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AN IMPLICIT ITERATIVE ALGORITHM FOR FINDING COMMON SOLUTIONS OF CONVEX MINIMIZATION, VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

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This paper is dedicated to the memory of Professor Kazimierz Goebel

Abstract. In this paper, we introduce an implicit iterative algorithm for finding a common element of the set of minimizers of a convex function, the set of solutions to a variational inequality problem for a continuous monotone mapping and the set of fixed points of a continuous pseudocontractive mapping in Hilbert spaces. Under suitable control conditions, we establish strong convergence of sequences generated by the proposed iterative algorithms to a common element of three sets, which is a solution of a certain variational inequality. As a direct consequence, we obtain the unique minimum-norm common element of three sets.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be self-mapping on C. We denote by $Fix(T)$ the set of fixed points of T.

The minimization problem (shortly, MP) is one of most import problems in nonlinear analysis and optimization theory. The MP is defined as follows: find $x \in H$ such that

$$F(x) = \min_{y\in H} F(y), \tag{1.1}$$

where $F : H \to (-\infty,\infty]$ is a proper convex and lower semi-continuous. The set of solutions to the MP(1.1), that is, the set of all minimizers of F is denoted by $\operatorname{argmin}_{y\in H} F(y)$. A successful and powerful tool for solving the MP is well-known proximal point algorithm (shortly, the PPA) which initiated by Martinet [8] and later studied by Rockafellar [12] in 1976. The PPA is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \operatorname{argmin}_{y\in H}\left[F(y) + \frac{1}{2\lambda_n}\|x_n - y\|^2\right], \end{cases}$$

where $\lambda_n > 0$ for all $n \geq 1$.

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Let $A : C \to H$ be a nonlinear mapping. The classical variational inequality problem (shortly, VIP) is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

This problem is called Hartman-Stampacchia variational inequality ([7, 14]). We denote the set of solutions to the VIP(1.2) by $VI(C, A)$. As we also know, variational inequality theory has emerged as an important tool in studying a lot of real-life problems, such as, in signal processing, resource allocation, image recovery and so on.

A fixed point problem (shortly, FPP) is to find a fixed point z of a nonlinear mapping T with property:

$$z \in C, \ Tz = z. \tag{1.3}$$

Fixed point theory is one of the most powerful and important tools of modern mathematics and may be considered a core subject of nonlinear analysis.

As we all know, the convex feasibility problem (shortly, CFP) is the problem of finding a point in the (nonempty) intersection $C = \cap_{i=1}^{m} C_i$ of a finite number of closed convex sets C_i $(i = 1, \cdots, m)$.

Recently, many authors considered iterative algorithms for finding a common element of solution sets of the MP(1.1), VIP(1.2), and the FPP(1.3) combined with some nonlinear problems as special cases of the CFP. For instance, we can refer to Takahashi and Toyoda [16] for the VIP(1.2) for an inverse strongly monotone mapping A and the FPP(1.3) for a nonexpansive mapping T, refer to Peng and Yao [11] for the VIP(1.2) for a monotone and Lipschitz continuous mapping A and the FPP(1.3) for a nonexpansive mapping T, refer to Jung [4] for the VIP(1.2) for an inverse strongly monotone mapping A and the FPP(1.3) for a strictly pseudocontractive mapping T, and refer to Jung [5] for the VIP(1.2) for a continuous monotone mapping A and the FPP(1.3) for a continuous pseudocontractive mapping T, and refer to Jung [6] for the the MP(1.1) for a real-valued convex (Fréchet) differentiable function F and the FPP(1.3) for a continuous pseudocontractive mapping T.

In particular, in 2020, Sow [13] considered an iterative algorithm for the MP(1.1) of a convex function F, the VIP(1.2) for an inverse strongly monotone mapping A and the FPP(1.3) for a demicontractive mapping T and a strictly pseudocontractive mapping mapping T.

In this paper, in order to study the MP(1.1) combined with the VIP(1.2) and the FPP(1.3), we introduce an implicit iterative algorithm for finding a common element of the set of solutions to the MP(1.1) for F, the set of solutions to the VIP(1.2) for A and the set of fixed points of T, where $F : C \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function, $A : C \to H$ are continuous monotone mapping and $T : C \to C$ is a continuous pseudocontractive mapping. Then we establish strong convergence of the sequence generated by the proposed iterative algorithm to a common element of three aforementioned sets, which is a solution of a certain variational inequality. As a direct consequence, we find the unique solution of the minimum-norm problem:

$$\|x^*\| = \min\{\|x\| : x \in \Phi\},$$

where $\Phi := \operatorname{argmin}_{y\in C} F(y) \cap VI(C,A) \cap Fix(T)$. The results in this paper develop and complement of the recent results announced by several authors in this direction.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

We recall ([1, 3]) that a mapping F of C into H is called

(i) *Lipschitzian* if there exists a constant $\kappa \geq 0$ such that

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \forall x, y \in C;$$

(ii) *monotone* if $\langle x - y, Fx - Fy\rangle \geq 0, \quad \forall x,\ y \in C;$

(iii) *α-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Fx - Fy\rangle \geq \alpha\|Fx - Fy\|^2, \quad \forall x,\ y \in C;$$

(iv) *η-strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Fx - Fy\rangle \geq \eta\|x - y\|^2, \quad \forall x,\ y \in C.$$

We note that if F is an α-inverse strongly monotone mapping of C into H, then it is obvious that F is $\frac{1}{\alpha}$-Lipschitz continuous, that is, $\|Fx - Fy\| \leq \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$. Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

We recall ([1]) that a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

and T is said to be *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x,\ y \in C,$$

where I is the identity mapping. The class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is *nonexpansive* (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$, $\forall x,\ y \in C$) if and only if T is 0-strictly pseudocontractive.

