

$$\rho(2x) \le M\rho(x),\ x \in X_\rho. \tag{2.1}$$

The authors of [12] considered a modular function space L_ρ (which is a particular case of a modular space) with a modular ρ satisfying a Δ_2-type condition. They showed that if T is a self-mapping of a closed subset K of L_ρ such that for some $c \in [0, 1)$,

$$\rho(T(x) - T(y)) \le c\rho(x - y) \text{ for all } x, y \in K$$

and such that there exists $x_0 \in K$ satisfying

$$\sup\{\rho(2T^p(x_0)) :\ p = 1, 2, \dots\} < \infty,$$

then T has a fixed point.

Assume that ρ is a modular defined on the vector space X. For each $x, y \in X$, define

$$d(x, y) := \rho(x - y).$$

It is easy to see that for each $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$ and that $d(x, y) = d(y, x)$.

Assume that ρ satisfies the Δ_2-type condition (2.1) with a number $M > 0$. Then for each $x, y, z \in X_\rho$, we have

$$\begin{aligned} d(x, z) &= \rho(x - z) = \rho((x - y) + (y - z)) \\ &= \rho(2(2^{-1}(x - y) + 2^{-1}(y - z))) \\ &\le M\rho(2^{-1}(x - y) + 2^{-1}(y - z)) \\ &\le M(\rho(x - y) + \rho(y - z)) \\ &\le Md(x, y) + Md(y, z). \end{aligned}$$

We say that a modular ρ is uniformly continuous (see Definition 5.4 of [11]) if for each $\epsilon > 0$ and each $L > 0$, there exists $\delta > 0$ such that

$$|\rho(x + y) - \rho(x)| \le \epsilon \tag{2.2}$$

for each pair $x, y \in X_\rho$ satisfying $\rho(y) < \delta$ and $\rho(x) < L$.

Assume that the modular ρ is uniformly continuous and that $\epsilon > 0$ and $L > 0$. Then there exists a number $\delta > 0$ such that (2.2) holds for each pair $x, y \in X_\rho$ satisfying $\rho(y) \le \delta$ and $\rho(x) \le L$.

Assume now that the points $x, y, z \in X_\rho$ satisfy

$$d(x, y) \le L,\ d(y, z) \le \delta.$$

Then

$$\rho(x - y) \le L,\ \rho(y - z) \le \delta,$$
$$d(x - z) = \rho(x - z) = \rho((x - y) + (y - z))$$

and in view of the choice of δ,

$$|d(x, z) - d(x, y)| = |\rho(x - z) - \rho(x - y)| \le \epsilon.$$

Thus we have shown that for each $\epsilon > 0$ and each $L > 0$, there exists $\delta > 0$ such that if $x, y, z \in X_\rho$ satisfy

$$d(x, y) \leq L, \ d(y, z) \leq \delta,$$

then

$$|d(x, z) - d(x, y)| \leq \epsilon.$$

In other words, d is uniformly continuous.

3. Generalized metric space

Assume that X is a nonempty set, $d : X \times X \to [0, \infty]$, $M > 0$, and that for each $x, y, z \in X$,

$$d(x, y) = 0 \text{ if and only if } x = y, \tag{3.1}$$

$$d(x, y) = d(y, x) \tag{3.2}$$

and

$$d(x, z) \leq Md(x, y) + Md(y, z). \tag{3.3}$$

We call the pair (X, d) a generalized metric space. For each point $x \in X$ and each number $r > 0$, set

$$B_d(x, r) := \{y \in X : \ d(x, y) \leq r\}.$$

Clearly, a generalized metric space is both a generalization of the concept of a modular space and a generalization of the concept of a metric space. It was introduced in [22]. By investigating generalized metric spaces we are able to unify the study of these two important classes of spaces. For specific examples of modular spaces, see [11, 14].

