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REFINEMENTS OF BOUNDS FOR ENTROPY AND RELATIVE ENTROPY

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Abstract. There are many generalizations of Hermite-Hadamard inequality for convex function f defined on $[a,b]$. Recently we gave several relations of upper bounds or lower bounds of refined Hermite-Hadamard inequality and apply to different types of inequalities under some conditions. In this article we give detailed lower and upper bounds for Tsallis entropy and Tsallis relative entropy. As applications we give bounds for Tsallis relative operator entropy in the case of operators satisfying the condition $\ell A \leq B \leq LA$, with $\ell < L$.

1. Introduction

A function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $[a,b]$ if the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.1}$$

holds for all $x, y \in [a,b]$. If inequality (1.1) reverses, then f is said to be concave on $[a,b]$. Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $[a.b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex functions. It has many applications in more different areas of pure and applied mathematics. Recently we obtained the following two refined Hermite-Hadamard inequalities.

Theorem 1.1 ([12]). *Let $f(x)$ be a convex function on $[a,b]$. Then for any $m, n \in \mathbb{N} \cup \{0\}$*

$$L^{(1)}_{f,n}(a,b) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq L^{(2)}_{f,m}(a,b),$$

where $h_n = \frac{b-a}{2^n}$,

$$L^{(1)}_{f,n}(a,b) = \frac{1}{2^n}\sum_{k=1}^{2^n} f(a+(2k-1)h_{n+1})$$

and

$$L^{(2)}_{f,m}(a,b) = \frac{1}{2^{m+1}}\left\{f(a)+f(b)+2\sum_{k=1}^{2^m-1} f(a+kh_m)\right\}.$$

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Theorem 1.2 ([12])**.** *Let $f(x)$ be a convex function on $[a,b]$. Then for any $v\in[0,1]$ and $m,n\in\mathbb{N}\cup\{0\}$,*

$$r^{(1)}_{f,v,n}(a,b)\leq\frac{1}{b-a}\int_a^b f(t)dt\leq r^{(2)}_{f,v,m}(a,b),$$

where $h_n=\frac{b-a}{2^n}$,

$$\begin{aligned} & r^{(1)}_{f,v,n}(a,b)\\ = & \frac{1}{2^n}\sum_{k=1}^{2^n}\{vf(a+(2k-1)vh_{n+1})\\ & +(1-v)f((1-v)a+vb+(2k-1)(1-v)h_{n+1})\}\end{aligned}$$

and

$$\begin{aligned} & r^{(2)}_{f,v,m}(a,b)\\ = & \frac{1}{2^{m+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\}\\ & +\frac{1}{2^m}\sum_{k=1}^{2^m-1}\{vf(a+kvh_m)+(1-v)f((1-v)a+vb+k(1-v)h_m)\}.\end{aligned}$$

For probability distributions $Q=(q_1,q_2,\dots,q_N)$ and $P=(p_1,p_2,\dots,p_N)$, Shannon entropy $S(P)$ and relative entropy $S(Q|P)$ are defined by

$$S(P)=-\sum_{i=1}^N p_i\log p_i,\ S(Q|P)=\sum_{i=1}^N q_i\log\frac{q_i}{p_i},$$

with convention $0\log 0=0$ and $q_i=0$ whenever $p_i=0$ for some i. C.Tsallis introduced the Tsallis entropy, which is one-parameter extension of Shannon entropy, for the analysis of statistical physics. We use the definitions of the Tsallis entropy and the Tsallis relative entropy as follows:

$$T_t(Q)=-\sum_{i=1}^N q_i^{1-t}\ln_t q_i,\ T_t(Q|P)=\sum_{i=1}^N q_i^{1-t}(\ln_t q_i-\ln_t p_i),$$

where $\ln_t$ is called r-logarithmic function and defined by $\ln_t(x)=\frac{x^t-1}{t}$ with the parameter $t\in\mathbb{R}$.They recover the usual entropy and relative entropy in the limit $t\to 0$, namely

$$\lim_{t\to 0}T_t(Q)=S(Q),\ \lim_{t\to 0}T_t(Q|P)=S(Q|P),$$

since r-logarithmic function $\ln_t$ umiformly converges to the usual logarithmic function log in the limit $t\to 0$. Analogically, quantum mechanical entropy and relative entropy and their one-parameter extended quantities were defined by

$$S(\rho)=-Tr[\rho\log\rho],\ S(\rho|\sigma)=Tr[\rho(\log\rho-\log\sigma)]$$

and

$$T_t(\rho)=-Tr[\rho^{1-t}\ln_t\rho],\ T_t(\rho|\sigma)=Tr[\rho^{1-t}(\ln_t\rho-\ln_t\sigma)]$$

for density operators ρ and σ. As J.I.Fujii and E.Kamei introduced the relative operator entropy by

$$S(X|Y) = X^{1/2}\log(X^{-1/2}YX^{-1/2})X^{1/2}$$

for $X, Y > 0$, the Tsallis relative operator entropy $T_t(X|Y)$ for $t \in \mathbb{R}$ with $t \neq 0$ was defined

$$T_t(X|Y) = X^{1/2}\ln_t(X^{-1/2}YX^{-1/2})X^{1/2}.$$

We remark that $\lim_{t\to 0} T_t(X|Y) = S(X|Y)$.

