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ON VON NEUMANN-JORDAN CONSTANT OF GENERALIZED BANAŚ-FRĄCZEK SPACES II

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Abstract. In this paper, we calculate von Neumann-Jordan constant of generalized Banaś-Frączek space $\mathbb{R}^2_{a,b,p}$ in the case where $1 \le p < 2$ by using the Banach-Mazur distance.

1. Introduction and preliminaries

The notion of the von Neumann-Jordan (NJ-) constant of Banach spaces was introduced by Clarkson in [3] and recently it has been studied by several authors (cf. [1, 4, 5, 6, 7, 8, 10, 11, 12, 13], etc.). The *NJ-constant* of a Banach space X is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x,y) \neq (0,0) \right\}.$$

It is known that $1 \le C_{\mathrm{NJ}}(X) \le 2$ for any Banach space X, X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$, and X is uniformly non-square if and only if $C_{\mathrm{NJ}}(X) < 2$. For some other results concerning the NJ-constant, we refer the reader to [5]. Recently, C. Yang and X. Yang introduced the Banaś-Frączek type space $X_{\lambda,p}$ in [11], i.e., $\mathbb{R}^2$ with the norm $\|\cdot\|_{\lambda,p}$ defined by

$$\|(x,y)\|_{\lambda,p} = \max\{\lambda|x|, \|(x,y)\|_p\},$$

where $\lambda > 1$, $p \ge 1$ and $\|\cdot\|_p$ is ℓ_p-norm. In the case where $p = 2$, $X_{\lambda,2}$ is called Banaś-Frączek space ([2, 10]). In [11], they showed that if $p \ge 2$ and $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \ge 1$, then

$$C_{\mathrm{NJ}}(X_{\lambda,p}) = 1 + \left(1 - \frac{1}{\lambda^p}\right)^{2/p}. \tag{1.1}$$

Also, Mitani, Saito and Takahashi [6] introduced the generalized Banaś-Frączek space $\mathbb{R}^2_{a,b,p}$, i.e., $\mathbb{R}^2$ with the norm

$$\|(x,y)\| = \max\{a|x|, b|y|, \|(x,y)\|_p\},$$

where $a \ge b \ge 1$ and $1 \le p < \infty$. In the case where $a = \lambda$ and $b = 1$, we have $\mathbb{R}^2_{a,b,p} = X_{\lambda,p}$. If $a = 1$, then $\|\cdot\| = \|\cdot\|_p$. If $a^{-p} + b^{-p} \le 1$, then $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 2$.

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Hence we may assume that $a > 1$ and $a^{-p} + b^{-p} > 1$. The authors showed that if $p \geq 2$ and $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 1 + b^2\Big(1 - \frac{1}{a^p}\Big)^{2/p},$$

and if $p \geq 2$ and $b > a(a^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = b^2\Big(1 + \Big(\frac{a}{b}\Big)^{\frac{2p}{p-2}}\Big)^{1-2/p}.$$

In particular, if $p \geq 2$ and $1 \leq \lambda(\lambda^p - 1)^{\frac{p-2}{2p}}$, then

$$C_{\mathrm{NJ}}(X_{\lambda,p}) = 1 + \Big(1 - \frac{1}{\lambda^p}\Big)^{2/p}.$$

This is an improvement of (1.1), since the inequality

$$\lambda(\lambda^p - 1)^{\frac{p-2}{2p}} \geq (\lambda^p - 1)^{\frac{p-2}{2p}}(\lambda^2 - 1)^{\frac{1}{2}} \geq 1$$

holds if $(\lambda^p - 1)^{p-2}(\lambda^2 - 1)^p \geq 1$.

In this paper, we consider the constant $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p})$ in the case where $1 \leq p < 2$ and show that if $1 \leq p < 2$ with $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$, then

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 1 + b^2\Big(1 - \frac{1}{a^p}\Big)^{2/p},$$

by using the Banach-Mazur distance.

We recall some notations and definitions on geometrical properties of Banach spaces. For isomorphic Banach spaces X and Y, the *Banach-Mazur distance* between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\| \cdot \|T^{-1}\|$ taken over all bicontinuous linear operators T from X onto Y.

Lemma 1.1 ([5])**.** *If X and Y are isomorphic Banach spaces, then*

$$\frac{C_{\mathrm{NJ}}(X)}{d(X,Y)^2} \leq C_{NJ}(Y) \leq C_{\mathrm{NJ}}(X)d(X,Y)^2.$$

In particular, if X and Y are isometric, then $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(Y)$.