In a real Hilbert space H, the following hold:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.1}$$

for all $x,\ y \in H$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

P_C is called the *metric projection* of H onto C. It is well known ([15])that P_C is nonexpansive and P_C is characterized by the property

$$u = P_C x \Longleftrightarrow \langle x - u, u - y\rangle \geq 0, \quad \forall x \in H,\ y \in C. \tag{2.2}$$

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([18]). *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. Then, for $\omega > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle Az, y - z\rangle + \frac{1}{\omega}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

For $\omega > 0$ and $x \in H$, define $A_\omega : H \to C$ by

$$A_\omega x = \left\{ z \in C : \langle Az, y - z\rangle + \frac{1}{\omega}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *A_ω is single-valued;*
(ii) *A_ω is firmly nonexpansive, that is,*

$$\|A_\omega x - A_\omega y\|^2 \leq \langle A_\omega x - A_\omega y, x - y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(A_\omega) = VI(C, A)$;*
(iv) *$V(I, A)$ is a closed convex subset of C.*

Lemma 2.2 ([18]). *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*
(iv) *$Fix(T)$ is a closed convex subset of C.*

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 2.3. *Let $V : C \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$, and $G : C \to H$ be a ρ-Lipschitzian and η-strongly monotone mapping with constants ρ and $\eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle (\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

We also need the following lemma (see [17] for the proof).

Lemma 2.4. *Let C be a nonempty closed subspace of a Hilbert space H. Let $G : C \to H$ be a ρ-Lipschizian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 < t \leq 1$. Then $I - t\mu G : C \to H$ is a contraction with contractive constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$.*

Lemma 2.5 ([1] (Demiclosedness principle))**.** *Let H be a real Hilbert space, let C be a closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed, that is,*

$$\{x_n\} \subset C, \quad x_n \rightharpoonup x \in C \text{ and } (I - T)x_n \to y \text{ implies that } (I - T)x = y.$$

Let $F : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of F in a real Hilbert space H as follows:

$$J_\lambda^F x = \operatorname*{argmin}_{y \in C}\left[F(y) + \frac{1}{2\lambda}\|x - y\|^2\right]$$

for all $x \in H$. It was shown in [2] that the set of fixed points of the resolvent associated with F coincides with the set of minimizers of F. Also the resolvent J_λ^F of F is single-valued and nonexpansive for all $\lambda > 0$.

Lemma 2.6. ([10]) *For any $r > 0$ and $\mu > 0$, the following holds:*

$$J_r^F x = J_\mu^F\left(\frac{\mu}{r}x + \left(1 - \frac{\mu}{r}\right)J_r^F x\right).$$

The following lemma is a variant of a Minty lemma (see [9]).

Lemma 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $G : C \to H$ is monotone and weakly continuous along segments, that is, $G(x + ty) \to G(x)$ weakly as $t \to 0$. Then the variational inequality*

$$\widetilde{x} \in C, \quad \langle G\widetilde{x}, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C,$$

is equivalent to the dual variational inequality

$$\widetilde{x} \in C, \quad \langle Gp, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C.$$

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subset of H;
- $F : C \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function;
- $\operatorname{argmin}_{y\in C} F(y)$ is the set of minimizers of F on C;
- $A : C \to H$ is a continuous monotone mapping ;
- $A_{\omega_n} : H \to C$ is a mapping defined by
$$A_{\omega_n}x = \left\{ z \in C : \langle y - z, Az\rangle + \frac{1}{\omega_n}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$
for $\omega_n \in (0, \infty)$ and $\liminf_{n\to\infty} \omega_n > 0$;
- $VI(C, A)$ is the set of the VIP(1.2) for A;
- $T : C \to C$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $T_{r_n} : H \to C$ is a mapping defined by
$$T_{r_n}x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \leq 0, \quad \forall y \in C \right\}$$
for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $V : C \to H$ is l-Lipschitzian mapping with constant $l \in [0, \infty)$;
- $G : C \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;
- Constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- $P_C : H \to C$ is the metric projection of H onto C.
- $\Phi := \operatorname{argmin}_{y\in C} F(y) \cap VI(C, A) \cap \cap Fix(T) \neq \emptyset$.

By Lemma 2.1 and Lemma 2.2, A_{ω_n} and T_{r_n} are nonexpansive and $VI(C, A) = Fix(A_{\omega_n})$, and $Fix(T) = Fix(T_{r_n})$.

First, we introduce the following iterative algorithm which generates a sequence $\{x_n\}$ in an implicit way:

$$\begin{cases} v_n = \operatorname{argmin}_{y\in C}\left[F(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2\right], \\ z_n = A_{\omega_n}v_n, \\ x_n = \ P_C[\alpha_n\gamma V x_n + (I - \alpha_n\mu G)(\beta_n x_n + (1 - \beta_n)T_{r_n}z_n)], \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$; $\{\lambda_n\}$, $\{\omega_n\}$, $\{r_n\} \subset (0, \infty)$; and $x_1 \in C$ is an arbitrary initial guess.

From Lemma 2.1, Lemma 2.2 and noxexpansivity of $J^F_{\lambda_n}$, it follows that $W_n := T_{r_n}A_{\omega_n}J^F_{\lambda_n}$ is nonexpansive.