2. Bounds for tsallis entropy and their applications

Let $f(x) = x^{t-1}$. Then $r^{(1)}_{f,v,n}(x,1)$ and $r^{(2)}_{f,v,m}(x,1)$ are represented as follows:

$$\begin{aligned}
& r^{(1)}_{f,v,n}(x,1) \\
= \ & \frac{1}{2^n}\sum_{k=1}^{2^n}\left\{ v\left(\left(1-\frac{(2k-1)v}{2^{n+1}}\right)x + \frac{(2k-1)v}{2^{n+1}}\right)^{t-1}\right. \\
& \left. +(1-v)\left(\left(1-v-\frac{(2k-1)(1-v)}{2^{n+1}}\right)x + \left(v+\frac{(2k-1)(1-v)}{2^{n+1}}\right)\right)^{t-1}\right\}
\end{aligned}$$

and

$$\begin{aligned}
& r^{(2)}_{f,v,m}(x,1) \\
= \ & \frac{1}{2^{m+1}}\{vx^{t-1} + (1-v) + ((1-v)x+1)^{t-1}\} \\
& +\frac{1}{2^m}\sum_{k=1}^{2^m-1}\left\{ v\left(\left(1-\frac{kv}{2^m}\right)x + \frac{kv}{2^m}\right)^{t-1}\right. \\
& \left. +(1-v)\left(\left(1-v-\frac{k(1-v)}{2^m}\right)x + \left(v+\frac{k(1-v)}{2^m}\right)\right)^{t-1}\right\}.
\end{aligned}$$

Then we have the following theorem.

Theorem 2.1. *Let $Q = (q_1, q_2, \ldots, q_N)$ be a probability distribution. Then for any $v \in [0,1]$ and $m, n \in \mathbb{N}\cup\{0\}$, if $t < 1$ or $t > 2$, then*

$$r^{(1)}_{f,v,n}(q_i,1)q_i^{1-t}(1-q_i) \le T_t(Q) \le r^{(2)}_{f,v,m}(q_i,1)q_i^{1-t}(1-q_i), \tag{2.1}$$

and if $1 < t < 2$, then

$$r^{(2)}_{f,v,m}(q_i,1)q_i^{1-t}(1-q_i) \le T_t(Q) \le r^{(1)}_{f,v,n}(q_i,1)q_i^{1-t}(1-q_i). \tag{2.2}$$

Proof. When $t < 1$ or $t > 2$, $f(x) = x^{t-1}$ is a convex function on $x > 0$. By Theorem 1.2, we get

$$r^{(1)}_{f,v,n}(x,1) \le \frac{1-x^t}{t(1-x)} = \frac{x^t-1}{t(x-1)} \le r^{(2)}_{f,v,m}(x,1). \tag{2.3}$$

When $1 < t < 2$, $f(x) = x^{t-1}$ is a concave function on $x > 0$. By Theorem 1.2, we get

$$r^{(2)}_{f,v,m}(x,1) \le \frac{1-x^t}{t(1-x)} = \frac{x^t-1}{t(x-1)} \le r^{(1)}_{f,v,n}(x,1). \tag{2.4}$$

When $t < 1$ or $t > 2$, we put $x = q_i (< 1)$. By (2.3), we have

$$r^{(1)}_{f,v,n}(q_i,1) \le \frac{1}{q_i-1}\ln_t q_i \le r^{(2)}_{f,v,m}(q_i,1).$$

Then $(q_i-1)r^{(2)}_{f,v,m}(q_i,1) \le \ln_t q_i \le (q_i-1)r^{(1)}_{f,v,n}(q_i,1)$. And also we have

$$q_i^{1-t}(1-q_i)r^{(1)}_{f,v,n}(q_i,1) \le -q_i^{1-t}\ln_t q_i \le q_i^{1-t}(1-q_i)r^{(2)}_{f,v,m}(q_i,1).$$

Therefore

$$\sum_{i=1}^N q_i^{1-t}(1-q_i)r^{(1)}_{f,v,n}(q_i,1) \le T_t(Q) \le \sum_{i=1}^N q_i^{1-t}(1-q_i)r^{(2)}_{f,v,m}(q_i,1).$$

Similarly when $1 < t < 2$, we put $x = q_i (< 1)$. By (2.4), we have

$$r^{(2)}_{f,v,m}(q_i,1) \le \frac{1}{q_i-1}\ln_t q_i \le r^{(1)}_{f,v,n}(q_i,1).$$

Then $(q_i-1)r^{(1)}_{f,v,n}(q_i,1) \le \ln_t q_i \le (q_i-1)r^{(2)}_{f,v,m}(q_i,1)$. And also we have

$$q_i^{1-t}(1-q_i)r^{(2)}_{f,v,m}(q_i,1) \le -q_i^{1-t}\ln_t q_i \le q_i^{1-t}(1-q_i)r^{(1)}_{f,v,n}(q_i,1).$$

Therefore

$$\sum_{i=1}^N q_i^{1-t}(1-q_i)r^{(2)}_{f,v,m}(q_i,1) \le T_t(Q) \le \sum_{i=1}^N q_i^{1-t}(1-q_i)r^{(1)}_{f,v,n}(q_i,1).$$

□

By putting $t = 0$ in (2.1), we have the following corollary.

Corollary 2.1. *Let $Q = (q_1, q_2, \ldots, q_N)$ be a probability distribution. Then for any $v \in [0,1]$ and $m, n \in \mathbb{N} \cup \{0\}$,*

$$\sum_{i=1}^N q_i(1-q_i)r^{(1)}_{f,v,n}(q_i,1) \le S(Q) \le \sum_{i=1}^N q_i(1-q_i)r^{(2)}_{f,v,m}(q_i,1), \tag{2.5}$$

where

$$\begin{aligned} & r^{(1)}_{f,v,n}(q_i,1) \\ = \; & 2\sum_{k=1}^{2^n}\left\{\frac{v}{(2^{n+1}-(2k-1)v)q_i+(2k-1)v}\right. \\ & \left. + \frac{1-v}{2^{n+1}\{(1-v)q_i+v\}+(2k-1)(1-v)(1-q_i)}\right\} \end{aligned}$$

and

$$r^{(2)}_{f,v,m}(q_i,1)$$

$$= \frac{1}{2^{m+1}}\left\{\frac{v}{q_i}+1-v+\frac{1}{(1-v)q_i+v}\right\}$$
$$+\sum_{k=1}^{2^m-1}\left\{\frac{v}{(2^m-kv)q_i+kv}+\frac{1-v}{2^m\{(1-v)q_i+v\}+k(1-v)(1-q_i)}\right\}.$$

We put $v=0$ or $v=1$ in (2.1) and (2.2). Since $r^{(1)}_{f,0,n}(q_i,1)=r^{(1)}_{f,1,n}(q_i,1)=L^{(1)}_{f,n}(q_i,1)$ and $r^{(2)}_{f,0,m}(q_i,1)=r^{(2)}_{f,1,m}(q_i,1)=L^{(2)}_{f,m}(q_i,1)$, we have the following corollary.

