Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 8, Number 3, 2022, 255–263

WELLPOSED SPACES FOR HYPERBOLIC EQUATIONS

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Dedicated to the late Professor Kazimierz Goebel

ABSTRACT. In the previous work we posed the problem of wellposed function spaces for nonparabolic equations and discussed the problem for Schrödinger equations. In this paper we intend to discuss "wellposed function spaces" for hyperbolic equations of the simplest type. We get minimum (or maximum) wellposed space containing (or contained in) any given function space. Some applications to nonlinear problems will be found to oblique boundary conditions.

INTRODUCTION

In the previous work [6] we posed the problem of wellposed function spaces for nonparabolic equations and discussed the problem for Schrödinger equations. In this paper we intend to discuss "wellposed function spaces" for hyperbolic equations of the simplest type. We get minimum (or maximum) wellposed space containing (or contained in) any given function space. Some applications to nonlinear problems will be found, and to oblique boundary conditions in [2] and [3]. An application of our theory to path integrals for Dirac equation will be given : the path integral is expressed by an L^2-valued measure.

1. WELLPOSED FUNCTION SPACE

1.1. **Equation.** We consider hyperbolic equations of the following type:

$$(1.1) \qquad \begin{cases} \frac{\partial}{\partial t}u(t,x) = & \sum_{j=1}^{N} A_j \frac{\partial}{\partial x_j} u(t,x) \\ u(0,x) = & u^0(x) \end{cases} \quad \text{for} \quad -\infty < t < \infty, \quad x \in \mathbb{R}^N,$$

where each $A_j = \left(a_{ik}^j\right)_{i,k=1}^n$ is an $n \times n$ Hermitian matrix. i.e. $a_{kl}^j = \overline{a_{lk}^j}$, $u =^t (u_1, \ldots, u_n)$ and $u^0 =^t (u_1^0, \ldots, u_n^0)$. We can rewright equation (1) as follows:

2020 *Mathematics Subject Classification.* 47D99.

Key words and phrases. C_0-semigroup, evolution equation, well-posed function spaces, hyperbolic equations, oblique boundary condition.

This work was supported by JSPS KAKENHI 20K03649.

$$\begin{pmatrix} \frac{\partial}{\partial t}u_1(t,x_1,\dots,x_N) \\ \frac{\partial}{\partial t}u_2(t,x_1,\dots,x_N) \\ \dots \\ \frac{\partial}{\partial t}u_n(t,x_1,\dots,x_N) \end{pmatrix}$$

$$= \sum_{j=1}^N \begin{pmatrix} a_{11}^j & a_{12}^j & \dots & a_{1n}^j \\ \overline{a_{12}^j} & a_{22}^j & \dots & a_{2n}^j \\ & & \dots & \\ \overline{a_{1n}^j} & \overline{a_{2n}^j} & \dots & a_{nn}^j \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_j}u_1(t,x_1,\dots,x_N) \\ \frac{\partial}{\partial x_j}u_2(t,x_1,\dots,x_N) \\ \dots \\ \frac{\partial}{\partial x_j}u_n(t,x_1,\dots,x_N) \end{pmatrix}$$

$$\begin{pmatrix} u_1(0,x_1,\dots,x_N) \\ u_2(0,x_1,\dots,x_N) \\ \dots \\ u_n(0,x_1,\dots,x_N) \end{pmatrix} = \begin{pmatrix} u_1^0(x_1,\dots,x_N) \\ u_2^0(x_1,\dots,x_N) \\ \dots \\ u_n^0(x_1,\dots,x_N) \end{pmatrix}.$$

The Fourier transform of (1.1) is

$$(1.2) \quad \begin{cases} \frac{d}{dt}\hat{u}(t,\xi) = & -i\left(\sum_{j=1}^N \xi_j A_j\right)\hat{u}(t,\xi) \\ \hat{u}(0,\xi) = & \hat{u}^0(\xi) \end{cases} \quad \text{for} \quad -\infty < t < \infty, \quad \xi \in \mathbb{R}^N.$$