Lemma 1.2 ([5])**.** *Let $X = (X, \|\cdot\|)$ be a non-trivial Banach space and $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $\alpha, \beta > 0$,*

$$\alpha\|x\| \leq \|x\|_1 \leq \beta\|x\|, \quad x \in X.$$

Then

$$\frac{\alpha^2}{\beta^2}C_{\mathrm{NJ}}(X) \leq C_{\mathrm{NJ}}(X_1) \leq \frac{\beta^2}{\alpha^2}C_{\mathrm{NJ}}(X).$$

Lemma 1.2 follows immediately from Lemma 1.1 and the fact that $d(X, X_1) \leq \beta/\alpha$. In particular, if $C_{\mathrm{NJ}}(X_1) = (\beta^2/\alpha^2)C_{\mathrm{NJ}}(X)$, then $d(X, X_1) = \beta/\alpha$.

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(|x|, |y|)\| = \|(x, y)\|$ for any $x, y \in \mathbb{R}$. By Lemma 1.2, we have the following formula on NJ-constant for absolute norm.

Lemma 1.3 ([6]). *Let $\|\cdot\|, \|\cdot\|_H$ be absolute norms on $\mathbb{R}^2$ satisfying the following conditions:*
(i) *$(\mathbb{R}^2, \|\cdot\|_H)$ is an inner product space.*
(ii) *$\|(x,y)\| \le \|(x,y)\|_H$ for any $(x,y) \in \mathbb{R}^2$.*
(iii) *$\|(1,0)\| = \|(1,0)\|_H$ and $\|(0,1)\| = \|(0,1)\|_H$.*
Then

$$C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|)) = \beta^2, \text{ where } \beta = \max\left\{\frac{\|(x,y)\|_H}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)\right\}.$$

Moreover, it follows that

$$d\left((\mathbb{R}^2, \|\cdot\|), (\mathbb{R}^2, \|\cdot\|_H)\right) = \beta = \sqrt{C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|))},$$

since $C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|)) = \beta^2 C_{\mathrm{NJ}}((\mathbb{R}^2, \|\cdot\|_H))$.

2. Results

Let $1 \le p < 2$. Let us now calculate the constant $C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p})$ by using Lemma 1.3. To do this we will need the following.

Lemma 2.1. *Let $a > 1$, $a \ge b \ge 1$ and $1 \le p < 2$ with $a^{-p} + b^{-p} > 1$. We define*

$$f(t) = \frac{(a^2t^2 + b^2)^{1/2}}{(t^p + 1)^{1/p}} \quad (t \ge 0).$$

Put $t_1 = (a^p - 1)^{-\frac{1}{p}}$, $t_2 = (\frac{b}{a})^{\frac{2}{2-p}}$, $t_3 = (b^p - 1)^{\frac{1}{p}}$ and $\beta_i = f(t_i)$ for each i. Then
(i) *f is non-increasing on $(0, t_2)$ and is non-decreasing on (t_2, ∞). Hence f has the minimum at $t = t_2$.*
(ii) *$\beta_1 \ge \beta_3$ holds.*
(iii) *If $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \le 1$, then $f(t) \ge 1$ for all $t \ge 0$.*

Proof. (i) Since the derivative of f is

$$f'(t) = (a^2t^2 + b^2)^{-1/2}(t^p + 1)^{-1/p-1}t(a^2 - b^2t^{p-2}),$$

we have (i).

(ii) It is easy to see that

$$\begin{aligned}\beta_1 = f(t_1) &= \frac{(a^2(a^p-1)^{-\frac{2}{p}} + b^2)^{\frac{1}{2}}}{((a^p-1)^{-1}+1)^{\frac{1}{p}}} = \frac{(a^2 + b^2(a^p-1)^{\frac{2}{p}})^{\frac{1}{2}}}{(1+(a^p-1))^{\frac{1}{p}}} \\ &= \left(1 + b^2\left(1 - \frac{1}{a^p}\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\beta_3 = f(t_3) &= \frac{(a^2(b^p-1)^{\frac{2}{p}} + b^2)^{1/2}}{((b^p-1)+1)^{1/p}} \\ &= \left(1 + a^2\left(1 - \frac{1}{b^p}\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}.\end{aligned}$$

As in the proof of Lemma 2.4 in [6] we have the inequality

$$b\Big(1-\frac{1}{a^p}\Big)^{1/p} \geq a\Big(1-\frac{1}{b^p}\Big)^{1/p},$$

because of the identity

$$b^p\Big(1-\frac{1}{a^p}\Big) - a^p\Big(1-\frac{1}{b^p}\Big) = (a^p-b^p)\Big(\frac{1}{a^p}+\frac{1}{b^p}-1\Big).$$

Thus $\beta_1 \geq \beta_3$.