Corollary 2.2. *Let $Q=(q_1,q_2,\dots,q_N)$ be a probability distribution. Then for any $m,n\in\mathbb{N}\cup\{0\}$, if $t<1$ or $t>2$, then*

$$\sum_{i=1}^{N}L^{(1)}_{f,n}(q_i,1)q_i^{1-t}(1-q_i)\le T_t(Q)\le\sum_{i=1}^{N}L^{(2)}_{f,m}(q_i,1)q_i^{1-t}(1-q_i),$$

and if $1<t<2$, then

$$\sum_{i=1}^{N}L^{(2)}_{f,m}(q_i,1)q_i^{1-t}(1-q_i)\le T_t(Q)\le\sum_{i=1}^{N}L^{(1)}_{f,n}(q_i,1)q_i^{1-t}(1-q_i),$$

where

$$L^{(1)}_{f,n}(q_i,1)=\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{\Big(1-\frac{2k-1}{2^{n+1}}\Big)q_i+\frac{2k-1}{2^{n+1}}\right\}^{t-1},$$

and

$$L^{(2)}_{f,m}(q_i,1)=\frac{1}{2^{m+1}}\left\{1+(q_i+1)^{t-1}\right\}+\frac{1}{2^m}\sum_{k=1}^{2^m-1}\left\{\Big(1-\frac{k}{2^m}\Big)q_i+\frac{k}{2^m}\right\}^{t-1}.$$

By putting $v=0$ or $v=1$ in (2.5), we have the following corollary.

Corollary 2.3. *Let $Q=(q_1,q_2,\dots,q_N)$ be a probability distribution. Then for any $m,n\in\mathbb{N}\cup\{0\}$,*

$$\sum_{i=1}^{N}q_i(1-q_i)L^{(1)}_{f,n}(q_i,1)\le S(Q)\le\sum_{i=1}^{N}q_i(1-q_i)L^{(2)}_{f,m}(q_i,1),\tag{2.6}$$

where

$$L^{(1)}_{f,n}(q_i,1)=2\sum_{k=1}^{2^n}\frac{1}{(2^{n+1}-(2k-1))q_i+2k-1}$$

and

$$L^{(2)}_{f,m}(q_i,1)=\frac{1}{2^{m+1}}\left(\frac{1}{q_i}+1\right)+\sum_{k=1}^{2^m-1}\frac{1}{(2^m-k)q_i+k}.$$

We put $n = m = 0$ in (2.6), Since

$$L^{(1)}_{f,0}(q_i,1) = \frac{2}{1+q_i},\ L^{(2)}_{f,0}(q_i,1) = \frac{1}{2}\left(\frac{1}{q_i}+1\right),$$

we have the well known result.

Corollary 2.4. *Let $Q = (q_1, q_2, \dots, q_N)$ be a probability distribution. Then*

$$\sum_{i=1}^{N} \frac{2q_i(1-q_i)}{1+q_i} \le S(Q) \le \sum_{i=1}^{N} \frac{1-q_i^2}{2}.$$

3. Bounds for Tsallis Relative Entropy and their Applications

Let $f(x) = x^{t-1}$. Then $r^{(1)}_{f,v,n}(a,b)$ and $r^{(2)}_{f,v,m}(a,b)$ are represented as follows:

$$\begin{aligned}
&r^{(1)}_{f,v,n}(a,b)\\
= &\frac{1}{2^n}\sum_{k=1}^{2^n}\Bigg\{v\left(\left(1-\frac{(2k-1)v}{2^{n+1}}\right)a+\frac{(2k-1)v}{2^{n+1}}b\right)^{t-1}\\
&+(1-v)\left(\left(1-v-\frac{(2k-1)(1-v)}{2^{n+1}}\right)a+\left(v+\frac{(2k-1)(1-v)}{2^{n+1}}\right)b\right)^{t-1}\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
&r^{(2)}_{f,v,m}(a,b)\\
= &\frac{1}{2^{m+1}}\{va^{t-1}+(1-v)b^{t-1}+((1-v)a+vb)^{t-1}\}\\
&+\frac{1}{2^m}\sum_{k=1}^{2^m-1}\Bigg\{v\left(\left(1-\frac{kv}{2^m}\right)a+\frac{kv}{2^m}b\right)^{t-1}\\
&+(1-v)\left(\left(1-v-\frac{k(1-v)}{2^m}\right)a+\left(v+\frac{k(1-v)}{2^m}\right)b\right)^{t-1}\Bigg\}.
\end{aligned}$$

Then we have the following theorem.