Now, consider the following mapping Q_n on C defined by, for $x \in C$,

$$\begin{aligned} Q_nx &= P_C[\alpha_n\gamma V x + (I - \alpha_n\mu G)(\beta_n x + (1 - \beta_n)W_n x)] \\ &= P_C[\alpha_n\gamma V x + (I - \alpha_n\mu G)(\beta_n x + (1 - \beta_n)T_{r_n}A_{\omega_n}J^F_{\lambda_n}x)], \ \forall \ n \geq 1. \end{aligned}$$

Let $R_nx = \beta_nx + (1-\beta_n)T_{r_n}A_{\omega_n}J^F_{\lambda_n}x = \beta_nx + (1-\beta_n)W_nx$. Since W_n is nonexpansive, we have for $x,\ y \in C$

$$\begin{aligned}\|R_nx - R_ny\| &\le \beta_n\|x-y\| + (1-\beta_n)\|W_nx - W_ny\|\\ &\le \beta_n\|x-y\| + (1-\beta_n)\|x-y\| = \|x-y\|.\end{aligned}$$

Then, by Lemma 2.4, we derive for $x, y \in C$,

$$\begin{aligned}\|Q_nx - Q_ny\| &= \|P_C(\alpha_nVx + (I+\alpha_n\mu G)R_nx] - P_C(\alpha_nVx + (I+\alpha_n\mu G)R_ny]\|\\ &\le \|\alpha_nVx + (I+\alpha_n\mu G)R_nx - (\alpha_nVx + (I+\alpha_n\mu G)R_ny)\|\\ &\le \alpha_n\gamma\|Vx - Vy\| + \|(I-\alpha_n\mu G)R_nx - (I-\alpha_n\mu G)R_ny\|\\ &\le \alpha_n\gamma l\|x-y\| + (1-\alpha_n\tau)\|R_nx - R_ny\|\\ &\le \alpha_n\gamma l\|x-y\| + (1-\alpha_n\tau)\|x-y\|\\ &= (1-\alpha_n(\tau-\gamma l)\|x-y\|.\end{aligned}$$

Since $0 < 1-\alpha_n(\tau-\gamma l) < 1$, Q_n is a contractive mapping. Therefor, by the Banach contraction principle, Q_n has a unique fixed point $x_n \in H$, which uniquely solves the fixed point equation

$$\begin{aligned}x_n &= P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)(\beta_nx_n + (1-\beta_n)T_{r_n}A_{\omega_n}J^F_{\lambda_n}x_n)]\\ &= P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)(\beta_nx_n + (1-\beta_n)W_nx_n)]\\ &= P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n],\end{aligned}$$

where $y_n = \beta_nx_n + (1-\beta_n)T_{r_n}A_{\omega_n}J^F_{\lambda_n}x_n$. We note that $J^F_{\lambda_n}x_n = v_n$ and $T_{r_n}z_n = T_{r_n}A_{\omega_n}v_n = T_{r_n}A_{\omega_n}J^F_{\lambda_n}x_n = W_nx_n$.

Now we prove strong convergence of the sequence $\{x_n\}$ and show the existence of $q \in \Phi$, which solves the variational inequality

$$\langle(\mu G - \gamma V)q, p-q\rangle \ge 0, \quad \forall\ p \in \Phi. \tag{3.2}$$

Equivalently, $q = P_\Phi(I-\mu G+\gamma V)q$ (by (2.2))

Theorem 3.1. *Let $\{x_n\}$ be a sequence defined by* (3.1). *Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\omega_n\}$, and $\{r_n\}$ be satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(iii) $0 < \lambda \le \lambda_n < \infty$;
(iv) $0 < \omega \le \omega_n < \infty$;
(v) $0 < r \le r_n < \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to a point $q \in \Phi$, which is the unique solution of the variational inequality (3.2).

Proof. First, we can show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \Longleftrightarrow \kappa \ge \eta$, it follows from Lemma 2.3 that

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x-y\rangle \ge (\mu\eta - \gamma l)\|x-y\|^2.$$

That is, $\mu G - \gamma V$ is strongly monotone for $0 \leq \gamma l < \tau \leq \mu\eta$. So the variational inequality (3.2) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the variational inequality (3.2).

Note that from the condition (i), without loss of generality, we assume that $\alpha_n(\tau - \gamma l) < 1$ for $n \geq 1$. From now, we put $v_n = J^F_{\lambda_n} x_n$, $z_n = A_{\omega_n} v_n$, $w_n = T_{r_n} z_n$ and $y_n = \beta_n x_n + (1-\beta_n) T_{r_n} A_{\omega_n} v_n = \beta_n x_n + (1-\beta_n) T_{r_n} z_n = \beta_n x_n + (1-\beta_n) W_n x_n$ for $n \geq 1$. Let $p \in \Phi$. Then, from Lemma 2.1 and Lemma 2.2, it follows that $p = A_{\omega_n} p$ and $p = T_{r_n} p$. Also we have $p = J^F_{\lambda_n} p$. In fact, since $F(p) \leq F(y)$ for all $y \in C$, this implies

$$F(p) + \frac{1}{2\lambda_n}\|p - p\|^2 \leq F(y) + \frac{1}{2\lambda_n}\|y - p\|^2,$$

and hence $J^F_{\lambda_n} p = p$ for all $n \geq 1$, where $J^F_{\lambda_n}$ is the Moreau-Yosida resolvent of F on C.