We denote by

$$A(\xi) = \sum_{j=1}^N \xi_j A_j = \left(\sum_{j=1}^N a_{ik}^j \xi_j\right)_{i,k=1}^n \quad \text{for} \quad \xi = (\xi_1,\dots,\xi_N), \quad \xi_j \in \mathbb{R}.$$

Since $A(\xi)$ is Hermitian, $e^{-iA(\xi)t}$ is a unitary matrix for $t \in \mathbb{R}$. Thus we get the solution to the equation (1.2):

$$\hat{u}(t,\xi) = e^{-itA(\xi)}\hat{u^0}(\xi) \quad \text{for} \quad -\infty < t < \infty, \quad \xi \in \mathbb{R}^N.$$

1.2. **Group representation.** We cite some basic results on group representations needed later. $M(n)$ denotes the set of $n \times n$ matrices, $\mathbb{U}(n)$ the set of $n \times n$ unitary matrices:

$$\mathbb{U}(n) = \left\{U \in M(n) \mid U \cdot {}^t\bar{U} = I\right\}$$

or

$$\|Ux\| = \|x\| \quad (Ux, Uy) = (x,y) \; = \sum_{i=1}^n x_i \bar{y}_i$$

for $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n) \in \mathbb{C}^n$.

The Lie algebra of $\mathbb{U}(n)$ is denoted by $L(\mathbb{U}(n))$:

$$L(\mathbb{U}(n)) = \left\{A \in M(n) \mid A^* \equiv {}^t\bar{A} = -A\right\}.$$

$$[X,Y] \equiv XY - YX \in L(\mathbb{U}(n)) \quad \text{for } X, Y \in L(\mathbb{U}(n)).$$

That is, $a_{ij} = -\bar{a}_{ji}$ for $(a_{ij})_{i,j=1}^n = A \in L(\mathbb{U}(n))$. A is an anti-Hermite matrix and the eigen values of A are pure imaginary.

Let G be the minimum subgroup of $\mathbb{U}(n)$ containing $\{e^{itA(\xi)} \mid \xi \in \mathbb{R}^N, t \in \mathbb{R}\}$ and L be the Lie algebra of G. L is the minimum Lie algebra containing $\{-iA(\xi) \mid \xi \in \mathbb{R}^N\}$. If G is reducible, it is completely reducible. Hence in a suitable coordinates, there exists $\tilde{G}$ such that

$$(1.3) \quad G \subset \tilde{G} \equiv \begin{pmatrix} \mathbb{U}(n_1) & 0 & \dots & 0 \\ 0 & \mathbb{U}(n_2) & \dots & 0 \\ \vdots & \cdot & \ddots & \vdots \\ 0 & \dots & 0 & \mathbb{U}(n_k) \end{pmatrix} \quad \text{for} \quad n = n_1 + n_2 + \dots + n_k.$$

We consider an m-dimensional commutative representation σ of G :

$$\sigma : \mathbb{U}(n) \longrightarrow \mathbb{U}(m),$$

$$\sigma(UV) = \sigma(U)\sigma(V) = \sigma(V)\sigma(U) \quad \text{for} \quad U, V \in G.$$

If $m = 1$, then σ is a 1-dimensional representation,

$$\forall U \in G, \quad \exists \theta \in \mathbb{R} \quad \text{such that} \quad \sigma(U) = e^{i\theta}.$$

In general($m > 1$), we have

$$(1.4) \qquad \sigma(U) = \begin{pmatrix} \sigma^1(U) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma^m(U) \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_m} \end{pmatrix},$$

where σ^k is a 1-dimensional representation. We denote $\sigma = (\sigma^k)$.

If all A_j are commutative : $A_j A_k = A_k A_j, 1 \le {}^\forall i, j \le N$, then G is commutative and hence every representation of G is commutative.