Theorem 3.1. *Let $Q = (q_1, q_2, \dots, q_N)$ and $P = (p_1, p_2, \dots, p_N)$ are probability distributions. Let $I = \{1 \le i \le N : p_i < q_i\}$ and $J = \{1 \le i \le N : p_i > q_i\}$. Then for any $v \in [0,1]$ and $m, n \in \mathbb{N} \cup \{0\}$, if $t < 1$ or $t > 2$, then*

$$\begin{aligned}
&\sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i) + \sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i)\\
\le\ &T_t(Q|P)\\
\le\ &\sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i) + \sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i),
\end{aligned} \tag{3.1}$$

and if $1 < t < 2$, then

$$\sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i) + \sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i)$$

$$\begin{aligned} &\le T_t(Q|P) \\ &\le \sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_{)} + \sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i). \end{aligned} \tag{3.2}$$

Proof. When $t<1$ or $t>2$, $f(x)=x^{t-1}$ is a convex function on $x>0$. By Theorem 1.2, we get

$$r^{(1)}_{f,v,n}(a,b) \le \frac{1}{b-a}\frac{b^t-a^t}{t} \le r^{(2)}_{f,v,m}(a,b). \tag{3.3}$$

Then we put $a=p_i$ and $b=q_i$ for $i\in I=\{1\le i\le N : p_i<q_i\}$. By (3.3), we have

$$\begin{aligned} &\sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i) \\ \le\ &\sum_{i\in I} q_i^{1-t}(\ln_t q_i-\ln_t p_i) \le \sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i). \end{aligned} \tag{3.4}$$

And we put $a=p_i$ and $b=q_i$ for $i\in J=\{1\le i\le N : p_i>q_i\}$. By (3.3), we have

$$\begin{aligned} &\sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i) \\ \le\ &\sum_{i\in J} q_i^{1-t}(\ln_t q_i-\ln_t p_i) \le \sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i). \end{aligned} \tag{3.5}$$

Hence by combining (3.4) and (3.5), we obtain (3.1).

Similarly when $1<t<2$, $f(x)=x^{t-1}$ is a concave function on $x>0$. By Theorem 1.2, we get

$$r^{(2)}_{f,v,m}(a,b) \le \frac{1}{b-a}\frac{b^t-a^t}{t} \le r^{(1)}_{f,v,n}(a,b). \tag{3.6}$$

Then we put $a=p_i$ and $b=q_i$ for $i\in I=\{1\le i\le N : p_i<q_i\}$. By (3.6), we have

$$\begin{aligned} &\sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i) \\ \le\ &\sum_{i\in I} q_i^{1-t}(\ln_t q_i-\ln_t p_i) \le \sum_{i\in I} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i). \end{aligned} \tag{3.7}$$

And we put $a=p_i$ and $b=q_i$ for $i\in J=\{1\le i\le N : p_i>q_i\}$. By Theorem 1.2, we get

$$\begin{aligned} &\sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(1)}_{f,v,n}(p_i,q_i) \\ \le\ &\sum_{i\in J} q_i^{1-t}(\ln_t q_i-\ln_t p_i) \le \sum_{i\in J} q_i^{1-t}(q_i-p_i)r^{(2)}_{f,v,m}(p_i,q_i). \end{aligned} \tag{3.8}$$

Hence by combining (3.7) and (3.8), we obtain (3.2). □

When $t=0$ in (3.1), we have the following corollary.

Corollary 3.1. *Ley $Q = (q_1, q_2, \ldots, q_N)$ and $P = (p_1, p_2, \ldots, p_N)$ are probability distributions. Then for any $v \in [0,1]$ and $m, n \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned}
&\sum_{i \in I} q_i(q_i - p_i) r^{(1)}_{f,v,n}(p_i, q_i) + \sum_{i \in J} q_i(q_i - p_i) r^{(2)}_{f,v,m}(p_i, q_i) \\
\leq\ & S(Q|P) \qquad (3.9) \\
\leq\ & \sum_{i \in I} q_i(q_i - p_i) r^{(2)}_{f,v,m}(p_i, q_i) + \sum_{i \in J} q_i(q_i - p_i) r^{(1)}_{f,v,n}(p_i, q_i),
\end{aligned}$$

where

$$\begin{aligned}
& r^{(1)}_{f,v,n}(p_i, q_i) \\
=\ & 2 \sum_{k=1}^{2^n} \left\{ \frac{v}{(2^{n+1} - (2k-1)v)p_i + (2k-1)vq_i} \right\} \\
& +2 \sum_{k=1}^{2^n} \left\{ \frac{1-v}{2^{n+1}\{(1-v)p_i + vq_i\} + (2k-1)(1-v)(q_i - p_i)} \right\}
\end{aligned}$$

and

$$\begin{aligned}
& r^{(2)}_{f,v,m}(p_i, q_i) \\
=\ & \frac{1}{2^{m+1}} \left\{ \frac{v}{p_i} + \frac{1-v}{q_i} + \frac{1}{(1-v)p_i + vq_i} \right\} \\
& + \sum_{k=1}^{2^m - 1} \left\{ \frac{v}{(2^m - kv)p_i + kvq_i} + \frac{1-v}{2^m\{(1-v)p_i + vq_i\} + k(1-v)(q_i - p_i)} \right\}.
\end{aligned}$$

We put $v = 0$ or $v = 1$ in (3.1) and (3.2). Since $r^{(1)}_{f,0,n}(p_i, q_i) = r^{(1)}_{f,1,n}(p_i, q_i) = L^{(1)}_{f,n}(p_i, q_i)$ and $r^{(2)}_{f,0,m}(p_i, q_i) = r^{(2)}_{f,1,m}(p_i, q_i) = L^{(2)}_{f,m}(p_i, q_i)$, we have the following corollary.

