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ON THE HYPERSTABILITY OF GENERAL LINEAR FUNCTIONAL EQUATIONS

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ABSTRACT. We prove that a function approximately satisfying a general linear condition must actually satisfy this condition. We point out a gap in the original proof of this result given by Piszczek [7]. Our result does not rely on the fixed point theorem of Brzdęk [4] as was the case in [7]. Some examples of control functions are given to show that our results are more general than the previous known ones.

1. Introduction

Suppose that X and Y are normed linear spaces over scalar fields $\mathbb{F}$ and $\mathbb{K}$, respectively. We assume from now on that $\mathbb{F}, \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A function $f : X \to Y$ is *general linear* if there exist $a, b \in \mathbb{F}$ and $A, B \in \mathbb{K}$ such that

$$f(ax + by) = Af(x) + Bf(y)$$

for all $x, y \in X$. Note that if f is general linear where $a = b = A = B = 1$, then it is said to be *additive*; and if f is general linear where $a = A = t$ and $b = B = 1 - t$ for some $t \in (0, 1)$, then it is said to be *t-affine*. We usually say that f is *Jensen* if it is $1/2$-affine. In this paper, we investigate a sufficient condition on $\varphi : X \times X \to [0, \infty)$ such that for any functions $f : X \to Y$ satisfying the following condition:

$$\|f(ax + by) - Af(x) - Bf(y)\| \le \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$ it must be the case that f is general linear. Such a phenomenon is called a *hyperstability* result for general linear functional equations. The interested reader is referred to an excellent exposition [5] for further information. The aforementioned problem is closely related to the *stability* result proposed by Ulam [10] and the first affirmative answer was given by Hyers [6]. In fact, Hyers obtained the following result for additive functions:

Theorem H. *Suppose that $\varphi(x, y) := \delta > 0$ for all $x, y \in X$ and Y is a Banach space. If $f : X \to Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y) \quad \text{for all } x, y \in X,$$

then there exists a unique function $F : X \to Y$ such that $F(x + y) = F(x) + F(y)$ for all $x, y \in X$ and $\|f(x) - F(x)\| \le \delta$ for all $x \in X$.

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Aoki [1] and Rassias [9] generalized Theorem H for $\varphi(x,y) := \delta(\|x\|^p + \|y\|^p)$ where $0 \le p < 1$. Note that if $p := 0$, then Theorem H is obtained from the results of Aoki and of Rassias.

Brzdęk [3] supplemented the results of Aoki and of Rassias where $p < 0$. In fact, the following result was proved. Note that the completeness of Y is not required as was the case in Theorem H.

Theorem B. *Suppose that* $\varphi(x,y) := \|x\|^p + \|y\|^p$ *for all* $x, y \in X \setminus \{0\}$ *where* $p < 0$. *If* $f : X \to Y$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y) \quad \text{for all } x, y \in X \setminus \{0\},$$

then $f(x+y) = f(x) + f(y)$ *for all* $x, y \in X$.

Inspired by Theorem B, Piszczek [7] proposed the following interesting result.

Theorem P. *Suppose that* $a, b \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K} \setminus \{0\}$, *and* $\varphi(x,y) := \|x\|^p \|y\|^q$ *for all* $x, y \in X \setminus \{0\}$ *where* $p, q \in \mathbb{R}$. *Then* f *is general linear if*

$$\|f(ax+by) - Af(x) - Bf(y)\| \le \varphi(x,y) \quad \text{for all } x, y \in X \setminus \{0\}$$

and one of the following conditions is satisfied:

(a) $p + q < 0$ *(see* [7, Theorem 2.1]*);*

(b) $p + q > 0$*; and either* (b1) $q > 0$ and $|a|^{p+q} \ne |A|$; or (b2) $p > 0$ and $|b|^{p+q} \ne |B|$ (see [7, Theorem 2.2]).