We equip the space X with the uniformity determined by the base

$$\mathcal{U}(\epsilon) := \{(x, y) \in X \times X : \ d(x, y) \leq \epsilon\}, \ \epsilon > 0. \tag{3.4}$$

This uniform space is metrizable (by a metric $\tilde{d}$). We also equip the space X with the topology induced by this uniformity and assume that the uniform space X is complete.

Consider a sequence $\{x_n\}_{n=1}^\infty \subset X$ and a point $x \in X$. Clearly,

$$\lim_{n\to\infty} x_n = x$$

if and only if

$$\lim_{n\to\infty} d(x_n, x) = 0$$

and $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence if and only if for each $\epsilon > 0$, there exists a natural number $n(\epsilon)$ such that $d(x_n, x_m) \leq \epsilon$ for every pair of integers $n, m \geq n(\epsilon)$.

A set $E \subset X$ is said to be bounded if

$$\sup\{d(x, y) : \ x, y \in E\} < \infty.$$

We say that the generalized metric d is uniformly continuous on bounded sets if for each nonempty bounded set $D \subset X$ and each $\epsilon > 0$, there exists a number $\delta > 0$ such that for each $x, y \in D$ and each $z \in X$ satisfying $d(y, z) \le \delta$, the inequality

$$|d(x, y) - d(x, z)| \le \epsilon$$

holds.

From now on we assume that the generalized metric d is uniformly continuous on bounded sets. For each mapping $T : S \to X$, where $S \subset X$ set $T^0(x) = x,\ x \in S$.

4. The main result

We use the notations and definitions introduced in Section 3 and assume that all the assumptions made there hold.

Theorem 4.1. *Let K be a nonempty and closed subset of X. Assume that $T : K \to X$ satisfies*

$$d(T(x), T(y)) \le \phi(d(x, y)) \text{ for each } x, y \in K, \tag{4.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous and satisfies

$$\phi(t) < t \text{ for all } t > 0. \tag{4.2}$$

Assume further that for each integer $n \ge 1$, $x_n \in K$, there exists

$$T^n(x_n) \in K \tag{4.3}$$

and the set

$$E := \{T^i(x_n) :\ n = 1, 2, \dots \text{ and } i \in \{0, \dots, n\}\}$$

is bounded. Then the following assertions hold.

(A) There exists $\bar{x} \in K$ such that $T(\bar{x}) = \bar{x}$ and this fixed point is unique if $d(x, y) < \infty$ for each pair $x, y \in K$.

(B) Let $M_0 > 0, \epsilon \in (0, 1)$. Then there exist $\delta > 0$ and a natural number k such that for each integer $n \ge k$ and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying

$$d(x_0, \bar{x}) \le M_0$$

and

$$d(x_{i+1}, T(x_i)) \le \delta,\ i = 0, \dots, n - 1,$$

the inequality $d(x_i, \bar{x}) \le \epsilon$ holds for $i = k, \dots, n$.

Proof. (A) The uniqueness of $\bar{x}$ is obvious. Let us establish its existence. Set

$$M_1 = \sup\{d(y, z) :\ y, z \in E\}. \tag{4.4}$$

Let $\epsilon \in (0, 1)$. We will show that there exists a natural number k such that the following property holds:

(P2) If n and i are integers such that $k \le i < n$, then

$$d(T^i(x_n), T^{i+1}(x_n)) \le \epsilon.$$

Assume the contrary. Then for each natural number k, there exist natural numbers n_k and i_k such that

$$k \le i_k < n_k \text{ and } d(T^{i_k}(x_{n_k}), T^{i_k+1}(x_{n_k})) > \epsilon. \tag{4.5}$$

Since the function $t - \phi(t)$ is positive for all $t > 0$ and lower semicontinuous, there is $\gamma > 0$ such that

$$t - \phi(t) \ge \gamma \text{ for all } t \in [\epsilon/2, 4M_1 + 4]. \tag{4.6}$$

