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VARIATIONS OF ORDERED FIXED POINT THEOREMS

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In memory of Art Kirk and Kaz Goebel

Abstract. There are many fixed point theorems on ordered spaces. For a long period, we noticed that certain maximal element theorems can be converted equivalently to certain types of fixed point theorems, and conversely. This can be applied to theorems due to Knaster-Tarski, Nadler, Zermelo, Zorn, Tarski-Kantorovitch, and Edelstein. Consequently, several existence theorems on maximal elements, fixed points, stationary points, common fixed points, and common stationary points are obtained for various ordered spaces.

1. Introduction

In 1982-2000, we had published several articles mainly related to the Caristi fixed point theorem, the Ekeland variational principle for approximate solutions of minimization problems, and their equivalent formulations with some applications; for example, see [21]–[27]. From the beginning of such study, we obtained a Metatheorem for some equivalent statements on maximality, fixed points, stationary points, common fixed points, common stationary points, and others. We applied the Metatheorem for various occasions.

Recently, we add up some statements to the previous versions of the Metatheorem and, by applying new Metatheorem, we obtain logically equivalent formulations of existence of maximal elements of preordered set, Zorn's lemma, Banach contraction principle, Nadler's fixed point theorem, Brézis-Browder principle, Caristi's fixed point theorem, Ekeland's variational principle, Takahashi's nonconvex minimization theorem, and other various results; see [28]. Consequently, several existence theorems on maximal elements, fixed points, stationary points, common fixed points, common stationary points are obtained for several ordered sets.

In 2001, Jachymski [16] showed that one of fundamental ordering principles — the Knaster-Tarski Theorem, Zermelo's Theorem or the Tarski-Kantorovitch Theorem — can be applied to deduce the existence of a fixed point. He emphasized that all the above principles are independent of the Axiom of Choice so the above approach to metric fixed point theory is wholly constructive. He listed previous works on such studies and noted that, on the other hand, authors have also studied a reciprocal of the above problem: Given a partially ordered set and a mapping on it, define a metric depending on this order so that some theorems of metric fixed point theory could be applied. He studied consequences of the Knaster-Tarski theorem and the

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famous theorem of Zermelo such as theorems due to Banach, Nadler, Caristi, and others.

In the present article, by applying our Metatheorem, we show that various order theoretic principles can be reformulated various types of fixed point theorems.

In Section 2, we introduce our Metatheorem with its proof for completeness. Section 3 is to introduce a particular form of Metatheorem for preordered sets and the useful Brøndsted Principle recently due to ourselves. In Sections 4–8, according to our Metatheorem, we introduce equivalent formulations of the Knaster-Tarski theorem, the new Nadler fixed point theorem, the Zermelo fixed point theorem, Zorn's Lemma, and the Tarski-Kantorovitch theorem, respectively. Finally, Section 9 devotes various formulations and applications of Edelstein's fixed point theorem.

2. A metatheorem related to the Ekeland principle

The well-known central result of I. Ekeland [8, 9] on the variational principle for approximate solutions of minimization problems runs as follows:

Theorem E (Ekeland). *Let V be a complete metric space, and $F : V \to \mathbb{R}\cup\{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ be given, and a point $u \in V$ such that $F(u) \leqq \inf_V F + \varepsilon$. Then for every $\lambda > 0$, there exists a point $v \in \overline{\mathrm{B}}(u, \lambda)$ such that $F(v) \leqq F(u)$ and $F(w) > F(v) - \varepsilon\lambda^{-1}d(v, w)$ for any $w \in V,\ w \neq v$.*

When $\lambda = 1$, this is called the ε-variational principle. In order to obtain some equivalents of this principle, we obtained a Metatheorem in [21]–[27]. Later we found more additional conditions and, consequently, we obtain a new extended version of Metatheorem [28, 29]. Now we add its simplified proof for the completeness.