Since $J^F_{\lambda_n}$, A_{ω_n} and T_{r_n} are nonexpansive as firmly nonexpansive, the following inequalities hold:

$$\|v_n - p\| = \|J^F_{\lambda_n} x_n - J^F_{\lambda_n} p\| \leq \|x_n - p\|, \tag{3.3}$$

$$\|z_n - p\| = \|A_{\omega_n} v_n - A_{\omega_n} p\| \leq \|v_n - p\| \quad (\leq \|x_n - p\|), \tag{3.4}$$

$$\|w_n - p\| = \|T_{r_n} z_n - T_{r_n} p\| \leq \|z_n - p\| (\leq \|v_n - p\| \leq \|x_n - p\|) \tag{3.5}$$

Now, we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in \Phi$. Then, from (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n - (1-\beta_n) W_n x_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|W x_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{3.6}$$

Therefore, by (3.6) and Lemma 2.4, we drive

$$\begin{aligned} \|x_n - p\| &= \|P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n] - P_C p\| \\ &\leq \|\alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n - p\| \\ &\leq \|\alpha_n(\gamma V x_n - \gamma V p) + (I - \alpha_n \mu G) y_n - (I - \alpha_n \mu G) p + \alpha_n(\gamma V p - \mu G p)\| \\ &\leq \alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|y_n - p\| + \alpha_n \|\gamma V p - \mu G p\| \\ &\leq \alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|x_n - p\| + \alpha_n(\gamma \|V p\| + \mu \|G p\|), \end{aligned}$$

and so

$$\|x_n - p\| \leq \frac{\gamma\|Vp\| + \mu\|Gp\|}{\tau - \gamma l}.$$

Thus, $\{x_n\}$ is bounded and $\{z_n\}$, $\{v_n\}$, $\{G y_n\}$, $\{V x_n\}$ and $\{T_{r_n} z_n\}$ are also bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_n - T_{r_n}z_n\| = 0$. Indeed, observing that

$$\begin{aligned}\|x_n - T_{r_n}z_n\| &= \|P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n] - P_C[T_{r_n}z_n]\|\\ &\le \|\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n] - P_C[T_{r_n}z_n]\|\\ &\le \alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|y_n - T_{r_n}u_n\|\\ &= \alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|\beta_n x_n + (1-\beta_n)T_{r_n}z_n - T_{r_n}z_n\|\\ &= \alpha_n\|\gamma Vx_n - \mu Gy_n\| + \beta_n\|x_n - T_{r_n}z_n\|,\end{aligned}$$

we obtain

$$\|x_n - T_{r_n}z_n\| \le \frac{\alpha_n}{1-\beta_n}\|\gamma Vx_n - \mu Gy_n\| \to 0 \quad \text{as } n\to\infty.$$

Step 3. We show that $\lim_{n\to\infty}\|v_n - x_n\| = \lim_{n\to\infty}\|J^F_{\lambda_n}x_n - x_n\| = 0$. In fact, using $v_n = J^F_{\lambda_n}x_n$, $p = J^F_{\lambda_n}p$ and firmly nonexpansivity of $J^F_{\lambda_n}$, we obtain from (2.1) that

$$\begin{aligned}\|v_n - p\|^2 &= \|J^F_{\lambda_n}x_n - p\|^2\\ &\le \langle J^F_{\lambda_n}x_n - J^F_{\lambda_n}p, x_n - p\rangle\\ &= \langle v_n - p, x_n - p\rangle\\ &= \frac{1}{2}(\|v_n - p\|^2 + \|x_n - p\|^2 - \|v_n - x_n\|^2).\end{aligned}$$

This implies

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|v_n - x_n\|^2. \tag{3.7}$$

Again, noting that $y_n = \beta_n x_n + (1-\beta_n)T_{r_n}z_n$ and $x_n = P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n]$, by (3.5) and (3.7), we induce that

(3.8)

$$\begin{aligned}\|x_n - p\|^2 &= \|P_C[\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n] - P_Cp\|^2\\ &\le \|\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n - p\|^2\\ &= \|\alpha_n(\gamma Vx_n - \mu Gy_n) + (y_n - p)\|^2\\ &= \|\alpha_n(\gamma Vx_n - \mu Gy_n) + \beta_n(x_n - T_{r_n}z_n) + (T_{r_n}z_n - p)\|^2\\ &\le [(\|\alpha_n(\gamma Vx_n - \mu Gy_n)\| + \|z_n - p\|) + \beta_n\|x_n - T_{r_n}z_n\|]^2\\ &\le [(\|\alpha_n(\gamma Vx_n - \mu Gy_n)\| + \|v_n - p\|) + \beta_n\|x_n - T_{r_n}z_n\|]^2\\ &= \alpha_n^2\|\gamma Vx_n - \mu Gy_n\|^2 + 2\alpha_n\|\gamma Vx_n - \mu Gy_n\|\|v_n - p\| + \|v_n - p\|^2\\ &\quad + \beta_n\|x_n - T_{r_n}z_n\|2(\alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|v_n - p\|)\\ &\quad + \beta_n^2\|x_n - T_{r_n}z_n\|^2\\ &\le \alpha_n\|\gamma Vx_n - \mu Gy_n\|^2 + \|v_n - p\|^2 + M_n\\ &\le \alpha_n\|\gamma Vx_n - \mu Gy_n\|^2 + \|x_n - p\|^2 - \|v_n - x_n\|^2 + M_n,\end{aligned}$$

where

(3.9)
$$
\begin{aligned}
M_n = {} & 2\alpha_n\|\gamma V x_n - \mu G y_n\|\|v_n - p\| \\
& + \beta_n\|x_n - T_{r_n} z_n\| 2(\alpha_n\|\gamma V x_n - \mu G y_n\| + \|v_n - p\|) + \beta_n^2\|x_n - T_{r_n} z_n\|^2.
\end{aligned}
$$

From (3.8), we obtain

$$\|v_n - x_n\|^2 \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + M_n.$$

Since $M_n \to 0$ as $n \to \infty$ by condition (i) and Step 2, we get

$$\lim_{n\to\infty} \|v_n - x_n\| = \lim_{n\to\infty} \|J^F_{\lambda_n} x_n - x_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|z_n - v_n\| = \lim_{n\to\infty} \|A_{\omega_n} v_n - v_n\| = 0$. Again, since $z_n = A_{\omega_n} v_n$, $p = A_{\omega_n} p$ and A_{ω_n} is firmly nonexpansive (Lemma 2.1 (ii)), from (2.1), we have

$$
\begin{aligned}
\|z_n - p\|^2 &= \|A_{\omega_n} v_n - p\|^2 \\
&\le \langle A_{\omega_n} v_n - A_{\omega_n} p, v_n - p\rangle \\
&= \langle z_n - p, v_n - p\rangle \\
&= \frac{1}{2}(\|z_n - p\|^2 + \|v_n - p\|^2 - \|z_n - v_n\|^2).
\end{aligned}
$$