Choose a natural number k such that

$$k > \gamma^{-1}(2 + 2M_1). \tag{4.7}$$

Then (4.5) holds. By (4.5),

$$d(T^i(x_{n_k}), T^{i+1}(x_{n_k})) > \epsilon,\ i = 0, \dots, i_k. \tag{4.8}$$

In view of (4.4) and (4.8), for all $i = 0, \dots, i_k$,

$$\epsilon < d(T^i(x_{n_k}), T^{i+1}(x_{n_k})) \le M_1. \tag{4.9}$$

It follows from (4.1), (4.3), (4.5), (4.6) and (4.9) that for all $i = 0, \dots, i_k - 1$,

$$\begin{aligned} d(T^{i+2}(x_{n_k}), T^{i+1}(x_{n_k})) &\le \phi(d(T^{i+1}(x_{n_k}), T^i(x_{n_k}))) \\ &\le d(T^{i+1}(x_{n_k}), T^i(x_{n_k})) - \gamma. \end{aligned}$$

When combined with (4.4) and (4.5), this implies that

$$\begin{aligned} -M_1 \le -d(x_{n_k}, T(x_{n_k})) &\le d(T^{i_k+1}(x_{n_k}), T^{i_k}(x_{n_k})) - d(x_{n_k}, T(x_{n_k})) \\ &= \sum_{i=0}^{i_k-1} [d(T^{i+2}(x_{n_k}), T^{i+1}(x_{n_k})) - d(T^{i+1}(x_{n_k}), T^i(x_{n_k}))] \\ &\le -\gamma i_k \le -k\gamma \end{aligned}$$

and

$$k \le \gamma^{-1} M_1.$$

This contradicts (4.7). The contradiction we have reached proves the existence of a natural number k such that property (P1) holds.

Now let $\delta > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P1) If n, i and j are integers such that $k \le i,\ j < n$, then

$$d(T^i(x_n), T^j(x_n)) \le \delta.$$

Assume to the contrary that there is no natural number k for which (P2) holds.

Then for each natural number k, there exist natural numbers n_k, i_k and j_k such that

$$k \le i_k < j_k < n_k \tag{4.10}$$

and

$$d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) > \delta. \tag{4.11}$$

We may assume without loss of generality that for each natural number k, the following property holds:

If an integer j satisfies $i_k < j \le j_k$, then

$$d(T^{i_k}(x_{n_k}), T^j(x_{n_k})) \le \delta. \tag{4.12}$$

Assume now that k is a natural number. It follows from (4.10)-(4.12) that

$$\begin{aligned}\delta &< d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) \\ &\le d(T^{i_k}(x_{n_k}), T^{j_k-1}(x_{n_k})) \\ &\quad + |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{i_k}(x_{n_k}), T^{j_k-1}(x_{n_k}))| \\ &\le \delta + |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k}(x_{n_k}), T^{j_k-1}(x_{n_k}))|.\end{aligned} \tag{4.13}$$

Property (P1) and (4.10) imply that

$$\lim_{k\to\infty} d(T^{j_k}(x_{n_k}), T^{j_k-1}(x_{n_k})) = 0.$$

When combined with the boundedness of E and the uniform continuity of d on bounded sets this implies that

$$\lim_{k\to\infty} |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{i_k}(x_{n_k}), T^{j_k-1}(x_{n_k}))| = 0. \tag{4.14}$$

By (4.13) and (4.14),

$$\lim_{k\to\infty} d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) = \delta. \tag{4.15}$$