The condition $|a|^{p+q} \ne |A|$ (and $|b|^{p+q} \ne |B|$) is not superfluous as shown by an example given in [5, 7].

First, we point out that there is a gap in the original proof of Theorem P. In fact, via the method used there [7], we can conclude only that

$$f(ax+by) = Af(x) + Bf(y) \quad \text{for all } x, y \ne 0 \ne ax+by.$$

It is clear that f satisfying the condition above is not necessarily general linear. To see this, let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(0) = 1$ and $f(x) = 0$ for all $x \ne 0$. In this case, we see that $f(x+y) = f(x) + f(y)$ for all $x, y \ne 0 \ne x+y$. In this paper, we use another approach to conclude Theorem P. The proof is given in Section 2. We do not use the fixed point theorem of Brzdęk [4] as was the case in [7].

2. Main results

Proof of Theorem P(a). We follow the idea of [2]. Let $x, y \in X$. Pick $z \in X$ such that $\|z\| > \max\{\|ax\|, \|a^2x\|, \|by\|, \|b^2y\|\}$. Put

$$x_n := x + \frac{nz}{a} \ne 0; \qquad x_n' := \frac{x}{a} + \frac{nz}{2a^2} \ne 0; \qquad x_n'' := \frac{nz}{2ab} \ne 0;$$
$$y_n := y - \frac{nz}{b} \ne 0; \qquad y_n' := -\frac{nz}{2ab} \ne 0; \qquad y_n'' := \frac{y}{b} - \frac{nz}{2b^2} \ne 0;$$

for all $n \ge 1$. Note that

$$\begin{aligned} ax + by &= ax_n + by_n \\ x &= ax_n' + by_n' \end{aligned}$$

$$y = ax''_n + by''_n$$
$$x_n = ax'_n + bx''_n$$
$$y_n = ay'_n + by''_n$$

for all $n \geq 1$. Moreover, it follows from $p + q < 0$ that

$$\begin{aligned}
\lim_{n\to\infty} \varphi(x_n, y_n) &= \lim_{n\to\infty} n^{p+q} \left\| \frac{x}{n} + \frac{z}{a} \right\|^p \left\| \frac{y}{n} - \frac{z}{b} \right\|^q = 0 \\
\lim_{n\to\infty} \varphi(x'_n, y'_n) &= \lim_{n\to\infty} n^{p+q} \left\| \frac{x}{na} + \frac{z}{2a^2} \right\|^p \left\| -\frac{z}{2ab} \right\|^q = 0 \\
\lim_{n\to\infty} \varphi(x''_n, y''_n) &= \lim_{n\to\infty} n^{p+q} \left\| \frac{z}{2ab} \right\|^p \left\| \frac{y}{nb} - \frac{z}{2b^2} \right\|^q = 0 \\
\lim_{n\to\infty} \varphi(x'_n, x''_n) &= \lim_{n\to\infty} n^{p+q} \left\| \frac{x}{na} + \frac{z}{2a^2} \right\|^p \left\| \frac{z}{2ab} \right\|^q = 0 \\
\lim_{n\to\infty} \varphi(y'_n, y''_n) &= \lim_{n\to\infty} n^{p+q} \left\| \frac{z}{2ab} \right\|^p \left\| \frac{y}{nb} - \frac{z}{2b^2} \right\|^q = 0.
\end{aligned}$$

Now we have the following

$$\begin{aligned}
\|f(ax+by) - Af(x) - Bf(y)\| \leq{}& \|f(ax+by) - Af(x_n) - Bf(y_n)\| \\
&+ |A|\|f(x) - Af(x'_n) - Bf(y'_n)\| \\
&+ |B|\|f(y) - Af(x''_n) - Bf(y''_n)\| \\
&+ |A|\|f(x_n) - Af(x'_n) - Bf(x''_n)\| \\
&+ |B|\|f(y_n) - Af(y'_n) - Bf(y''_n)\|.
\end{aligned}$$

Taking $n \to \infty$ gives the desired result. □

We can generalize Theorem P(a) as follows.