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following eight statements are equivalent:*

(i) *There exists an element $v \in A$ such that $G(v, w)$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If $f : A \to X$ is a map such that $\neg G(x, f(x))$ for each $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(v) *If $T : A \multimap X$ is a multimap such that $\neg G(x, y)$ holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $\neg G(x, f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ satisfying $\neg G(x, y)$ for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(viii) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $\neg G(x, z)$, then there exists a $v \in A \cap Y$.*

(ix) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for all $x \in A$ with $T(x) \neq \emptyset$, there exists $y \in X$ with $y \neq x$ and $\neg G(x, y)$ holds. Then there exists $v \in A$ such that $F(v) = \emptyset$ for all $T \in \mathfrak{F}$.*

Here, $\neg$ denotes the negation. This version will be called the 2022 Metatheorem later.

Proof. (i) $\Longrightarrow$ (ii): Suppose $v \notin T(v)$ in (ii). Then there exists a $y \in X\backslash\{v\}$ satisfying $\neg G(v, y)$. This contradicts (i).

(ii) $\Longrightarrow$ (iii): Clear.

(iii) $\Longrightarrow$ (iv): Clear.

(iv) $\Longrightarrow$ (v): Suppose T has no stationary element, that is, $T(x)\backslash\{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{T(x)\backslash\{x\} : x \in A\}$. Then f has no fixed element by its definition. However, for any $x \in A$, we have $\neg G(x, f(x))$. Therefore, by (iv), f has a fixed element, a contradiction.

(v) $\Longrightarrow$ (vi): Define a multimap $T : A \multimap X$ by $T(x) := \{f(x) : f \in \mathfrak{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, f(x))$ for any $x \in A$ and any $f \in \mathfrak{F}$, by (iv), T has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

(vi) $\Longrightarrow$ (i): Suppose that for any $x \in A$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$. Choose $f(x)$ to be one of such y. Then $f : A \to X$ has no fixed element by its definition. However, $\neg G(x, f(x))$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By (v), f has a fixed element, a contradiction.

(i)+(vi) $\Longrightarrow$ (vii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \in X\backslash\{v\}$. For each $i \in I$, by (vi), we have a $v_i \in A$ such that $\{v_i\} = T_i(v_i)$. Suppose $v \neq v_i$. Then $G(v, v_1)$ holds by (i) and $\neg G(v, v_1)$ holds by assumption on (vii). This is a contradiction. Therefore $v = v_i$ for all $i \in I$.

(vii) $\Longrightarrow$ (vi): Clear.

(i) $\Longrightarrow$ (viii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

(viii) $\Longrightarrow$ (i): For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y,\ \neg G(x, y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (viii) is satisfied. Therefore, by (viii), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $G(v, w)$ for all $w \neq v$. Hence (i) holds.

(i) $\Longrightarrow$ (ix): By (i) there exists $v \in A$ such that $G(v, x)$ holds for all $x \in X \backslash \{v\}$. Suppose to the contrary, there exists $T \in \mathfrak{F}$ such that $T(v) \neq \emptyset$. By hypothesis, there exists $w \in X$ with $w \neq v$ and $\neg G(v, w)$ holds. Therefore it leads a contradiction and $T(v) = \emptyset$ for all $T \in \mathfrak{F}$.

(ix) $\Longrightarrow$ (i): Suppose that, for each $x \in A$, there exists $y \in X \backslash \{x\}$ such that $\neg G(x, y)$ holds. For each $x \in A$, define a multimap $T : A \multimap X \backslash \{x\}$ by

$$T(x) = \{y \in X : \neg G(x, y)\} \neq \emptyset \text{ for all } x \in A.$$

Then, by (ix), there exists $v \in A$ such that $T(v) = \emptyset$. This is a contradiction.

This completes our proof. □

Note that (iv) $\Longrightarrow$ (v) adopted the Axiom of Choice, and that the element v is the same throughout (i)–(ix).

3. Preordered sets

Let $(X, \preceq)$ be a *preordered set*; that is, X is a nonempty set, $\preceq$ is reflexive and transitive. For each $x \in X$, we denote $S(x) = \{y \in X : x \preceq y\}$ and $G(x, y)$ means $x \preceq y$.