By (4.1) and (4.11), for each integer $k \ge 1$,

$$\begin{aligned}\delta &< d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) \\ &\le d(T^{i_k+1}(x_{n_k}), T^{j_k}(x_{n_k})) \\ &+ |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k}(x_{n_k}), T^{i_k+1}(x_{n_k})) \\ &\le |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k}(x_{n_k}), T^{i_k+1}(x_{n_k}))| \\ &\quad + d(T^{i_k+1}(x_{n_k}), T^{j_k+1}(x_{n_k})) \\ &\quad + |d(T^{i_k+1}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k+1}(x_{n_k}), T^{i_k+1}(x_{n_k}))| \\ &\le \phi(d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) \\ &+ |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k}(x_{n_k}), T^{i_k+1}(x_{n_k}))| \\ &\quad + |d(T^{i_k+1}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k+1}(x_{n_k}), T^{i_k+1}(x_{n_k}))|.\end{aligned} \tag{4.16}$$

Property (P1) and (4.10) imply that

$$\lim_{k\to\infty} d(T^{i_k}(x_{n_k}), T^{i_k+1}(x_{n_k})) = 0, \tag{4.17}$$

$$\lim_{k\to\infty} d(T^{j_k}(x_{n_k}), T^{j_k+1}(x_{n_k})) = 0. \tag{4.18}$$

It follows from (4.17), (4.18), the boundedness of the set E and the uniform continuity of d on bounded sets that

$$\lim_{k\to\infty} |d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k}(x_{n_k}), T^{i_k+1}(x_{n_k}))| = 0,$$

$$\lim_{k\to\infty} |d(T^{i_k+1}(x_{n_k}), T^{j_k}(x_{n_k})) - d(T^{j_k+1}(x_{n_k}), T^{i_k+1}(x_{n_k}))| = 0.$$

Together with (4.15), (4.16) and the upper semicontinuity of ϕ this implies that

$$\delta \le \liminf_{k\to\infty} \phi(d(T^{i_k}(x_{n_k}), T^{j_k}(x_{n_k})) \le \phi(\delta),$$

contradiction. The contradiction we have reached proves that there exists a natural number k such that (P2) holds.

Let $\epsilon > 0$ be given. We will show that there exists a natural number k such that the following property holds:

(P3) If the integers $n_1, n_2 > k$, then $d(T^k(x_{n_1}), T^k(x_{n_2})) \le \epsilon$.

Assume the contrary. Then for each integer $k \ge 1$, there are integers $n_1^{(k)}, n_2^{(k)} > k$ such that

$$d(T^k(x_{n_1^{(k)}}), T^k(x_{n_2^{(k)}})) > \epsilon. \tag{4.19}$$

Set

$$\delta = \limsup_{k\to\infty} d(T^k(x_{n_1^{(k)}}), T^k(x_{n_2^{(k)}})). \tag{4.20}$$

(Note that E is bounded.) By (4.20), there exists a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that

$$\delta = \lim_{i\to\infty} d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}})). \tag{4.21}$$

By (4.19) and (4.21),

$$\delta \ge \epsilon. \tag{4.22}$$

Clearly, for each natural number i,

$$\begin{aligned} & d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}})) \\ & \le d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_1^{(k_i)}})) \\ & \quad + |d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}})) - d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}}))| \\ & \le |d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}})) - d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}}))| \\ & \quad + d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i+1}(x_{n_2^{(k_i)}})) \\ & \quad + |d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i+1}(x_{n_2^{(k_i)}})) - d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}}))|. \end{aligned} \tag{4.23}$$

Property (P1) implies that

$$\lim_{i\to\infty} d(T^{k_i}(x_{n_2^{(k_i)}}), T^{k_i+1}(x_{n_2^{(k_i)}})) = 0, \ \lim_{i\to\infty} d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i+1}(x_{n_1^{(k_i)}})) = 0. \tag{4.24}$$

It follows from (4.1), (4.21)-(4.23), the boundedness of the set E, the uniform continuity of d on bounded sets and the upper semicontinuity of ϕ that

$$\begin{aligned} \epsilon \le \delta & \le \lim_{i\to\infty} d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}})) \\ & \le \liminf_{i\to\infty} d(T^{k_i+1}(x_{n_1^{(k_i)}}), T^{k_i+1} x_{n_2^{(k_i)}})) \end{aligned}$$

$$\leq \liminf_{i\to\infty} \phi(d(T^{k_i}(x_{n_1^{(k_i)}}), T^{k_i}(x_{n_2^{(k_i)}}))) \leq \phi(\delta),$$

contradiction. This contradiction implies that there is indeed a natural number k such that (P3) holds, as claimed.