Theorem 2.1. *Suppose that $\varphi : (X \setminus \{0\})^2 \to [0, \infty)$ satisfies the following conditions: For each $x, y \in X$ there exists $z \neq 0$ such that*

$$\lim_{n\to\infty} \varphi(x + nz, y - nz) = \lim_{n\to\infty} \varphi(x + nz, \pm nz) = \lim_{n\to\infty} \varphi(\pm nz, y - nz) = 0.$$

Then $f : X \to Y$ is general linear if

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq \varphi(x, y) \quad \text{for all } x, y \in X \setminus \{0\}.$$

Before we discuss Theorem P(b). We need the following lemmas.

Lemma 2.2 ([8])**.** *Suppose that $h : X \to Y$ satisfies the following condition:*

$$h(ax + by) = Ah(x) + Bh(y) \quad \text{for all } x, y \neq 0.$$

Then h is general linear.

Lemma 2.3. *Suppose that $h : X \to Y$ satisfies the following conditions: $h(0) = 0$ and*

$$h(ax + by) = Ah(x) + Bh(y) \quad \text{for all } x, y \neq 0 \neq ax + by.$$

The following statements are true for the odd part h_o and the even part h_e of h.

(i) *h_o is general linear, that is, $h_o(ax + by) = Ah_o(x) + Bh_o(y)$ for all $x, y \in X$.*

(ii) *h_e is constant on $X \setminus \{0\}$. If $A + B \neq 1$, then $h_e(x) = 0$ for all $x \neq 0$.*

Proof. Note that

$$h_o(x) := \frac{1}{2}(h(x) - h(-x)) \quad \text{and} \quad h_e(x) := \frac{1}{2}(h(x) + h(-x))$$

for all $x \in X$. In particular, we have

$$h_o(0) = 0, \quad h_o(-x) = -h_o(x), \quad h_e(-x) = h_e(x), \quad \text{and} \quad h(x) = h_o(x) + h_e(x)$$

for all $x \in X$. Moreover, we also have

$$h_o(ax + by) = Ah_o(x) + Bh_o(y) \quad \text{and} \quad h_e(ax + by) = Ah_e(x) + Bh_e(y)$$

for all $x, y \neq 0 \neq ax + by$.

To prove (i), let $x \neq 0$. Note that

$$x = a\frac{3x}{2a} + b\left(-\frac{x}{2b}\right) = a\frac{x}{2a} + b\frac{x}{2b}.$$

It follows that

$$h_o(x) = Ah_o\left(\frac{3x}{2a}\right) + Bh_o\left(-\frac{x}{2b}\right) = Ah_o\left(\frac{3x}{2a}\right) - Bh_o\left(\frac{x}{2b}\right);$$
$$h_o(2x) = Ah_o\left(\frac{3x}{2a}\right) + Bh_o\left(\frac{x}{2b}\right).$$

In particular,

$$h_o(2x) - h_o(x) = 2Bh_o\left(\frac{x}{2b}\right).$$

Similarly, we can prove that

$$h_o(2x) - h_o(x) = 2Ah_o\left(\frac{x}{2a}\right).$$

This implies that

$$h_o(2x) - h_o(x) = Ah_o\left(\frac{x}{2a}\right) + Bh_o\left(\frac{x}{2b}\right) = h_o(x)$$

and hence $h_o(2x) = 2h_o(x)$. Now, let $y \neq 0$ be such that $x + y \neq 0$. It follows that

$$h_o(x + y) = Ah_o\left(\frac{x}{a}\right) + Bh_o\left(\frac{y}{b}\right) = 2Ah\left(\frac{x}{2a}\right) + 2Bh_o\left(\frac{y}{2b}\right) = h_o(x) + h_o(y).$$