Now we apply Metatheorem to preordered sets:

Theorem 3.1. *Let $(X, \preceq)$ be a preordered set, $x_0 \in X$, and $A = S(x_0)$. Then the following eight statements are equivalent:*

(i) *There exists a maximal element $v \in A$, that is, $v \not\preceq w$ for any $w \in X \backslash \{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that, for any $x \in A \backslash T(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If $f : A \to X$ is a map such that $x \preceq f(x)$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(v) *If $T : A \multimap X$ is a multimap such that $x \preceq y$ holds for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that $x \preceq y$ holds for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(viii) *If Y is a subset of X such that, for each $x \in A \backslash Y$, there exists a $z \in X \backslash \{x\}$ satsfying $x \preceq z$, then there exists an element $v \in A \cap Y$.*

(ix) *Let* $\mathfrak{F}$ *be a family of multimaps* $T : A \multimap X$ *such that, for all* $x \in A$ *with* $T(x) \neq \emptyset$*, there exists* $y \in X \backslash \{x\}$ *satisfying* $x \preceq y$*. Then there exists* $v \in A$ *such that* $T(v) = \emptyset$ *for all* $T \in \mathfrak{F}$*.*

Proof. In Metatheorem, put $A = S(x_0)$ and let $G(v, w)$ be the statement $v \not\preceq w$. Then each of (i)–(ix) follows from the corresponding ones in Metatheorem.

This completes our proof. □

Remark 3.2. We claim that (i)–(ix) are equivalent in Theorem 3.1 and do not say that they are true. For a counter-example, the real line $\mathbb{R}$ does not have any maximal element in the natural order.

Now we borrow Jachymski [14, 16] as follows:

A *partially ordered set* is a pair $(P, \preceq)$, where P is a nonempty set and $\preceq$ is a relation in P which is *reflexive* ($p \preceq p$ for all $p \in P$), *weakly antisymmetric* (for $p, q \in P,\ p \preceq q$ and $q \preceq p$ imply $p = q$) and *transitive* (for $p, q, r \in P,\ p \preceq q$ and $q \preceq r$ imply $p \preceq r$). A nonempty subset C of P is called a *chain* if given $p, q \in C$, either $p \preceq q$ or $q \preceq p$. If every chain in $(P, \preceq)$ has a supremum, then $(P, \preceq)$ is said to be *chain-complete*. A map $f : P \to P$ is said to be *isotone* or *increasing* if it preserves order, i.e., given $p, q \in P,\ p \preceq q$ implies that $f(p) \preceq f(q)$.

Motivated by Brønsted [4], we established the following in [29]:

Brønsted Principle. *Let* $(E, \preceq)$ *be a preordered set and* $f : E \to E$ *be a map such that* $x \preceq f(x)$ *for all* $x \in E$*. Then a maximal element* $v \in E$ *is a fixed point of* f*.*

In most applications of this principle for partially ordered sets (posets), the existence of a maximal element is achieved by the upper bound of a chain in E.

From now on, by applying this principle, we are going to give examples of Theorem 3.1 on order theoretic fixed point theorems. These examples are mainly taken from Dugundji-Granas [7, 10] and Jachymski [16].

4. Knaster-Tarski theorem

The following is given as [16, Theorem 2.1]:

Theorem 4.1 (Knaster-Tarski)**.** *Let* $(P, \preceq)$ *be a partially ordered set in which every chain has a supremum. Assume that* $f : P \to P$ *is isotone and there is an element* $p_0 \in P$ *such that* $p_0 \preceq f(p_0)$*. Then* f *has a fixed point.*

According to Jachymski [16]: This theorem was proved in 1927 by Knaster [18J for increasing – under set-inclusion – mappings, on and to the family of all subsets of a set. In 1939 Tarski extended Knaster's result to increasing maps on a complete lattice and he gave its applications in set theory and topology, but his result was unpublished until 1955 (cf. Tarski [33, footnote no.2]). The version of the Knaster-Tarski theorem presented here is due to Abian and Brown [2] and, independently, Pelczar [30], and was established in 1961.