1.3. **Tensor Product Spaces.** In this section we discuss abstract groups of isometric operators on tensor product spaces. We simply call them isometric groups. The results are valid for semigroups of contractions, but we do not discuss them here. Tensor product produces some new function space other than Sobolev spaces. These new function spaces are sometimes useful for nonlinear problems. For the cross norm of tensor product spaces, see [1] or [8]. Let $S^{(j)}(t)$ be a isometric group on a Banach space X_j and B_j its generator for $1 \le j \le N$.

Lemma 1.1. *Let $X_1 = \dots = X_N = X$. If every pair of $\{B_j\}$ commutes, i.e. $B_j B_k = B_k B_j, 1 \le {}^\forall i, j \le N$, then the closure B of the sum $\sum_{j=1}^N B_j$ generates an isometric group $\{S(t)\}$ on X.*

Proof. Since $\{S(t) \equiv S^{(1)}(t) \dots S^{(N)}(t)\}$ is an isometric group. □

We define that

$$\mathcal{G}(1; X) \equiv \{S(t) \mid S(t) : X \to X : \text{ contraction semigroup}\},$$

$$G(S(t); X) \equiv \{B \mid B : X \to X : \text{ generator of } S(t)\}.$$

The following theorem is well known.

Theorem 1.2 ([5]P.502 Thorem 2.11). *Let $T \in G(e^{-tT}; X)$, $A \in G(e^{-tA}; X)$ and $e^{-tT}, e^{-tA} \in \mathcal{G}(1; X)$, let $D(T) \bigcap D(A)$ be dense in X and $T + A + \xi$ have a dense range $\mathcal{R}(T + A + \xi)$ for sufficiently large real $\xi \in \mathbb{R}$. If $T + A$ is closable, its closure $S \equiv \overline{T + A} \in G(e^{-tS}; A)$ and $e^{-tS} \in \mathcal{G}(1; X)$.*

The π-norm of the tensor product of two Banach spaces X and Y is defined by

$$\|z\|_\pi = inf \left\{ \sum_{j=1}^{N} \|x_j\| \cdot \|y_j\| \;\middle|\; z = \sum_{j=1}^{N} x_j \otimes y_j \right\}, \qquad z \in X \otimes Y.$$

The π-norm is the strongest cross norm and ε-norm, the dual norm of π-norm, is the weakest one:

$$\|z\|_\varepsilon \leq \|z\|_\alpha \leq \|z\|_\pi \quad \text{for} \quad \forall z \in X \otimes Y$$

for any compatible (= reasonable) norm $\| \cdot \|_\alpha$, that is, $\|x \otimes y\|_\alpha = \|x\|_X \cdot \|y\|_Y$. The completion of $X \otimes Y$ with respect to the norm $\| \cdot \|_\alpha$ denotes $X \hat{\underset{\alpha}{\otimes}} Y$.

Let X_0 be the tensor product of $\{X_j\}$ with the π-norm and X_π the completion of X_0 :

$$X_\pi = X_1 \hat{\underset{\pi}{\otimes}} \dots \hat{\underset{\pi}{\otimes}} X_N \supset X_0 = X_1 \underset{\pi}{\otimes} \dots \underset{\pi}{\otimes} X_N.$$

We define natural extensions of $\tilde{S}^{(j)}(t)$ and $\tilde{B}_j$ to X_0 by the following relations

$$\tilde{S}^{(j)}(t)(x_1 \otimes \cdots \otimes x_N) = x_1 \otimes \cdots \otimes S^{(j)}(t)x_j \otimes \cdots \otimes x_N,$$

$$\tilde{B}_j(x_1 \otimes \cdots \otimes x_N) = x_1 \otimes \cdots \otimes B_j x_j \otimes \cdots \otimes x_N.$$