It follows from [3] and $h_o(0) = 0$ that $h_o(x + y) = h_o(x) + h_o(y)$ for all $x, y \in X$. Moreover, we have

$$h_o(ax) = \frac{1}{2}h_o(2ax) = \frac{2A}{2}h_o\left(\frac{2ax}{2a}\right) = Ah_o(x).$$

Similarly, we have $h_o(bx) = Bh_o(x)$. Hence

$$h_o(ax + by) = h_o(ax) + h_o(by) = Ah_o(x) + Bh_o(y) \quad \text{for all } x, y \in X.$$

To prove (ii), we fix $x_0 \neq 0$. Let $x \neq 0$. We prove that $h_e(x) = h_e(x_0)$. If $x = x_0$ or $x = -x_0$, then we are done. We assume that $x \neq \pm x_0$. In particular, $\frac{x-x_0}{2a} \neq 0 \neq \frac{x+x_0}{2b}$. It follows that

$$h_e(x) = Ah_e\left(\frac{x - x_0}{2a}\right) + Bh_e\left(\frac{x + x_0}{2b}\right)$$

$$= Ah_e\left(\frac{-x+x_0}{2a}\right) + Bh_e\left(\frac{x+x_0}{2b}\right) = h_e(x_0).$$

Moreover, we also have

$$h_e(x_0) = Ah_e(x_0) + Bh_e(x_0).$$

If $A + B \neq 1$, then $h_e(x_0) = Ah_e(\frac{x_0}{2a}) + Bh_e(\frac{x_0}{2b}) = Ah_e(x_0) + Bh_e(x_0)$, that is, $h_e(x_0) = 0$. □

We are now ready to get rid of a gap in the original proof of Theorem P(b). The method we use here is different from the one in [7].

Proof of Theorem P(b). We assume that $p + q > 0$. It suffices to assume that $q > 0$ and $|a|^{p+q} \neq |A|$. (The assertion under the assumptions $p > 0$ and $|b|^{p+q} \neq |B|$ can be proved analogously.)

Case 1: $|a|^{p+q} < |A|$. Put $\alpha := |a|^{p+q}/|A| < 1$. The proof is broken into four steps.

Step 1: $\|f(ax + by) - Af(x) - Bf(y)\| \leq \alpha\|x\|^p\|y\|^q$ for all $x, y \neq 0 \neq ax + by$.

Let $x, y \in X \setminus \{0\}$ be such that $ax + by \neq 0$. For each integer $n \geq 1$, we put

$$x_n := \left(a + \frac{b}{n}\right)x; \qquad y_n := \left(a + \frac{b}{n}\right)y; \qquad z_n := \left(a + \frac{b}{n}\right)(ax + by);$$

$$x'_n := \frac{x}{n}; \qquad y'_n := \frac{y}{n}; \qquad z'_n := \frac{ax + by}{n}.$$

It follows that, for all sufficiently large n, $0 \notin \{x_n, y_n, z_n, x'_n, y'_n, z'_n\}$. In particular,

$$\lim_{n\to\infty} \|f(z_n) - Af(ax + by) - Bf(z'_n)\| \leq \lim_{n\to\infty} \frac{\|ax + by\|^{p+q}}{n^q} = 0;$$

$$\lim_{n\to\infty} \|f(x_n) - Af(x) - Bf(x'_n)\| \leq \lim_{n\to\infty} \frac{\|x\|^{p+q}}{n^q} = 0;$$

$$\lim_{n\to\infty} \|f(y_n) - Af(y) - Bf(y'_n)\| \leq \lim_{n\to\infty} \frac{\|y\|^{p+q}}{n^q} = 0;$$

$$\limsup_{n\to\infty} \|f(z_n) - Af(x_n) - Bf(y_n)\| \leq \limsup_{n} \left|a + \frac{b}{n}\right|^{p+q} \|x\|^p\|y\|^q = |a|^{p+q}\|x\|^p\|y\|^q;$$

$$\lim_{n\to\infty} \|f(z'_n) - Af(x'_n) - Bf(y'_n)\| \leq \lim_{n\to\infty} \frac{\|x\|^p\|y\|^q}{n^{p+q}} = 0.$$