Theorem 4.1 can be extended as follows:

Theorem 4.2. *Let $(X, \preceq)$ be a partially ordered set in which every chain has a supremum. Assume that $f : X \to X$ is isotone and there is an element $x_0 \in X$ such that $x_0 \preceq f(x_0)$ with $A = \{f^n(x_0) : n \in \mathbb{N}\} \cup \{its\ supremum\}$.*

Then Theorem 3.1(i)–(ix) hold including the following:

(i) *There exists a maximal element $v \in A$, that is, $v \not\preceq w$ for any $w \in X \backslash \{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that, for any $x \in A \backslash T(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $g : A \to X$ is a map such that, for any $x \in A$ with $x \neq g(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then g has a fixed element $v \in A$, that is, $v = g(v)$.*

(iv) *If $g : A \to X$ is a map such that $x \preceq g(x)$ for any $x \in A$, then g has a fixed element $v \in A$, that is, $v = g(v)$.*

Proof. Note that A is a chain with a supremum $v \in A$. Since f is isotone, $f|_A$ is progressive and has a fixed point $v = f(v) \in A$ by Theorem 4.1. Then by the Brøndsted Principle, $v \in A$ satisfies (i) in Theorem 3.1. This completes the proof as in Theorem 3.1. □

Note that (iii) and (iv) of Theorem 4.2 seem to be better than the Knaster-Tarski theorem and that all of (ii)–(vii) are its generalizations.

5. Nadler fixed point theorem

Following Jachymski [16], we shall give a new proof of Nadler's theorem [20] using partial ordering techniques. Let (X, d) be a metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\},\ \sup\{d(b, A) : b \in B\}\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Then H is called a generalized Hausdorff metric since it may have infinite values.

Jachymski recalled a more general form of Nadler's theorem established by Covitz and Nadler [6] as follows:

Theorem 5.1 (Nadler)**.** *Let (X, d) be a complete metric space and $T : X \to \mathrm{Cl}(X)$. Assume there is an $h \in [0, 1)$ such that*

$$H(T(x), T(y)) \leq hd(x, y) \quad \textit{for all } x, y \in X.$$

Then T has a fixed point.

Motivated by Theorem 5.1 and Metatheorem, we have the following extended form of [28, Theorem 3.2]:

Theorem 5.2. *Let X be a complete metric space, $T : X \to \mathrm{Cl}(X)$ be a multimap, and $0 < h < 1$. Then the following equivalent statements hold:*

(i) *There exists an element* $v \in X$ *such that* $H(T(v), T(w)) > hd(v, w)$ *for any* $w \in X \backslash \{v\}$.

(ii) *If* $T : X \multimap X$ *is a multimap such that, for any* $x \in X \backslash T(x)$, *there exists a* $y \in X \backslash \{x\}$ *satisfying* $H(T(x), T(y)) \leq hd(x, y)$, *then* T *has a fixed element* $v \in X$, *that is,* $v \in T(v)$.

(iii) *If* $f : X \to X$ *is a map such that, for any* $x \in X$ *with* $x \neq f(x)$, *there exists a* $y \in X \backslash \{x\}$ *satisfying* $d(f(x), f(y)) \leq hd(x, y)$, *then* f *has a fixed element* $v \in X$, *that is,* $v = f(v)$.

(iv) *If* $f : X \to X$ *is a map such that* $d(f(x), f^2(x)) \leq hd(x, f(x))$ *for any* $x \in X$, *then* f *has a fixed element* $v \in X$, *that is,* $v = f(v)$.

(v) *If* $T : X \multimap X$ *is a multimap such that* $H(T(x), T(y)) \leq hd(x, y)$ *holds for any* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$, *then* T *has a stationary element* $v \in X$, *that is,* $\{v\} = T(v)$.

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $d(f(x), f(y)) \leq hd(x, y)$ *for all* $x \in X$ *with* $x \neq f(x)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \to X$ *satisfying* $H(T(x), T(y)) \leq hd(x, y)$ *for all* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in X$, *that is,* $\{v\} = T(v)$ *for all* $T \in \mathfrak{F}$.

(viii) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $H(T(x), T(z)) \leq hd(x, z)$ *for a* $T : X \multimap X$, *then there exists a* $v \in X \cap Y = Y$.