Note that $\tilde{B}_j$ generates the semigroup $\tilde{S}^{(j)}(t)$ and every pair of $\{\tilde{B}_j\}$ commutes. Let

$$\tilde{S}(t) = \tilde{S}^{(1)}(t) \otimes \cdots \otimes \tilde{S}^{(j)}(t) \otimes \cdots \otimes \tilde{S}^{(N)}(t).$$

That is,

$$\tilde{S}(t)(x_1 \otimes \cdots \otimes x_N) = S^{(1)}(t)x_1 \otimes \cdots \otimes S^{(j)}(t)x_j \otimes \cdots \otimes S^{(N)}(t)x_N.$$

Using Lemma 1.1 we have the following theorem:

Theorem 1.3. *Let $S(t)$ and B be the minimal closed extensions of $\tilde{S}(t)$ and $\sum_{j=1}^{N} \tilde{B}_j$ respectively. Then B generates the isometric group $\{S(t)\}$ on X.*

Let X'_j be dual space of X_j. Since $S^{(j)}(t)$ is an isometric group, its dual ${}^tS^{(j)}(t)$ is also an isometric group in X'_j. Hence we can define ${}^t\tilde{S}(t)$ which is an isometric group on

$$X^* = X'_1 \hat{\underset{\pi}{\otimes}} \dots \hat{\underset{\pi}{\otimes}} X'_N.$$

Thus ${}^{tt}\tilde{S}(t)$ is an isometric group on

$$X_\varepsilon = X_1 \hat{\underset{\varepsilon}{\otimes}} \dots \hat{\underset{\varepsilon}{\otimes}} X_N.$$

The norm $\| \cdot \|_\alpha$ is said to be an *interpolation* of $\| \cdot \|_\pi$ and $\| \cdot \|_\varepsilon$ if the following condition is satisfied :
If a linear operator T of $X_1 \otimes \cdots \otimes X_N$ is bounded with respect to the two norms

π and ε then it is bounded with respect to the α-norm, that is there exist positive constants c_1, c_2 and c_3 such that

$$\|Tx\|_\pi \le c_1\|x\|_\pi \quad \text{and} \quad \|Tx\|_\varepsilon \le c_2\|x\|_\varepsilon, \Longrightarrow \|Tx\|_\alpha \le c_3\|x\|_\alpha.$$

Thus if a semigroup $\{T(t)\}$ is bounded with respect to the norms π and ε, it is bounded with respect to the norm $\|\cdot\|_\alpha$. In particular, $\tilde{S}(t)$ is a bounded group with respect to the norm $\|\cdot\|_\alpha$.

1.4. **Wellposed Spaces.** As in [6], the Schrödinger equation

$$\frac{\partial}{\partial t}u(t,x) = -i\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}u(t,x)$$

is wellposed in

$$X_\pi = X_1 \underset{\pi}{\hat{\otimes}} \dots \underset{\pi}{\hat{\otimes}} X_N, \quad \text{or} \quad X_\varepsilon = X_1 \underset{\varepsilon}{\hat{\otimes}} \dots \underset{\varepsilon}{\hat{\otimes}} X_N,$$

if $\frac{\partial}{\partial t}u(t,x) = -i\frac{\partial^2}{\partial x^2}u(t,x)$ is wellposed in every X_j, since every $\frac{\partial^2}{\partial x^2}$ commutes.

This is not the case for Dirac equations since that

$$A_j\frac{\partial}{\partial x_j}A_k\frac{\partial}{\partial x_k}u(t,x) \ne A_k\frac{\partial}{\partial x_k}A_j\frac{\partial}{\partial x_j}u(t,x) \quad \text{for} \quad j \ne k.$$

In order to treat Dirac equations in a similar way to Schrödinger equations, we make use of commutative representation of $\mathbb{U}(n)$.