Moreover, we have

$$\begin{aligned}
&|A|\|f(ax + by) - Af(x) - Bf(y)\| \\
&\leq \|f(z_n) - Af(ax + by) - Bf(z'_n)\| \\
&\quad + |A|\|f(x_n) - Af(x) - Bf(x'_n)\| \\
&\quad + |B|\|f(y_n) - Af(y) - Bf(y'_n)\| \\
&\quad + \|f(z_n) - Af(x_n) - Bf(y_n)\| \\
&\quad + |B|\|f(z'_n) - Af(x'_n) - Bf(y'_n)\|
\end{aligned}$$

Taking $n \to \infty$ completes the proof of Step 1.

Step 2: $f(ax+by) = Af(x)+Bf(y)$ for all $x, y \neq 0 \neq ax+by$. (A careful reading of the original proof [7] of Theorem P(b) reaches only this conclusion. It is worth mentioning that the proof technique used in [7] is the fixed point method while our method is different.) To see this, we put $f' := f/\alpha$. It follows from Step 1 that

$$\|f'(ax+by) - Af'(x) - Bf'(y)\| \leq \alpha\|x\|^p\|y\|^q \quad \text{for all } x, y \neq 0 \neq ax+by.$$

We proceed the same method as in Step 1 for f' and we obtain that

$$\|f'(ax+by) - Af'(x) - Bf'(y)\| \leq \alpha\|x\|^p\|y\|^q \quad \text{for all } x, y \neq 0 \neq ax+by.$$

This implies that

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq \alpha^2\|x\|^p\|y\|^q \quad \text{for all } x, y \neq 0 \neq ax+by.$$

For each $n \geq 1$, it follows by induction that

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq \alpha^n\|x\|^p\|y\|^q \quad \text{for all } x, y \neq 0 \neq ax+by.$$

Letting $n \to \infty$ completes the proof of Step 2.

Step 3: $f(0) = Af(0)+Bf(0)$. To see this, let $x \neq 0$. It follows from Step 2 that

$$\begin{aligned} f\left(\frac{x}{na}\right) &= Af\left(\frac{x}{2na^2}\right) + f\left(\frac{x}{2nab}\right) \\ f\left(-\frac{x}{nb}\right) &= Af\left(-\frac{x}{2nab}\right) + f\left(-\frac{x}{2nb^2}\right). \end{aligned}$$

In particular,

$$\begin{aligned} &\|f(0) - Af(0) - Bf(0)\| \\ &\leq \left\|f(0) - Af\left(\frac{x}{na}\right) - Bf\left(-\frac{x}{nb}\right)\right\| \\ &\quad + |A|\left\|f(0) - Af\left(\frac{x}{2na^2}\right) - Bf\left(-\frac{x}{2nab}\right)\right\| \\ &\quad + |B|\left\|f(0) - Af\left(\frac{x}{2nab}\right) - Bf\left(-\frac{x}{2nb^2}\right)\right\| \\ &\leq \frac{\|x\|^{p+q}}{n^{p+q}|a|^p|b|^q} + \frac{\|x\|^{p+q}}{2^{p+q}n^{p+q}|a|^{2p+q}|b|^q} + \frac{\|x\|^{p+q}}{2^{p+q}n^{p+q}|a|^p|b|^{p+2q}}. \end{aligned}$$

Letting $n \to \infty$ completes the proof of Step 3.