(ix) *Let* $\mathfrak{F}$ *be a family of multimaps* $T : X \multimap X$ *such that, for all* $x \in A$ *with* $T(x) \neq \emptyset$, *there exists* $y \in X \backslash \{x\}$ *such that* $H(T(x), T(y)) \leq hd(x, y)$ *holds. Then there exists* $v \in A$ *such that* $T(v) = \emptyset$ *for all* $T \in \mathfrak{F}$.

Proof. Note that, in Metatheorem, put $A = X$ and let $G(v, w)$ be the statement $H(T(v), T(w)) > hd(v, w)$. Then each of (i)–(ix) follows from the corresponding ones in Metatheorem. Note that, when the family consists of a single map f in (vi) holds, we have a maximal element in (i) by our Brøndsted principle. This completes our proof. □

Note that (ii) or (iv) extend Nadler's theorem and (iii) implies the Banach contraction principle. Therefore, in some sense, these two theorems are equivalent in view of Theorem 5.2 .

6. Zermelo fixed point theorem

The following is known; see [16]:

Theorem 6.1 (Zermelo). *Let $(P, \preceq)$ be a partially ordered set in which every chain has a supremum. Assume that $f : P \to P$ is such that*

$$p \preceq f(p) \quad for\ all \quad p \in P.$$

Then f has a fixed point.

The Zermelo fixed point theorem is also known as the Bourbaki fixed point theorem or the Bourbaki-Kneser fixed point theorem. It implies the Caristi fixed point theorem, the Bernstein-Cantor-Schröder theorem, the Ekeland variational principle, the Takahashi minimization theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma.

According to Jachymski [16]: "A map f satisfying $p \preceq f(p)$ is said to be progressive. The above theorem is attributed to Zermelo, although it does not appear explicitly in any of his papers. However, a proof of it can be derived from Zermelo's proofs of the well-ordering principle. This observation is due to Bourbaki [3], who was the first to formulate the theorem in the above form."

"Under the Axiom of Choice, the assumption of Theorem 6.1 can be weakened to 'each nonempty well-ordered subset has an upper bound.' This is Kneser's fixed point theorem [19], which turns out to be equivalent to the Axiom of Choice as shown by Abian [1]."

Theorem 6.2. *Let $(X, \preceq)$ be a partially ordered set in which either*

(a) *every chain has a supremum; or*

(b) *each nonempty well-ordered subset has an upper bound.*

Then the following equivalent statements hold:

(i) *There exists a maximal element $v \in X$ such that $v \not\preceq w$ for any $w \in X \backslash \{v\}$.*

(ii) *If $T : X \multimap X$ is a multimap such that for any $x \in X \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then T has a fixed element $v \in X$, that is, $v \in T(v)$.*

(iii) *If $f : X \to X$ is a map such that for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(iv) *If $f : X \to X$ is a map such that $x \preceq f(x)$ for any $x \in X$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(v) *If $T : X \multimap X$ is a multimap such that $x \preceq y$ holds for any $x \in X$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $x \preceq f(x)$ for all $x \in X$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T : X \multimap X$ satisfying $x \preceq y$ for any $x \in X$ and any $y \in T(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(viii) *If Y is a subset of X such that for each $x \in X \backslash Y$ there exists a $z \in X \backslash \{x\}$ such that $x \preceq z$, then there exists an element $v \in X \cap Y = Y$.*

(ix) *Let $\mathfrak{F}$ be a family of multimaps $T : X \multimap X$ such that, for all $x \in X$ with $T(x) \neq \emptyset$, there exists $y \in X \backslash \{x\}$ such that $x \preceq y$ holds. Then there exists $v \in X$ such that $T(v) = \emptyset$ for all $T \in \mathfrak{F}$.*

Proof. Note that (vi) for a single map f reduces to Theorem 6.1, which implies (i) the existence of a maximal elements $v \in X$ by our Brøndsted Principle [29], Now the conclusion follows from Theorem 3.1 or Metatheorem. □

Jachymski [16] showed without the Axiom of Choice that the Zermelo theorem implies directly a restriction of the Caristi fixed point theorem to continuous functions. In particular, Zermelo's theorem yields the Banach contraction principle. Under the Axiom of Choice, this restriction is proved to be equivalent to Caristi's theorem.

He also showed that Zermelo's theorem yields Nadler's theorem for closed-valued contraction [6]. He added more applications of Zermelo's theorem.