$-i\mathcal{A}\cdot\xi$ generates a semigroup $e^{-it\mathcal{A}\cdot\xi}$ for a fixed $\xi \in \mathbb{R}^N$. Hence $-i\mathcal{A}\cdot\xi$ generates a semigroup in the space $L^1_{loc}(\mathbb{R}^N, \mathbb{C}^n) \equiv (L^1_{loc})^n$, n-times product of L^1_{loc}, where $L^1_{loc} \equiv \{f(\cdot) \mid f \text{ is integrable on every compact set in } \mathbb{R}^N\}$.

We put

$$T(t)(\xi) = e^{-it\mathcal{A}\cdot\xi},\ T^{(j)}(t)(\xi) = e^{-it\xi_j\mathcal{A}_j} \quad \text{for} \quad j = 1, \dots, N.$$

$T(t)(\xi)$ and $T^{(j)}(t)(\xi)$ are transformations of $\mathbb{R}^N$ for fixed $\xi \in \mathbb{R}^N$.

If we consider these as transformations of a function space $Z \subset (L^1_{loc})^n$, or $Z_j \subset L^1_{loc}$, we denote these by $T(t)$ or $T^{(j)}(t)$. Z could be a space of distributions or generalized functions of some kind, but here we restrict Z to a subspace of $(L^1_{loc})^n$. When the space Z is not referred to, we call $\{T(t)\}$ a formal semigroup and $A = -i\mathcal{A}$ its formal generator.

Let σ be a 1-dimensional representation of G. We put

$$S^{(j)}(t) = \sigma\big(T^{(j)}(t)\big).$$

$\big\{S^{(j)}(t)\big\}_{j=1}^N$ has the semigroup property:

$$S^{(j)}(t)S^{(j)}(s) = \sigma\left(T^{(j)}(t)\right)\sigma\left(T^{(j)}(s)\right) = \sigma\left(T^{(j)}(t+s)\right) = S^{(j)}(t+s).$$

Since σ is commutative, we have

$$S^{(j)}(t)S^{(i)}(t) = S^{(i)}(t)S^{(j)}(t) \quad \text{for} \quad i,j = 1 \dots N.$$

We denote the generator of $\{S^{(j)}(t)\}$ by B_j.
Let $Y_j = \{\ f \mid f(\xi_j) \in \mathbb{C},\ \xi_j \in \mathbb{R}\ \}$ be a Banach space such that

$$S^{(j)}(t):\ Y_j \longrightarrow Y_j \ \text{ is isometric } \ i.e.\ \|S^{(j)}(t)f\|_{Y_j} = \|f\|_{Y_j}.$$

Let

$$Y = Y_1 \hat{\underset{\pi}{\otimes}} \dots \hat{\underset{\pi}{\otimes}} Y_N.$$

We define two operators of Y :

$$B = B_1 \otimes I \otimes \cdots \otimes I + I \otimes B_2 \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes B_N,$$
$$S(t) = S^{(1)}(t) \otimes \cdots \otimes S^{(N)}(t).$$

Lemma 1.4. *The formal generator B generates an isometric group $\{S(t)\}$ on Y.*

We denote by $\mathbb{U}(n)$ the group of unitary matrices and by $S\mathbb{U}(n)$ the group of special unitary matrices.

Definition 1.5. We denote by $\mathcal{U}_n$ the set of the $\mathbb{U}(n)$-valued measurable functions and by $\mathcal{T}$ the set of the unitary group of hyperbolic type:

$$\mathcal{U}_n = \left\{\ U \mid U(\xi) \in \mathbb{U}(n), \forall \xi \in \mathbb{R}^N \right\}, \tag{1.5}$$

$$\mathcal{T} = \left\{ e^{-it\sum_{j=1}^{N}\xi_j A_j} \mid A_j = Hermitian \right\}. \tag{1.6}$$

We denote

$$\mathcal{V}_0 = \text{ the minimum subgroup of } \mathcal{U}_n \text{ containing } \mathcal{T} \text{ and } S\mathbb{U}(n). \tag{1.7}$$

$$\mathcal{V}_{\mathcal{A}} = \text{ the minimum subgroup of } \mathcal{U}_n \text{ containing } \{e^{it\mathcal{A}\cdot\xi}\} \text{ and } S\mathbb{U}(n). \tag{1.8}$$