Let $\epsilon > 0$ be given. By (P3), there exists a natural number k_1 such that

$$d(T^{k_1}(x_{n_1}), T^{k_1}(x_{n_2})) \leq (\epsilon/4)(M+1)^{-2}$$

$$\text{for all integers } n_1, n_2 \geq k_1. \tag{4.25}$$

Property (P2) implies that that there exists a natural number k_2 such that

$$d(T^i(x_n), T^j(x_n)) \leq (\epsilon/4)(M+1)^{-2}$$

$$\text{for all natural numbers } n, i, j \text{ satisfying } k_2 \leq i, j < n. \tag{4.26}$$

Assume that the natural numbers n_1, n_2, i and j satisfy

$$n_1, n_2 > k_1 + k_2,\ i, j \geq k_1 + k_2,\ i < n_1,\ j < n_2. \tag{4.27}$$

We claim that $d(T^i(x_{n_1}), T^j(x_{n_2})) \leq \epsilon$. By (4.1), (4.25) and (4.27),

$$d(T^{k_1+k_2}(x_{n_1}), T^{k_1+k_2}(x_{n_2})) \leq d(T^{k_1}(x_{n_1}), T^{k_1}(x_{n_2})) \leq (\epsilon/4)(M+1)^{-2}. \tag{4.28}$$

In view of (4.26) and (4.27),

$$d(T^{k_1+k_2}(x_{n_1}), T^i(x_{n_1})) \leq (\epsilon/4)(M+1)^{-2}$$

$$\text{and } d(T^{k_1+k_2}(x_{n_2}), T^j(x_{n_2})) \leq (\epsilon/4)(M+1)^{-2}. \tag{4.29}$$

Inequalities (4.28) and (4.29) imply that

$$\begin{aligned} d(T^i(x_{n_1}), T^j(x_{n_2})) &\leq M(d(T^i(x_{n_1}), T^{k_1+k_2}(x_{n_1})) + d(T^{k_1+k_2}(x_{n_1}), T^j(x_{n_2}))] \\ &\leq M(M+1)^{-2}\epsilon/4 + Md(T^{k_1+k_2}(x_{n_1}), T^j(x_{n_2})) \\ &\leq M(M+1)^{-2}\epsilon/4 + M^2(d(T^{k_1+k_2}(x_{n_1}), T^{k_1+k_2}(x_{n_2})) \\ &\quad + d(T^{k_1+k_2}(x_{n_2}), T^j(x_{n_2}))) \\ &\leq M(M+1)^{-2}\epsilon/4 + M^2(M+1)^{-2}\epsilon/4 \\ &\quad + M^2 d(T^{k_1+k_2}(x_{n_1}), T^{k_1+k_2}(x_{n_2})) < \epsilon. \end{aligned}$$

Thus we have shown that the following property holds:

(P4) For each $\epsilon > 0$, there exists a natural number $k(\epsilon)$ such that

$$d(T^i(x_{n_1}), T^j(x_{n_2})) < \epsilon \text{ for all natural numbers } n_1, n_2, i \text{ and } j$$

such that

$$n_1, n_2 > k(\epsilon),\ i \in [k(\epsilon), n_1) \text{ and } j \in [k(\epsilon), n_2).$$

Consider now the sequences $\{T^{n-2}(x_n)\}_{n=3}^{\infty}$ and $\{T^{n-1}(x_n)\}_{n=3}^{\infty}$. Property (P4) implies that both of them are Cauchy sequences and that

$$\lim_{n\to\infty} d(T^{n-2}(x_n), T^{n-1}(x_n)) = 0.$$

Hence there exists $\bar{x} \in K$ such that

$$\lim_{n\to\infty} d(\bar{x}, T^{n-2}(x_n)) = \lim_{t\to\infty} d(\bar{x}, T^{n-1}(x_n)) = 0.$$

Since the mapping T is continuous, it follows that $T(\bar{x}) = \bar{x}$. Thus part (A) of our theorem is proved.