Step 4: f is general linear. To see this, let $g(x) := f(x) - f(0)$ for all $x \in X$. It follows that $g(0) = 0$ and

$$g(ax+by) = Ag(x) + Bg(y) \quad \text{for all } x, y \neq 0 \neq ax+by.$$

Note that $0 = g(0) = g_o(0) + g_e(0) = g_e(0)$. It follows from Lemma 2.3 that the even part g_e of g is constant on $X \setminus \{0\}$. Fix $x_0 \neq 0$, we have $g_e(x) = g_e(x_0)$ for all $x \neq 0$. We now prove that $g_e(x_0) = 0$. If $A + B \neq 1$, then we are done. Now, we assume that $A + B = 1$. In this case, we have

$$\|g_e(0) - Ag_e(bx_0) - Bg_e(-ax_0)\| = \left\|g_e(0) - Ag_e\left(\frac{x_0}{n}\right) - Bg_e\left(\frac{x_0}{n}\right)\right\| \leq \frac{\|x_0\|^{p+q}}{n^{p+q}}.$$

Letting $n \to \infty$ gives $0 = g_e(0) = (A+B)g_e(x_0) = g_e(x_0)$. Now, we can conclude that $g_e(x) = 0$ for all $x \in X$. Finally, let $x, y \neq 0$ such that $ax + by = 0$. Then

$$\begin{aligned} f(ax+by) &= g(ax+by) + f(0) \\ &= g_o(ax+by) + f(0) \\ &= Ag_o(x) + Bg_o(y) + f(0) \\ &= Ag(x) + Bg(y) + Af(0) + Bf(0) \\ &= Af(x) + Bf(y). \end{aligned}$$

Hence the conclusion follows from Lemma 2.2. The proof of Case 1 is finished.

Case 2: $|a|^{p+q} > |A|$. Put $\beta := |A|/|a|^{p+q} < 1$. The proof of this case is very similar to that of Case 1. We only give a sketch proof. Let $x, y \in X$ such that $x, y \neq 0 \neq ax + by$. For each $n \geq 1$, we put

$$\begin{aligned} x_n &:= \left(\frac{1}{a} + \frac{1}{na}\right)(ax+by); & x_n' &:= \left(\frac{1}{a} + \frac{1}{na}\right)x; & x_n'' &:= \left(\frac{1}{a} + \frac{1}{na}\right)y; \\ y_n &:= -\frac{1}{nb}(ax+by); & y_n' &:= -\frac{1}{nb}x; & y_n'' &:= -\frac{1}{nb}. \end{aligned}$$

It follows that $0 \notin \{x_n, x_n', x_n'', y_n, y_n', y_n''\}$ for all sufficiently large n. Note that

$$\begin{aligned} ax + by &= ax_n + by_n \\ x &= ax_n' + by_n' \\ y &= ax_n'' + by_n'' \\ x_n &= ax_n' + bx_n'' \\ y_n &= ay_n' + by_n'' \end{aligned}$$

for all $n \geq 1$. It follows that

$$\begin{aligned} \lim_{n\to\infty} \|f(ax+by) - Af(x_n) - Bf(y_n)\| &\leq \lim_{n\to\infty} \left|\frac{1}{a} + \frac{1}{na}\right|^p \left|\frac{1}{na}\right|^q \|ax+by\|^{p+q} = 0; \\ \lim_{n\to\infty} \|f(x) - Af(x_n') - Bf(y_n')\| &\leq \lim_{n\to\infty} \left|\frac{1}{a} + \frac{1}{na}\right|^p \left|\frac{1}{na}\right|^q \|x\|^{p+q} = 0; \\ \lim_{n\to\infty} \|f(y) - Af(x_n'') - Bf(y_n'')\| &\leq \lim_{n\to\infty} \left|\frac{1}{a} + \frac{1}{na}\right|^p \left|\frac{1}{na}\right|^q \|y\|^{p+q} = 0; \\ \limsup_{n} \|f(x_n) - Af(x_n') - Bf(x_n'')\| &\leq \lim_{n\to\infty} \left|\frac{1}{a} + \frac{1}{na}\right|^{p+q} \|x\|^p\|y\|^q = \frac{\|x\|^p\|y\|^q}{|a|^{p+q}}; \\ \lim_{n\to\infty} \|f(y_n) - Af(y_n') - Bf(y_n'')\| &\leq \lim_{n\to\infty} \left|\frac{1}{na}\right|^{p+q} \|x\|^p\|y\|^q = 0. \end{aligned}$$