Let $Y \subset L^1_{loc}$ be a Banach space such that

$$\|f(\xi)\| = \|e^{i\lambda\cdot\xi} f(\xi)\| \quad \text{for} \quad \lambda, \xi \in \mathbb{R}^N \text{ and } f \in Y.$$

For the unit ball B_Y of Y and a unit vector $e_0 \in \mathbb{C}^n$, we put

$$B_X = con\left\{ \|f\|_Y U\sigma^{-1}\left(\frac{f}{\|f\|_Y}\right) \mid f \in B_Y, f \not\equiv 0,\ U \in \mathcal{V}_0 \right\} \cdot e_0,$$

where $con\{\cdot\}$ is the convex hull of a set. Let X be the Banach space which is the completion of the normed space with the unit ball B_X. This Banach space X does not depend on the choice of e_0. We evidently have $X \supset Y$.

Definition 1.6. The Banach space X above is denoted by $\bar{Y}^{\mathcal{V}_0}$.

Theorem 1.7. *The Banach space $\bar{Y}^{\mathcal{V}_0}$ is the minimum wellposed spce containing Y. (A formal generator $A = -i\mathcal{A}$ generates an isometric group $\{T(t)\}$ on $\bar{Y}^{\mathcal{V}_0}$.)*

The proof is clear by the preceding lemma.

Definition 1.8.

$$\breve{Y}^{\mathcal{V}_0} = \bigcap_{U \in \mathcal{V}_0} UX.$$

Theorem 1.9. *The Banach space $\breve{Y}^{\mathcal{V}_0}$ is the maximum wellposed space contained in Y.*

2. Examples

2.1. **0-dimensional representation.** Let σ be a 0-dimensional representation : $\sigma(U) = 1,\ \forall U \in G$. In this case the solution to the equation (1.1) is

$$|f(\xi)|\sigma(T(t)U_0(\xi))1 = |f(\xi)|.$$

Since

$$|f(\xi)| = |\hat{u}(0,\xi)| \left(\equiv \left(\sum_{j=1}^{n} |\hat{u_j}(0,\xi)|^2 \right)^{\frac{1}{2}} \right),$$

we have

$$\left(\sum_{j=1}^{n} |\hat{u_j}(t,\xi)|^2 \right)^{\frac{1}{2}} = \sigma\big(u(t,\xi)\big) = \sigma\big(u^0(\xi)\big) = \left(\sum_{j=1}^{n} |u_j^0(\xi)|^2 \right)^{\frac{1}{2}}.$$

Theorem 2.1. *The equation (1.2) is wellposed on the following Banach space :*

$$X = \left\{ v = {}^t(v_1, \dots, v_n) \;\middle|\; \left(\sum_{j=1}^{n} |v_j(\cdot)|^2 \right)^{\frac{1}{2}} \in Y \right\},$$

where Y is any function space with N-variables.

For example, the equation (1.2) is well-posed , or the semigroup is isometric, in the norm

$$\|f\|_{2,p} = \left(\int \|f(\xi)\|_2^p d\xi \right)^{1/p} \quad \text{for} \quad 1 \le p < \infty.$$

In this case for $X = {}^t(X_1, \dots, X_N)$, each X_j is equal to L^p. The most simple and usefull case is $Y = M(1)$. For an application of this space to semilinear equations see [6], or to oblique boundary conditions see [2].