We now turn to the proof of part (B). Fix $\theta \in K$. Set

$$M_1 = MM_0 + Md(\theta, \bar{x}) + 2. \tag{4.30}$$

In view of upper semicontinuity of ϕ,

$$\gamma_0 := \inf\{t - \phi(t) :\ t \in [\epsilon/4, (M+1)(M_1+1)]\} > 0. \tag{4.31}$$

By the uniform continuity of d on bounded sets, there exists a positive number

$$\delta \in (0, \min\{\gamma_0, 2^{-1}\}) \tag{4.32}$$

such that the following property holds:

(P5) for each $y, z \in B_d(\bar{x}, (M+1)(M_1+1))$ satisfying $d(y,z) \le \delta$, we have

$$|d(\bar{x}, z) - d(\bar{x}, y)| \le \gamma_0/4.$$

Assume that

$$y \in B_d(\theta, M_0).$$

Then

$$d(y, \bar{x}) \le M(d(y,\theta) + d(\theta, \bar{x})) \le MM_0 + Md(\theta, \bar{x})$$

and in view of (4.30),

$$B_d(\theta, M_0) \subset B(\bar{x}, M_1). \tag{4.33}$$

Assume that

$$y \in K \cap B(\bar{x}, M_1),\ z \in X \text{ and } d(z, T(y)) \le \delta. \tag{4.34}$$

By (4.1) and the equation $T(\bar{x}) = \bar{x}$,

$$d(\bar{x}, z) \le d(\bar{x}, T(y)) + |d(\bar{x}, T(y)) - d(\bar{x}, z)|$$

$$\le \phi(d(\bar{x}, y)) + |d(\bar{x}, T(y)) - d(\bar{x}, z)|. \tag{4.35}$$

It follows from (4.1), (4.32), (4.34) and the equation $T(\bar{x}) = \bar{x}$ that

$$d(\bar{x}, T(y)) \le d(\bar{x}, y) \le M_1, \tag{4.36}$$

$$d(x, z) \le M(d(\bar{x}, T(y)) + d(T(y), z))$$

$$\le M(d(\bar{x}, y) + 1) \le M(M_1 + 1). \tag{4.37}$$

Property (P5) and equations (4.34), (4.36) and (4.37) imply that

$$|d(\bar{x}, z) - d(\bar{x}, T(y))| \le \gamma_0/4. \tag{4.38}$$

There are two cases:

$$d(y, \bar{x}) \le M_1/4; \tag{4.39}$$

$$d(y, \bar{x}) > M_1/4. \tag{4.40}$$

Assume that (4.39) holds. By (4.1), (4.30), (4.31), (4.38), (4.39) and the equation $T(\bar{x}) = \bar{x}$,

$$d(\bar{x}, z) \leq d(\bar{x}, y) + \gamma_0 \leq M_1/4 + \gamma_0/4 \leq M_1.$$

Assume that (4.40) holds. Then by (4.1), (4.30), (4.31), (4.34), (4.38) and the equation $T(\bar{x}) = \bar{x}$,

$$d(\bar{x}, z) \leq \gamma_0 + \phi(d(\bar{x}, y)) \leq d(\bar{x}, y) \leq M_1.$$

Thus the following property holds:

(P6) if $y \in K \cap B(\bar{x}, M_1)$, $z \in X$ and $d(z, T(y)) \leq \delta$, then $d(\bar{x}, z) \leq M_1$.