This implies

$$\|z_n - p\|^2 \le \|v_n - p\|^2 - \|z_n - v_n\|^2 \le \|x_n - p\|^2 - \|z_n - v_n\|^2. \tag{3.10}$$

Now, from (3.8), (3.9) and (3.10), we derive

$$
\begin{aligned}
\|x_n - p\|^2 &\le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|u_n - p\|^2 + M_n \\
&\le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|z_n - p\|^2 + M_n \\
&\le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|x_n - p\|^2 - \|z_n - v_n\|^2 + M_n,
\end{aligned}
$$

where M_n is of (3.9) and so

$$\|z_n - v_n\|^2 \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + M_n.$$

From $\lim_{n\to\infty} M_n = 0$ and condition (i), it follows that

$$\lim_{n\to\infty} \|z_n - v_n\| = \lim_{n\to\infty} \|A_{\omega_n} v_n - v_n\| = 0.$$

Step 5. We show that $\lim_{n\to\infty} \|z_n - T_{r_n} z_n\| = \lim_{n\to\infty} \|w_n - z_n\| = 0$. In fact, since $\|z_n - T_{r_n} z_n\| \le \|z_n - v_n\| + \|v_n - x_n\| + \|x_n - T_{r_n} z_n\|$, by Step 2, Step 3 and Step 4, we conclude

$$\lim_{n\to\infty} \|z_n - T_{r_n} z_n\| = \lim_{n\to\infty} \|z_n - w_n\| = 0.$$

Step 6. We show that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Indeed, by Step 2 and Step 5,

$$\|x_n - z_n\| \le \|x_n - T_{r_n} z_n\| + \|T_{r_n} z_n - z_n\| \to \infty \ \text{ as } n \to \infty.$$

Step 7. We show that $\{x_n\}$ converges strongly to $q \in \Phi$ as $n \to \infty$, where q is the unique solution of variational inequality (3.2). To this end, consider a subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Since $\{x_n\}$ is bounded, there exists a subsequence

$\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly $q \in C$. Without loss of generality, we can assume $x_{n_i} \rightharpoonup q \in C$.

First of all, by (3.1) and Lemma 2.4, we induce for $p \in \Phi$

$$\begin{aligned}
&\|x_n - p\|^2 \\
&= \|P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n] - P_C p\|^2 \\
&\leq \|\alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n - p\|^2 \\
&= \|(I - \alpha_n \mu G) y_n - (I - \alpha_n \mu G) p - \alpha_n (\mu G - \gamma V) p + \alpha_n \gamma (V x_n - V p)\|^2 \\
&= \|(I - \mu G) y_n - (I - \mu G) p\|^2 \\
&\quad - 2\alpha_n [\langle (\mu G - \gamma V) p, y_n - p\rangle - \alpha_n \langle (\mu G - \gamma V) p, \mu G y_n - \mu G p\rangle] \\
&\quad + 2\alpha_n \gamma [\langle V x_n - V p, y_n - p\rangle - \alpha_n \langle V x_n - V p, \mu G y_n - \mu G p\rangle] \\
&\quad - 2\alpha_n^2 \gamma \langle (\mu G - \gamma V) p, V x_n - V p\rangle \\
&\quad + \alpha_n^2 \|(\mu G - \gamma V) p\|^2 + \alpha_n^2 \gamma^2 \|V x_n - V p\|^2 \\
&\leq (1 - \alpha_n \tau)^2 \|y_n - p\|^2 - 2\alpha_n \langle (\mu G - \gamma V) p, y_n - p\rangle \\
&\quad + 2\alpha_n \gamma l \|x_n - p\| \|y_n - p\| + 2\alpha_n^2 \|(\mu G - \gamma V) p\| (\mu \|G y_n\| + \mu \|G p\|) \\
&\quad + 2\alpha_n^2 \gamma l \|x_n - p\| ((\mu \|G y_n\| + \mu \|G p\|) + 2\alpha_n^2 \gamma l \|(\mu G - \gamma V) p\| \|x_n - p\| \\
&\quad + \alpha_n^2 \|(\mu G - \gamma V) p\|^2 + \alpha_n^2 \gamma^2 l^2 \|x_n - p\|^2 \\
&= (1 - 2\alpha_n \tau + \alpha_n^2 \tau^2) \|y_n - p\|^2 - 2\alpha_n \langle (\mu G - \gamma V) p, y_n - p\rangle \\
&\quad + 2\alpha_n \gamma l \|x_n - p\| \|y_n - p\| + 2\alpha_n^2 \|(\mu G - \gamma V) p\| (\mu \|G y_n\| + \mu \|G p\|) \\
&\quad + 2\alpha_n^2 \gamma l \|x_n - p\| (\mu \|G y_n\| + \mu \|G p\|) + 2\alpha_n^2 \gamma l \|(\mu G - \gamma V) p\| \|x_n - p\| \\
&\quad + \alpha_n^2 (\|(\mu G - \gamma V) p\|^2 + \gamma^2 l^2 \|x_n - p\|^2) \\
&\leq (1 - 2\alpha_n \tau) \|y_n - p\|^2 + 2\alpha_n \langle (\mu G - \gamma V) p, p - y_n\rangle \\
&\quad + \alpha_n \tau l (\|x_n - p\|^2 + \|y_n - p\|^2) + \alpha_n^2 M,
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
M = \sup\{&\tau^2 \|y_n - p\|^2 + 2(\|(\mu G - \gamma V) p\| + \gamma l \|x_n - p\|)(\mu \|G y_n\| + \mu \|G p\|) \\
&+ 2\gamma l \|(\mu G - \gamma V) p\| \|x_n - p\| + \|(\mu G - \gamma V) p\|^2 + \gamma^2 l^2 \|x_n - p\|^2 : n \geq 1\}.
\end{aligned}$$