7. Zorn's lemma

Motivated by Zorn's Lemma and Theorem 3.1, we have the following:

Theorem 7.1. *Let $(X, \preceq)$ be a preordered set in which each nonempty well-ordered subset has an upper bound. Let $x_0 \in X$ and $A = S(x_0) = \{y \in X : x_0 \preceq y\}$.*

Then the following nine statements are equivalent:

(i) *There exists a maximal element $v \in A$, that is, $v \not\preceq w$ for any $w \in X \backslash \{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If $f : A \to X$ is a map such that $x \preceq f(x)$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(v) *If $T : A \multimap X$ is a multimap such that $x \preceq y$ holds for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ satisfying $x \preceq y$ for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(viii) *If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ such that $x \preceq z$, then there exists an element $v \in A \cap Y$.*

(ix) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for all $x \in A$ with $T(x) \neq \emptyset$, there exists $y \in X\backslash\{x\}$ such that $x \preceq y$ holds. Then there exists $v \in A$ such that $T(v) = \emptyset$ for all $T \in \mathfrak{F}$.*

Proof. In Metatheorem, put $A := S(x_0)$ and let $G(v, w)$ be the statement $v \not\preceq w$. Then each of (i)–(ix) follows from the corresponding ones in Theorem 3.1. This completes our proof. □

Note that if $(X, \preceq)$ is partially ordered set, then (i) is Zorn's Lemma and (iii) extends Zermelo's Theorem 6.1. Moreover, in Theorem 7.1, it is enough to assume $S(x_0)$ has an upper bound.

There are several forms of Zorn's Lemma equivalent to the Axiom of Choice; see Suppes [31]. For the consequences of the Axiom of Choice, see Howard-Rubin [11].

8. Tarski-Kantorovitch theorem

Jachymski [16] introduced the following after enough preparation:

Theorem 8.1 (Tarski-Kantorovitch)**.** *Let $(P, \preceq)$ be a $\preceq$-complete partially ordered set and a mapping $f : P \to P$ be $\preceq$-continuous. If there exists $p_0 \in P$ such that $p_0 \preceq f(p_0)$, then f has a fixed point; moreover, $p_* = \sup\{f^n(p_0) : n \in \mathbb{N}\}$ is fixed under f.*

This also can have equivalent formulations by applying our Metatheorem.

In 1998, Jachymski [13] showed that the Tarski-Kantorovitch Principle for continuous maps on a partially ordered set yields some fixed point theorems for contractive maps on a uniform space. His proofs do not depend on the Axiom of Choice.

In 2000, Jachymski et al. [15] applied a slightly different version of the Tarski-Kantorovitch principle to derive some results of the theory of iterated function system. See also [33].

9. Edelstein fixed point theorem

In this section, we apply Metatheorem to a particular situation:

Theorem 9.1. *Let X be a compact metric space, $f : X \to X$ be a continuous map and $G(x, y)$ be $d(x, f(x)) \leq d(y, f(y))$ for $x, y \in X$.*

Then the following eight equivalent statements hold:

(i) *There exists a point $v \in X$ such that $d(v, f(v)) \leq d(w, f(w))$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : X \multimap X$ is a multimap such that for any $x \in X\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $d(x, f(x)) > d(y, f(y))$, then T has a fixed point $v \in X$, that is, $v \in T(v)$.*

(iii) *If $g : X \to X$ is a map such that for any $x \in X$ with $x \neq g(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $d(x, f(x)) > d(y, f(y))$, then g has a fixed point $v \in X$, that is, $v = g(v)$.*

(iv) *If* $g : X \to X$ *is a map such thatd*$(x, f(x)) > d(g(x), f(g(x)))$ *for any* $x \in X$*, then* g *has a fixed point* $v \in X$*, that is,* $v = g(v)$*.*