Choose a natural number k such that

$$k > 2(M_1 + 1)\gamma_0^{-1} + 2. \tag{4.41}$$

Assume that n is a natural number such that $n \geq k$ and that

$$\{x_i\}_{i=0}^n \subset K,\ d(x_0, \theta) \leq M_0 \tag{4.42}$$

and that

$$d(x_{i+1}, T(x_i)) \leq \delta,\ i = 0, \dots, n-1. \tag{4.43}$$

It follows from (3.3), (4.30) and (4.42),

$$d(x_0, \bar{x}) \leq M_1.$$

Together with property (P6), (4.42) and (4.43) this implies that

$$d(x_i, \bar{x}) \leq M_1,\ i = 0, \dots, n. \tag{4.44}$$

We claim that

$$d(x_i, \bar{x}) \leq \epsilon,\ i = k, \dots, n.$$

First we show that there exists $j \in \{0, \dots, k-1\}$ such that

$$d(x_j, \bar{x}) \leq \epsilon/2.$$

Assume the contrary. Then

$$d(x_i, \bar{x}) > \epsilon/2,\ i = 0, \dots, k-1. \tag{4.45}$$

Let $i \in \{0, \dots, k-1\}$. By (4.1), (4.30), (4.31), (4.44) and (4.45) and the equation $(T(\bar{x}) = \bar{x}$,

$$d(T(x_i), \bar{x}) \leq \phi(d(x_i, \bar{x})) \leq d(x_i, \bar{x}) - \gamma_0. \tag{4.46}$$

Equations (4.43), (4.44), (4.46) and property (P5) imply that

$$d(x_{i+1}, \bar{x}) - d(T(x_i), \bar{x}) \leq \gamma_0/4. \tag{4.47}$$

In view of (4.46) and (4.47),

$$d(x_{i+1}, \bar{x}) \leq d(x_i, \bar{x}) - \gamma_0/2.$$

Together with (4.44) this implies that

$$M_1 \geq d(x_0, \bar{x}) \geq d(x_0, \bar{x}) - d(x_k, \bar{x})$$

$$= \sum_{i=0}^{k-1} (d(x_i, \bar{x}) - d(x_{i+1}, \bar{x})) \geq k\gamma_0/2$$

and

$$k \leq 2\gamma_0^{-1} M_1.$$

This contradicts (4.41). The contradiction we have are reached proves that there exists an integer $j \in \{0, \ldots, k-1\}$ such that

$$d(x_j, \bar{x}) \leq \epsilon/2. \tag{4.48}$$

Assume that there is an integer p such that

$$p \in \{j, \ldots, n\} \text{ and } d(x_p, \bar{x}) > \epsilon.$$

In view of (4.48), we may assume without loss of generality that

$$d(x_i, \bar{x}) \leq \epsilon, \ i = j, \ldots, p-1. \tag{4.49}$$

There are two cases:

$$d(x_{p-1}, \bar{x}) \leq \epsilon/2; \tag{4.50}$$

$$d(x_{p-1}, \bar{x}) > \epsilon/2. \tag{4.51}$$

Assume that (4.50) holds. By (4.1) and (4.50),

$$d(T(x_{p-1}), \bar{x}) \leq \epsilon/2. \tag{4.52}$$

Assume that (4.51) holds. By (4.1), (4.31), (4.44), (4.49) and (4.51),

$$d(T(x_{p-1}), \bar{x}) \leq \phi(d(x_{p-1}, \bar{x})) \leq d(x_{p-1}, \bar{x}) - \gamma_0 \leq \epsilon - \gamma_0. \tag{4.54}$$

Property (P5) and equations (4.1), (4.43) and (4.44) imply that

$$|d(x_p, \bar{x}) - d(T(x_{p-1}), \bar{x})| \leq \gamma_0/4. \tag{4.55}$$

It follows from (4.52), (4.54) and (4.55) that

$$d(x_p, \bar{x}) \leq \epsilon,$$

a contradiction. The contradiction we have reached proves that

$$d(x_i, \bar{x}) \leq \epsilon, \ i = j, \ldots, n.$$

Thus part (B) of our theorem is also proved. This completes the proof of Theorem 4.1.