As we proved Theorem P(a), we obtain that

$$\|f(ax+by) - Af(x) - Bf(y)\| \leq \beta\|x\|^p\|y\|^q \quad \text{for all } x, y \neq 0 \neq ax+by.$$

Since $\beta < 1$, we repeat the proof above obtain that

$$f(ax+by) = Af(x) + Bf(y) \quad \text{for all } x, y \neq 0 \neq ax+by.$$

We can follow Step 3 and Step 4 of the proof of Theorem P(b) to obtain that f is indeed general linear. □

A simple inspection of the proof of Theorem P(b) yields the following more general results.

Theorem 2.4. *Suppose that $\varphi : (X \setminus \{0\})^2 \to (0,\infty)$ satisfies one of the following conditions:*

(a) $\lim_{n\to\infty} \varphi\left(x, \frac{x}{n}\right) = 0$ *for all* $x \neq 0$*; and* $\lim_{n\to\infty} \varphi\left(\frac{x}{n}, \frac{y}{n}\right) = 0$ *for each* $x, y \neq 0 \neq ax + by$*; and*

$$|A| \sup \left\{ \frac{\limsup_n \varphi\left(\left(a + \frac{b}{n}\right)x, \left(a + \frac{b}{n}\right)y\right)}{\varphi(x,y)} : x, y \neq 0 \neq ax + by \right\} < 1;$$

(b) $\lim_{n\to\infty} \varphi\left(\left(1 + \frac{1}{n}\right)x, -\frac{ax}{nb}\right) = 0$ *for all* $x \neq 0$*; and* $\lim_{n\to\infty} \varphi\left(\frac{x}{n}, \frac{y}{n}\right) = 0$ *for all* $x, y \neq 0 \neq ax + by$*; and*

$$\frac{1}{|A|} \sup \left\{ \frac{\limsup_n \varphi\left(\left(\frac{1}{a} + \frac{1}{na}\right)x, \left(\frac{1}{a} + \frac{1}{na}\right)y\right)}{\varphi(x,y)} : x, y \neq 0 \neq ax + by \right\} < 1.$$

Then $f : X \to Y$ *is general linear if*

$$\|f(ax + by) - Af(x) - Bf(y)\| \leq \varphi(x,y) \quad \textit{for all } x, y \in X \setminus \{0\}.$$

Theorem 2.5. *Suppose that $\varphi : (X \setminus \{0\})^2 \to (0,\infty)$ satisfies one of the following conditions:*

(a) $\lim_{n\to\infty} \varphi\left(\frac{x}{n}, x\right) = 0$ *for all* $x \neq 0$*; and* $\lim_{n\to\infty} \varphi\left(\frac{x}{n}, \frac{y}{n}\right) = 0$ *for each* $x, y \neq 0 \neq ax + by$*; and*

$$|B| \sup \left\{ \frac{\limsup_{n\to\infty} \varphi\left(\left(\frac{a}{n} + b\right)x, \left(\frac{a}{n} + b\right)y\right)}{\varphi(x,y)} : x, y \neq 0 \neq ax + by \right\} < 1;$$

(b) $\lim_{n\to\infty} \varphi\left(-\frac{bx}{na}, \left(\frac{1}{b} + \frac{1}{n}\right)x\right) = 0$ *for all* $x \neq 0$*; and* $\lim_{n\to\infty} \varphi\left(\frac{x}{n}, \frac{y}{n}\right) = 0$ *for all* $x, y \neq 0 \neq ax + by$*; and*

$$\frac{1}{|B|} \sup \left\{ \frac{\limsup_{n\to\infty} \varphi\left(\left(\frac{1}{b} + \frac{1}{nb}\right)x, \left(\frac{1}{b} + \frac{1}{nb}\right)y\right)}{\varphi(x,y)} : x, y \neq 0 \neq ax + by \right\} < 1.$$