2.2. **1-dimensional representation.** Let σ be a 1-dimensional representation. As is already discussed,

$$\sigma\left(T^j(t)\right) = e^{it\theta_j}, \quad \sigma\big(T(t)\big) = e^{it\theta} \quad \text{for} \quad \theta = \theta_1 + \cdots + \theta_n.$$

Fourier transform of $e^{it\theta}$ means the translation:

$$\mathcal{F}^{-1}(e^{it\theta}\hat{u}) = u(x + t\theta).$$

Hence for a translation invariant space Y, we have $T(t)\sigma^{-1}(Y) \subset Y$. For instance we let $Y = \{f \mid \lim_{x\to\pm\infty} f(x) = 0\}$ and $\{T(t)\}$ is an isometric semigroup on $X = \sigma^{-1}(Y)$. In this case each X_j is somewhat ambiguous. Nevertheless it is useful to discuss the path integral for Dirac equations. Some other results will be published elsewhere.

2.3. **Commutative G.** If $\{\mathcal{A}_j\}$ in (1.2) are commutative:

$$\mathcal{A}_j\mathcal{A}_k = \mathcal{A}_k\mathcal{A}_j \quad \text{for} \quad 1 \le j, k \le N,$$

then they are diagonal matrices in a suitable coordinates. Hence for the formal generator $A = -i\mathcal{A}(\xi) = -i\sum_{j=1}^{N} \xi_j\mathcal{A}_j$ there exists $T \in \mathbb{U}(n)$ such that

$$TA(\xi)T^{-1} = \begin{pmatrix} i\displaystyle\sum_{j=1}^{N} a_1^j\xi_j & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & i\displaystyle\sum_{j=1}^{N} a_N^j\xi_j \end{pmatrix} \quad \text{for} \quad a_k^j \in \mathbb{R}.$$

In this case G is commutative and we have

$$\bar{Y}^{\mathcal{V}_{\mathcal{A}}} = Y, \text{ if } \|f(\xi)\|_Y = \|e^{i\lambda\cdot\xi}f(\xi)\|_Y \quad \text{for} \quad f \in Y.$$

We can pick up $\sigma = \sigma(n)$ such that $\sigma(U) = TUT^{-1}$. The semigroup is

$$\tilde{T}(t) = TT(t)T^{-1} = \begin{pmatrix} exp(t\displaystyle\sum_{j=1}^{N} a_1^j\xi_j) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & exp(t\displaystyle\sum_{j=1}^{N} a_N^j\xi_j) \end{pmatrix}.$$

In this case the problem is reduced to first order partial differential equations

$$\frac{\partial v_j}{\partial t}(t,x) = \sum_{l=1}^{N} b_l^j \frac{\partial v_j}{\partial x_l}(t,x) \quad \text{for} \quad 0 \le j \le n,$$

or equivalently, ordinary differential equations

$$\frac{d\tilde{v}_j}{dt}(t,x) = 0 \quad \text{for} \quad 0 \le j \le n,$$

where $\tilde{v}_j(t,x) = v_j\left(x_1 - b_j^1 t, \dots, x_N - b_j^N t\right)$.

This is known as a singular case. Though our theory brings nothing new, it unifies this singular case and general cases. A family of solutions to (1.1) is usually considered as a $\mathbb{C}^n$-valued function space, and hence we shall identify a $\mathbb{C}^n$-valued function and the first column $(\hat{f}u_{j1})_{j=1}^n$ of our matrx-valued function $\hat{f}U$. For a $\mathbb{C}^n$-valued function $G = {}^t(g^1, \dots, g^n)$, there exist a function g and a unitary matrix valued function $V = (v_{jk})$ such that $g^j = g * \hat{v}_{j1}$. Using these, we define a matrix valued function

$$\check{G} = g * \hat{V} = (g_{jk}), \qquad g_{jk} = g * \hat{v}_{jk}.$$

The map $\check{}: G \longmapsto \check{G}$ is multi-valued. In this sense $\hat{X}_n^{\mathcal{V}}$ is the minimum (in our class) wellposed space containing $\hat{X}^n$ since $e^{-itA(\xi)}\mathcal{B} = \mathcal{B}$

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Manuscript received 31 October 2022
revised 31 December 2022

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