(v) *If* $T : A \multimap X$ *is a multimap such that* $d(x, f(x)) > d(y, f(y))$ *holds for any* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$*, then* T *has a stationary point* $v \in X$*, that is,* $\{v\} = T(v)$*.*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $g : X \to X$ *satisfying* $d(x, f(x)) > d(f(x), f^2(x))$ *for all* $x \in X$ *with* $x \neq g(x)$*, then* $\mathfrak{F}$ *has a common fixed point* $v \in X$*, that is,* $v = g(v)$ *for all* $g \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T : X \multimap X$ *such that* $d(x, f(x)) > d(y, f(y))$ *holds for any* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary point* $v \in X$*, that is,* $\{v\} = T(v)$ *for all* $T \in \mathfrak{F}$*.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in X\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $d(x, f(x)) > d(z, f(z))$*, then there exists a* $v \in X \cap Y = Y$*.*

(ix) *Let* $\mathfrak{F}$ *be a family of multimaps* $T : X \multimap X$ *such that, for all* $x \in X$ *with* $T(x) \neq \emptyset$*, there exists* $y \in X\backslash\{x\}$ *such that* $d(x, f(x)) > d(y, f(y))$ *holds. Then there exists* $v \in X$ *such that* $T(v) = \emptyset$ *for all* $T \in \mathfrak{F}$*.*

Proof. Let a map $\varphi : X \to \mathbb{R}^+$ by putting

$$\varphi(x) = d(x, f(x)), \quad x \in X.$$

Then φ is continuous and bounded below, so it has a minimum value at a point $v \in X$. Hence (i) holds.

Moreover, (ii)–(ix) also hold by Metatheorem. □

Theorem 9.1 has many consequences. The first one is the well-known Edelstein fixed point theorem.

Definition 9.2. A map $f : X \to X$ on a metric space (X, d) is said to be *contractive* if

$$d(f(x), f(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$.

Theorem 9.3 (Edelstein)**.** *Let* (X, d) *be a compact metric space and* $f : X \to X$ *be a contractive map. Then* f *has a unique fixed point* $v \in X$*, and moreover, for each* $x \in X$*, we have* $\lim_{n\to\infty} f^n(x) = v$*.*

Proof. As in Theorem 9.1(i), $\varphi(x) = d(x, f(x))$ has a minimum at $v \in X$. If $v \neq f(v)$, then

$$\varphi(f(v)) = d(f(v), f^2(v)) < d(v, f(v)) = \varphi(v),$$

and hence $v = f(v)$. For the proof of $\lim_{n\to\infty} f^n(x) = v$ for any $x \in X$, see W. Kirk [17]. □

From Theorem 9.1, we can deduce several fixed point theorems on a compact metric space (X, d) as follows:

(ii) *If* $T : X \multimap X$ *is a multimap such that for any* $x \in X \backslash T(x)$ *there exists a* $y \in X \backslash \{x\}$ *satisfying* $d(x, f(x)) > d(y, f(y))$ *for a continuous selfmap* f *of* X*, then* T *has a fixed point* $v \in X$*, that is,* $v \in T(v)$.

(iii) *If* $g : X \to X$ *is a continuous map such that for any* $x \in X$ *with* $x \neq g(x)$*, there exists a* $y \in X \backslash \{x\}$ *satisfying* $d(x, g(x)) > d(y, g(y))$*, then* g *has a fixed point* $v \in X$*, that is,* $v = g(v)$.

(v) *If* $T : A \multimap X$ *is a multimap and* $f : X \to X$ *is a continuous selfmap such that* $d(x, f(x)) > d(y, f(y))$ *holds for any* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$*, then* T *has a stationary point* $v \in X$*, that is,* $\{v\} = T(v)$.

(vi) *If* $\mathfrak{F}$ *is a family of continuous maps* $f : X \to X$ *satisfying* $d(x, f(x)) > d(f(x), f^2(x))$ *for all* $x \in X$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed point* $v \in X$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : X \multimap X$ *for* $i \in I$ *with an index set* I *and a continuous selfmap* $f : X \to X$ *such that* $d(x, f(x)) > d(y, f(y))$ *holds for any* $x \in X$ *and any* $y \in T_i(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary point* $v \in X$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

Very recently, Kirk and Shahzad gave one open question on Edelstein's fixed point theorem. In 2018, Suzuki [32] gave a negative answer to this question, and extended Edelstein's theorem to semimetric spaces.

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