Hence by (3.6) and (3.11), we obtain

$$\begin{aligned}
\|x_n - p\|^2 &\leq \frac{1 - 2\alpha_n \tau + \alpha_n \gamma l}{1 - \alpha_n \gamma l} \|y_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma l} \langle (\mu G - \gamma V) p, p - y_n\rangle \\
&\quad + \frac{\alpha_n^2}{1 - \alpha_n \gamma l} M \\
&\leq \frac{1 - 2\alpha_n \tau + \alpha_n \gamma l}{1 - \alpha_n \gamma l} \|x_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \gamma l} \langle (\mu G - \gamma V) p, p - y_n\rangle \\
&\quad + \frac{\alpha_n^2}{1 - \alpha_n \gamma l} M.
\end{aligned} \tag{3.12}$$

Observe that

$$
\begin{aligned}
\langle(\mu G-\gamma V)p, p-y_n\rangle &= \langle(\mu G-\gamma V)p, p-(\beta_n x_n+(1-\beta_n)T_{r_n}z_n)\rangle \\
&= \langle(\mu G-\gamma V)p, p-T_{r_n}z_n\rangle \\
&\quad + \beta_n\langle(\mu G-\gamma V)p, T_{r_n}z_n-x_n\rangle \\
&= \langle(\mu G-\gamma V)p, p-z_n\rangle \\
&\quad + \langle(\mu G-\gamma V)p, u_n-T_{r_n}z_n\rangle \\
&\quad + \beta_n\langle(\mu G-\gamma V)p, T_{r_n}z_n-x_n\rangle \\
&\le \langle(\mu G-\gamma V)p, p-z_n\rangle \\
&\quad + \|(\mu G-\gamma V)p\|\|u_n-T_{r_n}z_n\| \\
&\quad + \beta_n\|(\mu G-\gamma V)p\|\|T_{r_n}u_n-x_n\| \\
&\le \langle(\mu G-\gamma V)p, p-z_n\rangle + L_n,
\end{aligned}
\tag{3.13}
$$

where $L_n = \|(\mu G-\gamma V)p\|\|z_n-T_{r_n}z_n\|+\beta_n\|(\mu G-\gamma V)p\|\|T_{r_n}z_n-x_n\|$. Then, from (3.12) and (3.13), we derive

$$
\|x_n-p\|^2 \le \frac{1}{\tau-\gamma l}\langle\mu G-\gamma Vp, p-z_n\rangle + \frac{\alpha_n M}{2(\tau-\gamma l} + \frac{L_n}{\tau-\gamma l}. \tag{3.14}
$$

Now, we show that $q \in \Phi$. For this purpose, we divide its proof into three steps.

(i) We prove that $q \in \operatorname{argmin}_{y\in C} F(y)$. Using $v_n = J_{\lambda_n}^F x_n$, Lemma 2.6 and condition (iii), we derive

$$
\begin{aligned}
\|x_n-J_\lambda^F x_n\| &\le \|v_n-J_\lambda^F x_n\|+\|v_n-x_n\| \\
&= \|J_{\lambda_n}^F x_n-J_\lambda^F x_n\|+\|v_n-x_n\| \\
&= \|v_n-x_n\| + \left\|J_\lambda^F\left(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}^F x_n+\frac{\lambda}{\lambda_n}x_n\right)-J_\lambda^F x_n\right\| \\
&\le \|v_n-x_n\| + \left\|\left(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}^F x_n+\frac{\lambda}{\lambda_n}x_n\right)-x_n\right\| \\
&= \|v_n-x_n\| + \left(1-\frac{\lambda}{\lambda_n}\right)\|v_n-x_n\| \\
&= \left(2-\frac{\lambda}{\lambda_n}\right)\|v_n-x_n\| \\
&\le L\|v_n-x_n\|
\end{aligned}
$$

for some $L > 0$. Hence it follows from Step 3 that

$$
\lim_{n\to\infty}\|x_n-J_\lambda^F x_n\| = 0 \tag{3.15}
$$

Since J_λ^F is single-valued and nonexpansive and $x_{n_i} \rightharpoonup q$ as $i\to\infty$, using (3.15)and Lemma 2.5, we have

$$
q \in Fix(J_\lambda^F) = \operatorname*{argmin}_{y\in C} F(y).
$$

(ii) We prove that $q \in VI(C,A)$. In fact, from $z_n = A_{\omega_n} v_n$ and Lemma 2.1, we obtain

$$\langle y - z_n, Az_n \rangle + \left\langle y - z_n, \frac{z_n - v_n}{\omega_n} \right\rangle \geq 0, \quad \forall y \in C. \tag{3.16}$$

Set $v_\epsilon = \epsilon v + (1-\epsilon)q$ for $\epsilon \in (0,1]$ and $v \in C$. Then $v_\epsilon \in C$, and it follows from (3.16) that

$$\langle v_\epsilon - z_n, Av_\epsilon \rangle \geq \langle v_\epsilon - z_n, Av_\epsilon \rangle - \langle v_\epsilon - z_n, Az_n \rangle - \left\langle v_\epsilon - z_n, \frac{z_n - v_n}{\omega_n} \right\rangle. \tag{3.17}$$