(iii) Let $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$. By $p<2$,

$$\beta_2 = f(t_2) = \frac{\left(a^2\left(\frac{b}{a}\right)^{\frac{4}{2-p}} + b^2\right)^{\frac{1}{2}}}{\left(\left(\frac{b}{a}\right)^{\frac{2p}{2-p}}+1\right)^{\frac{1}{p}}} = \frac{b\left(\left(\frac{b}{a}\right)^{\frac{2p}{2-p}}+1\right)^{\frac{1}{2}}}{\left(\left(\frac{b}{a}\right)^{\frac{2p}{2-p}}+1\right)^{\frac{1}{p}}}$$

$$= b\left(\left(\frac{b}{a}\right)^{\frac{2p}{2-p}}+1\right)^{\frac{p-2}{2p}} = \left(a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \geq 1.$$

Thus it follows from (i) that $f(t) \geq f(t_2) \geq 1$ for all $t \geq 0$.

□

Let $a>1$, $a \geq b \geq 1$ and $1 \leq p < 2$ with $a^{-p}+b^{-p}>1$. We define the norms $\|\cdot\|$ and $\|\cdot\|_H$ on $\mathbb{R}^2$ by

$$\|(x,y)\| = \max\{a|x|, b|y|, \|(x,y)\|_p\} \tag{2.1}$$

and

$$\|(x,y)\|_H = \|(ax,by)\|_2. \tag{2.2}$$

It is clear that $\|\cdot\|$ and $\|\cdot\|_H$ are absolute norms and satisfy the conditions (i) and (iii) in Lemma 1.3.

Lemma 2.2. *Let $a>1, a \geq b \geq 1$ and $1 \leq p < 2$ with $a^{-p}+b^{-p}>1$ and $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$. Let $\|\cdot\|$ and $\|\cdot\|_H$ be the norms defined by (2.1) and (2.2), respectively.*
(i) $\|(x,y)\| \leq \|(x,y)\|_H$ *for any* $(x,y) \in \mathbb{R}^2$.
(ii) *Put*

$$\beta = \max\Big\{\frac{\|x\|_H}{\|x\|} : x \in \mathbb{R}^2, x \neq 0\Big\}.$$

Then

$$\beta = \Big(1 + b^2\Big(1-\frac{1}{a^p}\Big)^{\frac{2}{p}}\Big)^{\frac{1}{2}}.$$

Proof. Let t_i and β_i $(i=1,2,3)$ be elements as in Lemma 2.1. We first show $\|(x,y)\| \leq \|(x,y)\|_H \leq \beta_1\|(x,y)\|$ for all $(x,y) \in \mathbb{R}^2$. In the case where $x=0$ or $y=0$, since $\beta_1 \geq 1$, this is true. Let $x \neq 0$ and $y \neq 0$. Put $t = |x|/|y|$.

We first consider the case $\|(x,y)\| = a|x|$. It is clear that $\|(x,y)\| \le \|(x,y)\|_H$. Since $a|x| \ge \|(x,y)\|_p$, we have $a \ge \|(1,1/t)\|_p$ and so $t \ge (a^p-1)^{-1/p} = t_1$. Hence

$$\frac{\|(x,y)\|_H}{\|(x,y)\|} = \frac{\|(ax,by)\|_2}{a|x|} = \left\|\left(1, \frac{b}{at}\right)\right\|_2 \le \left\|\left(1, \frac{b}{at_1}\right)\right\|_2$$
$$= \left(1 + \frac{b^2}{a^2}(a^p-1)^{\frac{2}{p}}\right)^{\frac{1}{2}} = \beta_1,$$

that is, $\|(x,y)\|_H \le \beta_1\|(x,y)\|$.