Corollary 3.2. *Ley $Q = (q_1, q_2, \ldots, q_N)$ and $P = (p_1, p_2, \ldots, p_N)$ are probability distributions. Then for any $m, n \in \mathbb{N} \cup \{0\}$, if $t < 1$ or $t > 2$, then*

$$\begin{aligned}
&\sum_{i \in I} q_i^{1-t}(q_i - p_i) L^{(1)}_{f,n}(p_i, q_i) + \sum_{i \in J} q_i^{1-t}(q_i - p_i) L^{(2)}_{f,m}(p_i, q_i) \\
\leq\ & T_t(Q|P) \\
\leq\ & \sum_{i \in I} q_i^{1-t}(q_i - p_i) L^{(2)}_{f,m}(p_i, q_i) + \sum_{i \in J} q_i^{1-t}(q_i - p_i) L^{(1)}_{f,n}(p_i, q_i),
\end{aligned}$$

and if $1 < t < 2$, then

$$\begin{aligned}
&\sum_{i \in I} q_i^{1-t}(q_i - p_i) L^{(2)}_{f,m}(p_i, q_i) + \sum_{i \in J} q_i^{1-t}(q_i - p_i) L^{(1)}_{f,n}(p_i, q_i) \\
\leq\ & T_t(Q|P) \\
\leq\ & \sum_{i \in I} q_i^{1-t}(q_i - p_i) L^{(1)}_{f,n}(p_i, q_i) + \sum_{i \in J} q_i^{1-t}(q_i - p_i) L^{(2)}_{f,m}(p_i, q_i),
\end{aligned}$$

where

$$L_{f,n}^{(1)}(p_i,q_i)=\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{\Big(1-\frac{2k-1}{2^{n+1}}\Big)p_i+\frac{2k-1}{2^{n+1}}q_i\right\}^{t-1}$$

and

$$L_{f,m}^{(2)}(p_i,q_i)=\frac{1}{2^{m+1}}(p_i^{t-1}+q_i^{t-1})+\frac{1}{2^m}\sum_{k=1}^{2^m-1}\left\{\Big(1-\frac{k}{2^m}\Big)p_i+\frac{k}{2^m}q_i\right\}^{t-1}.$$

We put $v=0$ or $v=1$ in (3.9), we have the following corollary.

Corollary 3.3. *Let $Q=(q_1,q_2,\dots,q_N)$ and $P=(p_1,p_2,\dots,p_N)$ are probability distributions. Then for any $m,n\in\mathbb{N}\cup\{0\}$,*

$$\begin{aligned}&\sum_{i\in I}q_i(q_i-p_i)L_{f,n}^{(1)}(p_i,q_i)+\sum_{i\in J}q_i(q_i-p_i)L_{f,m}^{(2)}(p_i,q_i)\\ \le\ &S(Q|P)\\ \le\ &\sum_{i\in I}q_i(q_i-p_i)L_{f,m}^{(2)}(p_i,q_i)+\sum_{i\in J}q_i(q_i-p_i)L_{f,n}^{(1)}(p_i,q_i),\end{aligned}\tag{3.10}$$

where

$$L_{f,n}^{(1)}(p_i,q_i)=\sum_{k=1}^{2^n}\frac{2}{(2^{n+1}-(2k-1))p_i+(2k-1)q_i}$$

and

$$L_{f,m}^{(2)}(p_i,q_i)=\frac{1}{2^{m+1}}\left(\frac{1}{p_i}+\frac{1}{q_i}\right)+\sum_{k=1}^{2^m-1}\frac{1}{(2^m-k)p_i+kq_i}.$$

We put $n=m=0$ in (3.10). Since

$$L_{f,0}^{(1)}(p_i,q_i)=\frac{2}{p_i+q_i},\ L_{f,0}^{(2)}(p_i,q_i)=\frac{1}{2}\left(\frac{1}{p_i}+\frac{1}{q_i}\right)=\frac{p_i+q_i}{2p_iq_i},$$

we have the well known result.

Corollary 3.4. *Let $Q=(q_1,q_2,\dots,q_N)$ and $P=(p_1,p_2,\dots,p_N)$ are probability distributions. Then*

$$\sum_{i\in I}\frac{2q_i(q_i-p_i)}{p_i+q_i}+\sum_{i\in J}\frac{q_i^2-p_i^2}{2p_i}\le S(Q|P)\le\sum_{i\in I}\frac{q_i^2-p_i^2}{2p_i}+\sum_{i\in J}\frac{2q_i(q_i-p_i)}{p_i+q_i}.$$

We give a remark on Pinsker inequality and its related result. The Pinsker inequality is well known as the following inequality.

$$\frac{1}{2}\|Q-P\|_1^2\le S(Q|P),\tag{3.11}$$

where

$$\|Q-P\|_1=\sum_{i\in I}(q_i-p_i)+\sum_{i\in J}(p_i-q_i).$$

We can give an example where (3.10) is a stronger lower bound of $S(Q|P)$ than (3.11). Let $Q = (\frac{3}{4}, \frac{1}{4})$ and $P = (\frac{1}{2}, \frac{1}{2})$ and we put $n = m = 1$ in (3.10). Since

$$L_{f,1}^{(1)}(p_i, q_i) = \frac{2}{3p_i + q_i} + \frac{2}{p_i + 3q_i} = \frac{8(p_i + q_i)}{(3p_i + q_i)(p_i + 3q_i)}$$

and

$$L_{f,1}^{(2)}(p_i, q_i) = \frac{1}{4}\left(\frac{1}{p_i} + \frac{1}{q_i}\right) + \frac{1}{p_i + q_i},$$

we have

$$\sum_{i \in I} q_i(q_i - p_i) L_{f,1}^{(1)}(p_i, q_i) + \sum_{i \in J} q_i(q_i - p_i) L_{f,1}^{(2)}(p_i, q_i) = 0.12594 \cdots$$

and

$$\frac{1}{2}\|Q - P\|_1^2 = 0.125, \ S(Q|P) = 0.13084 \cdots .$$

Then

$$\frac{1}{2}\|Q - P\|_1^2 < \sum_{i \in I} q_i(q_i - p_i) L_{f,1}^{(1)}(p_i, q_i) + \sum_{i \in J} q_i(q_i - p_i) L_{f,1}^{(2)}(p_i, q_i) < S(Q|P).$$