□

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[2] A. Betiuk-Pilarska and T. Domínguez Benavides, *Fixed points for nonexpansive mappings and generalized nonexpansive mappings on Banach lattices*, Pure Appl. Func. Anal. **1** (2016), 343–359.

[3] Y. Censor and M. Zaknoon, *Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review*, Pure Appl. Func. Anal. **3** (2018), 565–586.

[4] F. S. de Blasi and J. Myjak, *Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach*, C. R. Acad. Sci. Paris **283** (1976), 185–187.

[5] F. S. de Blasi, J. Myjak, S. Reich and A. J. Zaslavski, *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set-Valued and Variational Analysis **17** (2009) , 97–112.

[6] A. Gibali, *A new split inverse problem and an application to least intensity feasible solutions*, Pure Appl. Funct. Anal. **2** (2017), 243–258.

[7] A. Gibali, S. Reich and R. Zalas, *Outer approximation methods for solving variational inequalities in Hilbert space*, Optimization **66** (2017), 417–437.

[8] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.

[9] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, New York and Basel, 1984.

[10] J. Jachymski, *Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points*, Pure Appl. Funct. Anal. **2** (2017), 657–666.

[11] M. A. Khamsi and W. M. Kozlowski, *Fixed point theory in modular function spaces*, Birkhäser/Springer, Cham, 2015.

[12] M. A. Khamsi, W. M. Kozlowski and S. Reich, *Fixed point theory in modular function spaces*, Nonlinear Anal. **14** (1990), 935–953.

[13] W. A. Kirk, *Contraction mappings and extensions*, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 1–34.

[14] W. M. Kozlowski, *An introduction to fixed point theory in modular function spaces*, Topics in fixed point theory, 15–222, Springer, Cham, 2014.

[15] R. Kubota, W. Takahashi and Y. Takeuchi, *Extensions of Browder's demiclosedness principle and Reich's lemma and their applications*, Pure Appl. Func. Anal. **1** (2016), 63–84.

[16] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459–465.

[17] S. Reich and A. J. Zaslavski, *Generic aspects of metric fixed point theory*, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 557–575.

[18] S. Reich and A. J.Zaslavski, *Convergence of iterates for a class of mappings of contractive type*, JP J. Fixed Point Theory Appl., 2, 2007, 69–78.

[19] S. Reich and A. J. Zaslavski, *Genericity in nonlinear analysis*, Developments in Mathematics, **34**, Springer, New York, 2014.

[20] S. Reich and A. J. Zaslavski, *Contractivity and genericity results for a class of nonlinear mappings*, J. Nonlinear Convex Anal. **16** (2015), 1113–1122.

[21] S. Reich and A. J. Zaslavski, *On a class of generalized nonexpansive mappings*, Mathematics, **8**, 1085, 2020, https://doi.org/10.3390/math8071085.

[22] S. Reich and A. J. Zaslavski, *A fixed point result in generalized metric spaces,* Journal of Analysis, accepted for publication.

[23] W. Takahashi, *The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces*, Pure Appl. Funct. Anal. **2** (2017), 685–699.

[24] W. Takahashi, *A general iterative method for split common fixed point problems in Hilbert spaces and applications*, Pure Appl. Funct. Anal. **3** (2018), 349–369.

[25] A. J. Zaslavski, *Approximate solutions of common fixed point problems*, Springer Optimization and Its Applications, Springer, Cham, 2016.
[26] A. J. Zaslavski, *Algorithms for solving common fixed point problems*, Springer Optimization and Its Applications, Springer, Cham, 2018.

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