We next consider the case $\|(x,y)\| = b|y|$. It is clear that $\|(x,y)\| \le \|(x,y)\|_H$. Since $b|y| \ge \|(x,y)\|_p$, we have $b \ge \|(t,1)\|_p$ and so $t \le (b^p-1)^{1/p} = t_3$. Hence

$$\frac{\|(x,y)\|_H}{\|(x,y)\|} = \frac{\|(ax,by)\|_2}{b|y|} = \left\|\left(\frac{a}{b}t, 1\right)\right\|_2 \le \left\|\left(\frac{a}{b}t_3, 1\right)\right\|_2$$
$$= \left(\frac{a^2}{b^2}(b^p-1)^{\frac{2}{p}} + 1\right)^{\frac{1}{2}} = \beta_3.$$

Since $\beta_1 \ge \beta_3$ holds by Lemma 2.1 (ii), we obtain $\|(x,y)\|_H \le \beta_1\|(x,y)\|$.

Finally we consider the case $\|(x,y)\| = \|(x,y)\|_p$. Since $\|(x,y)\|_p \ge a|x|$ and $\|(x,y)\|_p \ge b|y|$, it follows that $|x| \le (a^p-1)^{-\frac{1}{p}}|y|$ and $(b^p-1)^{\frac{1}{p}}|y| \le |x|$. Hence $t_3 \le t \le t_1$. Then

$$\frac{\|(x,y)\|_H}{\|(x,y)\|} = \frac{\|(ax,by)\|_2}{\|(x,y)\|_p} = \frac{\|(at,b)\|_2}{\|(t,1)\|_p} = \frac{(a^2t^2+b^2)^{1/2}}{(t^p+1)^{1/p}} =: f(t).$$

It follows from Lemma 2.1 that $f(t) \ge 1$ and hence $\|(x,y)\|_H \ge \|(x,y)\|$. By Lemma 2.1 (i), (ii) we have

$$\max\{f(t) : t_3 \le t \le t_1\} = f(t_1) = \beta_1$$

and so $\|(x,y)\|_H \le \beta_1\|(x,y)\|$.

Moreover, we have equality in above for $(x,y) = (t_1, 1)$. Hence $\beta = \beta_1$. Thus we obtain (ii).

□

By Lemma 1.3 and Lemma 2.2 we obtain the main theorem.

Theorem 2.3. *Let* $a > 1, a \ge b \ge 1$ *and* $1 \le p < 2$ *with* $a^{-p} + b^{-p} > 1$ *and* $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \le 1$. *Then*

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = 1 + b^2\left(1 - \frac{1}{a^p}\right)^{\frac{2}{p}}.$$

Moreover,

$$C_{\mathrm{NJ}}(\mathbb{R}^2_{a,b,p}) = d(\mathbb{R}^2_{a,b,p}, H)^2,$$

where H *is a two-dimensional inner product space.*

Remark 2.4. (i) Let $a > b = 1$ and $1 \le p < 2$. Then it is clear that $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} > 1$. Hence, using Theorem 2.3 we can not obtain the value of $C_{\mathrm{NJ}}(X_{\lambda,p})$ for this case.
(ii) The unit sphere of $\mathbb{R}^2_{\sqrt{2},\sqrt{2},1}$ is a regular octagon. From Theorem 2.3, we obtain $C_{\mathrm{NJ}}(\mathbb{R}^2_{\sqrt{2},\sqrt{2},1}) = 4 - 2\sqrt{2}$ (cf. [9]).

(iii) Takahashi [9] showed that for any Banach space X,

$$1 + \frac{\varepsilon_0(X)^2}{4} \leq C_{\mathrm{NJ}}(X), \tag{2.3}$$

where $\varepsilon_0(X) = \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}$ is the characteristic of convexity of X. We consider the case $X = \mathbb{R}^2_{a,b,p}$. Let $a > 1, a \geq b \geq 1$ with $a^{-p} + b^{-p} > 1$. In [6], it was shown that if $p \geq 2$, then we have equality in (2.3) if and only if $b \leq a(a^p - 1)^{\frac{p-2}{2p}}$ holds. Let $1 \leq p < 2$. It is easy to see that

$$\varepsilon_0(\mathbb{R}^2_{a,b,p}) \geq 2b\Big(1 - \frac{1}{a^p}\Big)^{1/p}.$$

This inequality and Theorem 2.3 give that if $a^{\frac{2p}{p-2}} + b^{\frac{2p}{p-2}} \leq 1$, then we have equality in (2.3).

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