4. Bounds for tsallis relative operator entropy and their applications

When $t < 1$ or $t > 2$, $f(x) = x^{t-1}$ is a convex function on $x > 0$. By Theorem 1.1, we get

$$L_{f,n}^{(1)}(a, b) \le \frac{1}{b - a} \frac{b^t - a^t}{t} \le L_{f,m}^{(2)}(a, b), \tag{4.1}$$

where

$$L_{f,n}^{(1)}(a, b) = \frac{1}{2^n} \sum_{k=1}^{2^n} \left\{ \left(1 - \frac{2k-1}{2^{n+1}}\right) a + \frac{2k-1}{2^{n+1}} b \right\}^{t-1}$$

and

$$L_{f,m}^{(2)}(a, b) = \frac{1}{2^{m+1}} \left\{ a^{t-1} + b^{t-1} + 2 \sum_{k=1}^{2^m - 1} \left(\left(1 - \frac{k}{2^m}\right) a + \frac{k}{2^m} b \right)^{t-1} \right\}.$$

When $0 < a < b$, we have the following inequality.

$$(b - a) L_{f,n}^{(1)}(a, b) \le \frac{b^t - a^t}{t} \le (b - a) L_{f,m}^{(2)}(a, b). \tag{4.2}$$

We put $a = 1$ and $b = y$ in (4.2). Then

$$(y - 1) L_{f,n}^{(1)}(1, y) \le \frac{y^t - 1}{t} \le (y - 1) L_{f,m}^{(2)}(1, y). \tag{4.3}$$

Next we put $a = y$ and $b = 1$ in (4.2). Then

$$(1 - y) L_{f,n}^{(1)}(y, 1) \le \frac{1 - y^t}{t} \le (1 - y) L_{f,m}^{(2)}(y, 1). \tag{4.4}$$

On the other hand when $0 < b < a$, we have the following inequality.

$$(b-a)L^{(2)}_{f,m}(a,b) \le \frac{b^t - a^t}{t} \le (b-a)L^{(1)}_{f,n}(a,b). \tag{4.5}$$

We put $a = 1$ and $b = y$ in (4.5). Then

$$(y-1)L^{(2)}_{f,m}(1,y) \le \frac{y^t - 1}{t} \le (y-1)L^{(1)}_{f,n}(a,b).$$

That is

$$(1-y)L^{(1)}_{f,n}(a,b) \le \frac{1-y^t}{t} \le (1-y)L^{(2)}_{f,m}(1,y). \tag{4.6}$$

Next we put $a = y$ and $b = 1$ in (4.5). Then

$$(1-y)L^{(2)}_{f,m}(y,1) \le \frac{1-y^t}{t} \le (1-y)L^{(1)}_{f,n}(y,1).$$

That is

$$(y-1)L^{(1)}_{f,n}(y,1) \le \frac{y^t-1}{t} \le (y-1)L^{(2)}_{f,m}(y,1). \tag{4.7}$$

We remark that $L^{(1)}_{f,n}(1,y) = L^{(1)}_{f,n}(y,1)$ and $L^{(2)}_{f,m}(1,y) = L^{(2)}_{f,m}(y,1)$. Then we obtain the following inequality by (4.3) and (4.7) for $y > 1$.

$$(y-1)L^{(1)}_{f,n}(1,y) \le \frac{y^t-1}{t} \le (y-1)L^{(2)}_{f,m}(1,y). \tag{4.8}$$

And also we obtain the following inequality by (4.4) and (4.6) for $y < 1$.

$$(1-y)L^{(1)}_{f,n}(1,y) \le \frac{1-y^t}{t} \le (1-y)L^{(2)}_{f,m}(1,y). \tag{4.9}$$

Similarly when $1 < t < 2$, $f(x) = x^{t-1}$ is a concave function on $x > 0$. We obtain the following two inequalities by the same as above method. For $y > 1$

$$(y-1)L^{(2)}_{f,m}(1,y) \le \frac{y^t-1}{t} \le (y-1)L^{(1)}_{f,n}(1,y) \tag{4.10}$$

and for $y < 1$

$$(1-y)L^{(2)}_{f,m}(1,y) \le \frac{1-y^t}{t} \le (1-y)L^{(1)}_{f,n}(1,y). \tag{4.11}$$

We need the following lemma in order to obtain the bounds of Tsallis relative operator entropy.

Lemma 4.1. *Let A, B be two positive invertible operators and the constants $0 < \ell < L$ with the property that $\ell A \le B \le LA$. When $t < 1$ or $t > 2$, we have the following inequalities for any $m, n \in \mathbb{N} \cup \{0\}$. If $\ell > 1$, then*

$$(B-A)\max_{\ell \le y \le L} L^{(1)}_{f,n}(1,y) \le T_t(A|B) \le (B-A)\min_{\ell \le y \le L} L^{(2)}_{f,m}(1,y).$$

If $L < 1$, then

$$(B-A)\min_{\ell \le y \le L} L^{(2)}_{f,m}(1,y) \le T_t(A|B) \le (B-A)\max_{\ell \le y \le L} L^{(1)}_{f,n}(1,y).$$

When $1 < t < 2$*, we have the following inequalities for any* $m, n \in \mathbb{N} \cup \{0\}$*. If* $\ell > 1$*, then*

$$(B-A) \max_{\ell \le y \le L} L^{(2)}_{f,m}(1,y) \le T_t(A|B) \le (B-A) \min_{\ell \le y \le L} L^{(1)}_{f,n}(1,y).$$

If $L < 1$*, then*

$$(B-A) \min_{\ell \le y \le L} L^{(1)}_{f,n}(1,y) \le T_t(A|B) \le (B-A) \max_{\ell \le y \le L} L^{(2)}_{f,m}(1,y).$$

Proof. Let $t < 1$ or $t > 2$. When $\ell > 1$, since $1 < \ell \le y \le L$, we have

$$(y-1) \max_{\ell \le y \le L} L^{(1)}_{f,n}(1,y) \le \frac{y^t - 1}{t} \le (y-1) \min_{\ell \le y \le L} L^{(2)}_{f,m}(1,y). \tag{4.12}$$