Then $f : X \to Y$ *is general linear if*

$$\|f(ax + by) - Af(x) - Bf(y)\| \leq \varphi(x,y) \quad \textit{for all } x, y \in X \setminus \{0\}.$$

We end the paper with the following two examples which are beyond the scope of Theorem B and Theorem P.

Example 2.6. Suppose that $\varphi(x,y) := \frac{|x|}{|y|^2} + \frac{|y|}{|x|^2}$ for all $x, y \in \mathbb{R} \setminus \{0\}$. It follows that

$$\lim_{n\to\infty} \varphi(x + nz, y - nz) = \lim_{n\to\infty} \varphi(x + nz, \pm nz) = \lim_{n\to\infty} \varphi(\pm nz, y - nz) = 0$$

for all $x, y \in \mathbb{R}$ and for all $z \neq 0$. Hence our Theorem 2.1 can be applicable. Moreover, $\varphi(x,y)$ is neither of the form $|x|^p + |y|^p$ where $p < 0$ nor $|x|^p|y|^q$ where $p + q < 0$.

Example 2.7. Suppose that

$$\varphi(x,y) := \begin{cases} |x||y| & \text{if } 0 < |x| \leq 1 \text{ and } 0 < |y| \leq 1; \\ |x|^2|y| & \text{if } |x| \geq 1 \text{ and } 0 < |y| \leq 1; \\ |x||y|^2 & \text{if } 0 < |x| \leq 1 \text{ and } |y| \geq 1; \\ |x|^2|y|^2 & \text{if } |x| \geq 1 \text{ and } |y| \geq 1. \end{cases}$$

Our Theorem 2.4(a) is applicable where $a = b = A = B = 1/2$. Because $\lim_{n\to\infty} \varphi\left(x, \frac{x}{n}\right) = 0$ for all $x \neq 0$; and $\lim_{n\to\infty} \varphi\left(\frac{x}{n}, \frac{y}{n}\right) = 0$ for each $x, y \neq 0 \neq \frac{1}{2}x + \frac{1}{2}y$; and

$$\frac{1}{2}\sup\left\{\frac{\limsup_{n\to\infty} \varphi\left(\left(\frac{1}{2} + \frac{1}{2n}\right)x, \left(\frac{1}{2} + \frac{1}{2n}\right)y\right)}{\varphi(x,y)} : x, y \neq 0 \neq \frac{1}{2}x + \frac{1}{2}y\right\} \leq \frac{1}{8} < 1.$$

Note that φ is not smooth at $(1,1)$ and hence $\varphi(x,y)$ is not of the form $|x|^p|y|^q$ where $p, q \in \mathbb{R}$.

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References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] A. Bahyrycz, Zs. Páles, and M. Piszczek, *Asymptotic stability of the Cauchy and Jensen functional equations*, Acta Math. Hungar. **150** (2016), 131–141.

[3] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar. **141** (2013), 58–67.

[4] J. Brzdęk, *Stability of additivity and fixed point methods*, Fixed Point Theory Appl. **2013**:285 (2013), 9 pp.

[5] J. Brzdęk and K. Ciepliński, *Hyperstability and superstability*, Abstr. Appl. Anal. **2013**, Art. ID 401756 (2013), 13 pp.

[6] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[7] M. Piszczek, *Hyperstability of the general linear functional equation*, Bull. Korean Math. Soc. **52** (2015), 1827–1838.

[8] Th. Phochai and S. Saejung, *The hyperstability of general linear equation via that of Cauchy equation*, Aequationes Math. **93** (2019), 781–789.

[9] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[10] S.M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York-London 1960.

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