By Step 4, we have $\frac{\|z_n - v_n\|}{\omega_n} \leq \frac{\|z_n - v_n\|}{\omega} \to 0$ as $n \to \infty$. Moreover, by Step 6, $\{x_n\}$ and $\{z_n\}$ have the same asymptotic behavior. So, since $x_{n_i} \rightharpoonup q$ as $i \to \infty$, we also induce $z_{n_i} \rightharpoonup q$ as $i \to \infty$. And, from monotonicity of A, it follows that

$$\langle v_\epsilon - z_n, Av_\epsilon - Az_n \rangle \geq 0.$$

Thus, replacing n by n_i and letting $i \to \infty$, from (3.17), we obtain

$$0 \leq \langle v_\epsilon - q, Av_\epsilon \rangle,$$

and hence

$$\langle v - q, Av_\epsilon \rangle \geq 0, \quad \forall v \in C.$$

If $\epsilon \to 0$, then the continuity of A yields that

$$\langle v - q, Aq \rangle \geq 0, \quad \forall v \in C.$$

This means that $q \in VI(C,A)$.

(iii) We prove that $q \in Fix(T)$. In fact, noting $w_n = T_{r_n} z_n$, by Lemma 2.2, we induce

$$\langle y - w_n, Tw_n \rangle - \frac{1}{r_n} \langle y - w_n, (1+r_n)w_n - z_n \rangle \leq 0, \quad \forall y \in C. \tag{3.18}$$

Put $v_\epsilon = \epsilon v + (1-\epsilon)q$ for $0 < \epsilon \leq 1$ and $v \in C$. Then $v_\epsilon \in C$, and from (3.18) and pseudocontractivity of T, it follows that

$$\begin{aligned}
\langle w_n - v_\epsilon, Tv_\epsilon \rangle &\geq \langle w_n - v_\epsilon, Tv_\epsilon \rangle + \langle v_\epsilon - w_n, Tw_n \rangle \\
&\quad - \frac{1}{r_n} \langle v_\epsilon - w_n, (1+r_n)w_n - z_n \rangle \\
&= -\langle v_\epsilon - w_n, Tv_\epsilon - Tw_n \rangle - \frac{1}{r_n} \langle v_\epsilon - w_n, w_n - z_n \rangle \\
&\quad - \langle v_\epsilon - w_n, w_n \rangle \\
&\geq -\|v_\epsilon - w_n\|^2 - \frac{1}{r_n} \langle v_\epsilon - w_n, w_n - z_n \rangle - \langle v_\epsilon - w_n, w_n \rangle \\
&= -\langle v_\epsilon - w_n, v_\epsilon \rangle - \left\langle v_\epsilon - w_n, \frac{w_n - z_n}{r_n} \right\rangle.
\end{aligned} \tag{3.19}$$

By Step 6, $\{x_n\}$ and $\{z_n\}$ have the same asymptotic behavior. Also, by Step 5, we have $\frac{\|w_n - z_n\|}{r_n} \le \frac{\|w_n - z_n\|}{r} \to 0$ as $n \to \infty$. From Step 2, it follows that $w_{n_i} \rightharpoonup q$ as $i \to \infty$. So, replacing n by n_i and letting $i \to \infty$, we derive from (3.19)

$$\langle q - v_\epsilon, Tv_\epsilon\rangle \ge \langle q - v_\epsilon, v_\epsilon\rangle$$

and

$$-\langle v - q, Tv_\epsilon\rangle \ge -\langle v - q, v_\epsilon\rangle, \quad \forall v \in C.$$

Letting $\epsilon \to 0$ and using the fact that T is continuous, we obtain

$$-\langle v - q, Tq\rangle \ge -\langle v - q, q\rangle, \quad \forall v \in C. \tag{3.20}$$

Let $v = Tq$ in (3.20). Then we have $q = Tq$, that is, $q \in Fix(T)$. This along with (i) and (ii) obtains $q \in \Phi$.

Now, replacing n by n_i, we substitute q for p in (3.14) to obtain

$$\|x_{n_i} - q\|^2 \le \frac{1}{\tau - \gamma l}\langle \mu G - \gamma V q, q - z_{n_i}\rangle + \frac{\alpha_n M}{2(\tau - \gamma l)} + \frac{L_{n_i}}{\tau - \gamma l}. \tag{3.21}$$

Note that $z_{n_i} \rightharpoonup q$ as $i \to \infty$ by Step 6 and $\lim_{n\to\infty} L_n = 0$ by Step 2 and Step 5. This fact and the inequality (3.21) along with condition (i) imply that $x_{n_i} \to q$ strongly as $i \to \infty$.

Next, we show that q solves the the variational inequality (3.2). Indeed, taking the limit in (3.14) as $i \to \infty$, we get

$$\|q - p\|^2 \le \frac{1}{\tau - \gamma l}\langle (\mu G - \gamma V)p, p - q\rangle, \quad \forall p \in \Phi.$$

In particular, q solves the following variational inequality

$$q \in \Phi \quad \langle (\mu G - \gamma V)p, p - q\rangle \ge 0, \quad p \in \Phi,$$

or the equivalent dual variational inequality(Lemma 2.7).

$$q \in \Phi \quad \langle (\mu G - \gamma V)q, p - q\rangle \ge 0, \quad p \in \Phi. \tag{3.22}$$

Finally we show that the sequence $\{x_n\}$ converges strongly to q. Indeed, let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ and assume $x_{n_k} \to \widehat{q}$. By the same method as the proof above, we have $\widehat{q} \in \Phi$. Moreover, it follows from (3.22) that

$$\langle (\mu G - \gamma V)q, q - \widehat{q}\rangle \le 0. \tag{3.23}$$

Interchanging q and $\widehat{q}$, we obtain

$$\langle (\mu G - \gamma V)\widehat{q}, \widehat{q} - q\rangle \le 0. \tag{3.24}$$

Lemma 2.3 and adding these two inequalities (3.23) and (3.24) yields

$$(\mu\eta - \gamma l)\|q - \widehat{q}\|^2 \le \langle (\mu G - \gamma V)q - (\mu G - \gamma V)\widehat{q}, q - \widehat{q}\rangle \le 0.$$

Hence $q = \widehat{q}$. Therefore we conclude that $x_n \to q$ as $n \to \infty$.