When $L < 1$, since $\ell \le y \le L < 1$, we have

$$(1-y) \max_{\ell \le y \le L} L^{(1)}_{f,n}(1,y) \le \frac{1 - y^t}{t} \le (1-y) \min_{\ell \le y \le L} L^{(2)}_{f,m}(1,y).$$

That is

$$(y-1) \min_{\ell \le y \le L} L^{(2)}_{f,m}(1,y) \le \frac{y^t - 1}{t} \le (y-1) \max_{\ell \le y \le L} L^{(1)}_{f,n}(1,y). \tag{4.13}$$

Let $1 < t < 2$. When $\ell > 1$, since $1 < \ell \le y \le L$, we have

$$(y-1) \max_{\ell \le y \le L} L^{(2)}_{f,m}(1,y) \le \frac{y^t - 1}{t} \le (y-1) \min_{\ell \le y \le L} L^{(1)}_{f,n}(1,y). \tag{4.14}$$

When $L < 1$, since $\ell \le y \le L < 1$, we have

$$(1-y) \max_{\ell \le y \le L} L^{(2)}_{f,m}(1,y) \le \frac{1 - y^t}{t} \le (1-y) \min_{\ell \le y \le L} L^{(1)}_{f,n}(1,y).$$

That is

$$(y-1) \min_{\ell \le y \le L} L^{(1)}_{f,n}(1,y) \le \frac{y^t - 1}{t} \le (y-1) \max_{\ell \le y \le L} L^{(2)}_{f,m}(1,y). \tag{4.15}$$

Since $\ell A \le B \le LA$ and A is invertible, then by multiplying both sides with $A^{-1/2}$ we get $\ell 1_H \le A^{-1/2} B A^{-1/2} \le L 1_H$. Denote $X = A^{-1/2} B A^{-1/2}$ and by using the functional calculus for X that has its spectrum contained in the interval $[\ell, L]$ and the inequalities (4.12), (4.13), (4.14) and (4.15), we get the results. □

Theorem 4.1. *Let* A, B *be two positive invertible operators and the constants* $0 < \ell < L$ *with the property that* $\ell A \le B \le LA$*. When* $t < 1$*, we have the following inequalities for any* $m, n \in \mathbb{N} \cup \{0\}$*. If* $\ell > 1$*, then*

$$(B-A) L^{(1)}_{f,n}(1,\ell) \le T_t(A|B) \le (B-A) L^{(2)}_{f,m}(1,L).$$

If $L < 1$*, then*

$$(B-A) L^{(2)}_{f,m}(1,L) \le T_t(A|B) \le (B-A) L^{(1)}_{f,n}(1,\ell).$$

When $t > 1$*, we have the following inequalities for any* $m, n \in \mathbb{N} \cup \{0\}$*. If* $\ell > 1$*, then*

$$(B-A) L^{(2)}_{f,m}(1,L) \le T_t(A|B) \le (B-A) L^{(1)}_{f,n}(1,\ell).$$

If $L < 1$, then

$$(B-A)L^{(1)}_{f,n}(1,\ell) \le T_t(A|B) \le (B-A)L^{(2)}_{f,m}(1,L).$$

Proof. If $t < 1$, then

$$\max_{\ell\le y\le L} L^{(1)}_{f,n}(1,y) = L^{(1)}_{f,n}(1,\ell), \quad \min_{\ell\le y\le L} L^{(1)}_{f,n}(1,y) = L^{(1)}_{f,n}(1,L),$$

and

$$\max_{\ell\le y\le L} L^{(2)}_{f,m}(1,y) = L^{(2)}_{f,m}(1,\ell), \quad \min_{\ell\le y\le L} L^{(2)}_{f,m}(1,y) = L^{(2)}_{f,m}(1,L).$$

If $1 < t < 2$ or $t > 2$,then

$$\max_{\ell\le y\le L} L^{(1)}_{f,n}(1,y) = L^{(1)}_{f,n}(1,L), \quad \min_{\ell\le y\le L} L^{(1)}_{f,n}(1,y) = L^{(1)}_{f,n}(1,\ell),$$

and

$$\max_{\ell\le y\le L} L^{(2)}_{f,m}(1,y) = L^{(2)}_{f,m}(1,L), \quad \min_{\ell\le y\le L} L^{(2)}_{f,m}(1,y) = L^{(2)}_{f,m}(1,\ell).$$

Since it is trivial for $t = 2$, we have the results by Lemma 4.1. □

Let $m = n = 0$ in Theorem 4.1. Then we have the following corollary.

Corollary 4.1. *Let A, B be two positive invertible operators and the constants $0 < \ell < L$ with the property that $\ell A \le B \le LA$. When $t < 1$, we have the following inequalities. If $\ell > 1$, i.e. $A \le B$, then*

$$(B-A)\left(\frac{1+\ell}{2}\right)^{t-1} \le T_t(A|B) \le (B-A)\left(\frac{1+L^{t-1}}{2}\right).$$

If $L < 1$, i.e. $B \le A$, then

$$(B-A)\left(\frac{1+L^{t-1}}{2}\right) \le T_t(A|B) \le (B-A)\left(\frac{1+\ell}{2}\right)^{t-1}.$$

When $t > 1$, we have the following inequalities. If $\ell > 1$, i.e. $A \le B$, then

$$(B-A)\left(\frac{1+L^{t-1}}{2}\right) \le T_t(A|B) \le (B-A)\left(\frac{1+\ell}{2}\right)^{t-1}.$$

If $L < 1$, i.e. $B \le A$, then

$$(B-A)\left(\frac{1+\ell}{2}\right)^{t-1} \le T_t(A|B) \le (B-A)\left(\frac{1+L^{t-1}}{2}\right).$$

Proof. By $m = n = 0$, we have

$$L^{(1)}_{f,0}(1,\ell) = \left(\frac{1+\ell}{2}\right)^{t-1}, \; L^{(2)}_{f,0}(1,L) = \frac{1+L^{t-1}}{2}.$$