The variational inequality (3.2) can be rewritten as

$$\langle (I - \mu G + \gamma V)q - q, q - p\rangle \ge 0, \quad \forall p \in \Phi.$$

By (2.2), this is equivalent to the fixed point equation

$$P_{\Phi}(I - \mu G + \gamma V)q = q.$$

□

From Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} v_n = \operatorname{argmin}_{v \in C}\left[F(v) + \frac{1}{2\lambda_n}\|v - x_n\|^2\right], \\ z_n = A_{\omega_n} v_n, \\ x_n = \ P_C[(1-\alpha_n)(\beta_n x_n + (1-\beta_n)T_{r_n} z_n)], \quad \forall n \geq 1. \end{cases}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, $\{\omega_n\}$ and $\{r_n\}$ be satisfy the conditions (i), (ii), (iii), (iv) *and* (v) *in Theorem 3.1. Then $\{x_n\}$ converges strongly as $n \to \infty$ to a point $q \in \Phi$, which solves the following minimum-norm problem: find $x^* \in \Phi$ such that*

$$\|x^*\| = \min_{x \in \Phi} \|x\|. \tag{3.25}$$

Proof. Take $G = I$, $\mu = 1$, $\tau = 1$, $V = 0$ and $l = 0$ in Theorem 3.1. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q, p - q\rangle \geq 0, \quad \forall p \in \Phi.$$

This is equivalent to $\|q\|^2 \leq \langle p, q\rangle \|p\|\|q\|$ for all $p \in \Phi$. It turns out that $\|q\| \leq \|p\|$ for all $p \in \Phi$ and q is the minimum-norm point of Φ. □

If in Theorem 3.1, we take $T \equiv I$, the identity mapping on C, then we obtain the following result.

Corollary 3.3. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} v_n = \operatorname{argmin}_{y \in C}\left[F(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2\right], \\ z_n = A_{\omega_n} v_n, \\ x_n = \ P_C[\alpha_n \gamma V x_n + (I - \alpha_n \mu G)(\beta_n x_n + (1-\beta_n) z_n)], \quad \forall n \geq 1, \end{cases}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{\omega_n\}$ be satisfy the conditions (i), (ii), (iii) *and* (iv) *in Theorem 3.1. Then $\{x_n\}$ converges strongly as $n \to \infty$ to a point $q \in \Gamma :=$ $\operatorname{argmin}_{y \in C} F(y) \cap VI(C, A)$, which is the unique solution of the variational inequality*

$$\langle (\mu G - \gamma V)q, p - q\rangle \geq 0, \quad \forall\ p \in \Gamma.$$

By taking $V \equiv 0, G \equiv I, \mu = 1$ in Corollary 3.3, we also obtain the following result.

Corollary 3.4. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} v_n = \operatorname{argmin}_{y\in C}\left[F(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2\right], \\ z_n = A_{\omega_n} v_n, \\ x_n = \ P_C[(1-\alpha_n)(\beta_n x_n + (1-\beta_n) z_n)], \quad \forall n \geq 1, \end{cases}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$ and $\{\omega_n\}$ be satisfy the conditions (i)*,* (ii)*,* (iii) *and* (iv) *in Theorem 3.1. Then $\{x_n\}$ converges strongly as $n \to \infty$ to a point $q \in \Gamma$, which is the minimum-norm element of* Γ.

Remark 3.5. 1) For finding a point in $\Phi = \operatorname{argmin}_{y\in C} F(y) \cap VI(C,A) \cap Fix(T)$, where $F : C \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function, $A : C \to H$ is a continuous monotone mapping and $T : C \to C$ is a continuous pseudocontractive mapping, Theorem 3.1 is a new one different from previous those introduced by several authors. Consequently, as a new result for convex minimization problem combined with some nonlinear problems, Theorem 3.1 develops and complements the corresponding results, which were obtained recently by several authors in references; for instance, see [5, 6, 11, 13, 16] and the references therein.

2) Theorem 3.1 supplements and develops the corresponding result in [13] in following aspect:
   (a) The VIP(1.2) for an inverse strongly monotone mapping A in [13] is extended to the case of the VIP (1.2) for a continuous monotone mapping A.
   (b) The FPP(1.3) for a demicontractive mapping T and a strictly pseudocontractive mapping T in [13] is extended to the case of the FPP(1.3) of a continuous pseudocontractive mapping T.
   (c) The contractive mapping f with a constant $b \in (0,1)$ is extended to case of Lipschitizian V with a constant $l \geq 0$.
   (d) The ρ-Lipschitzian and η-strongly monotone mapping G with constants $\rho > 0$ and $\eta > 0$ was utilized in comparison to [13].

3) Corollary 3.3 is also a new ones for finding common solutions of MP(1.1) and VIP(1.2).

3) We point out that Corollary 3.2 and Corollary 3.4 for finding the minimum-norm element of $\Phi = \operatorname{argmin}_{y\in C} F(y) \cap VI(C,A) \cap Fix(T)$ and $\Gamma = \operatorname{argmin}_{y\in C} F(y) \cap VI(C,A)$, respectively, are also new ones different from previous those introduced by several authors.

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