Then we have the result. □

Let $f(x) = \frac{b}{x} - 1$. Then $L^{(1)}_{f,n}(a,b)$ and $L^{(2)}_{f,m}(a,b)$ are represented as follows:

$$L^{(1)}_{f,n}(a,b) = \frac{b-a}{2^n} \sum_{k=1}^{2^n} \frac{1 - \frac{2k-1}{2^{n+1}}}{(1 - \frac{2k-1}{2^{n+1}})a + \frac{2k-1}{2^{n+1}}b}$$

and

$$L^{(2)}_{f,m}(a,b) = \frac{b-a}{2^{m+1}} \left\{ \frac{1}{a} + 2 \sum_{k=1}^{2^m-1} \frac{1 - \frac{k}{2^m}}{(1 - \frac{k}{2^m})a + \frac{k}{2^m}b} \right\}.$$

By Hermite-Hadamard inequality, we have

$$L^{(1)}_{f,n}(a,b) \le \frac{1}{b-a}\{b \log \frac{b}{a} - (b-a)\} \le L^{(2)}_{f,m}(a,b).$$

When $0 < a < b$, we have the following inequality.

$$\begin{aligned} & \frac{(b-a)^2}{2^n} \sum_{k=1}^{2^n} \frac{1 - \frac{2k-1}{2^{n+1}}}{(1 - \frac{2k-1}{2^{n+1}})a + \frac{2k-1}{2^{n+1}}b} \\ \le\ & b \log \frac{b}{a} - (b-a) \\ \le\ & \frac{(b-a)^2}{2^{m+1}} \left\{ \frac{1}{a} + 2 \sum_{k=1}^{2^m-1} \frac{1 - \frac{k}{2^m}}{(1 - \frac{k}{2^m})a + \frac{k}{2^m}b} \right\}. \end{aligned} \tag{4.16}$$

Similarly when $0 < b < a$, we have the following inequality.

$$\begin{aligned} & \frac{(b-a)^2}{2^{m+1}} \left\{ \frac{1}{a} + 2 \sum_{k=1}^{2^m-1} \frac{1 - \frac{k}{2^m}}{(1 - \frac{k}{2^m})a + \frac{k}{2^m}b} \right\}. \\ \le\ & b \log \frac{b}{a} - (b-a) \\ \le\ & \frac{(b-a)^2}{2^n} \sum_{k=1}^{2^n} \frac{1 - \frac{2k-1}{2^{n+1}}}{(1 - \frac{2k-1}{2^{n+1}})a + \frac{2k-1}{2^{n+1}}b} \end{aligned} \tag{4.17}$$

We put $a = 1$ and $b = y$ in (4.16). Then

$$\begin{aligned} & \frac{(y-1)^2}{2^n y} \sum_{k=1}^{2^n} \frac{1}{1 + \frac{2k-1}{2^{n+1}-(2k-1)}y} \\ \le\ & \log y - \frac{y-1}{y} \\ \le\ & \frac{(y-1)^2}{2^{m+1}y} \left\{ 1 + 2 \sum_{k=1}^{2^m-1} \frac{1}{1 + \frac{k}{2^m-k}y} \right\}. \end{aligned} \tag{4.18}$$

Similarly we put $a = y$ and $b = 1$ in (4.16). Then

$$\begin{aligned} & \frac{(y-1)^2}{2^n} \sum_{k=1}^{2^n} \frac{1}{y + \frac{2k-1}{2^{n+1}-(2k-1)}} \\ \le\ & y - 1 - \log y \end{aligned} \tag{4.19}$$

$$\leq \frac{(y-1)^2}{2^{m+1}}\left\{\frac{1}{y}+2\sum_{k=1}^{2^m-1}\frac{1}{y+\frac{k}{2^m-k}}\right\}.$$

On the other hand, we put $a=y$ and $b=1$ in (4.17). Then

$$\begin{aligned}
&\frac{(y-1)^2}{2^{m+1}}\left\{\frac{1}{y}+2\sum_{k=1}^{2^m-1}\frac{1}{y+\frac{k}{2^m-k}}\right\}\\
\leq\ & y-1-\log y \qquad (4.20)\\
\leq\ & \frac{(y-1)^2}{2^n}\sum_{k=1}^{2^n}\frac{1}{y+\frac{2k-1}{2^{n+1}-(2k-1)}}.
\end{aligned}$$

Similarly we put $a=1$ and $b=y$ in (4.17). Then

$$\begin{aligned}
&\frac{(y-1)^2}{2^{m+1}y}\left\{1+2\sum_{k=1}^{2^m-1}\frac{1}{1+\frac{k}{2^m-k}y}\right\}\\
\leq\ & \log y-\frac{y-1}{y} \qquad (4.21)\\
\leq\ & \frac{(y-1)^2}{2^n y}\sum_{k=1}^{2^n}\frac{1}{1+\frac{2k-1}{2^{n+1}-(2k-1)}y}.
\end{aligned}$$

We define

$$\alpha(n,y)=\frac{1}{2^n}\sum_{j=1}^{2^n}\left(1+\frac{2j-1}{2^{n+1}-(2j-1)}y\right)^{-1},$$

$$\beta(m,y)=\frac{1}{2^{m+1}}\left\{1+2\sum_{j=1}^{2^m-1}\left(1+\frac{j}{2^m-j}y\right)^{-1}\right\},$$

$$\gamma(n,y)=\frac{1}{2^n}\sum_{j=1}^{2^n}\left(y+\frac{2j-1}{2^{n+1}-(2j-1)}\right)^{-1},$$

$$\delta(m,y)=\frac{1}{2^{m+1}}\left\{\frac{1}{y}+2\sum_{j=1}^{2^m-1}\left(y+\frac{j}{2^m-j}\right)^{-1}\right\}.$$

Since $\alpha(n,y)\leq\beta(m,y),\gamma(n,y)\geq\delta(m,y)$ for $y>1$ and $\alpha(n,y)\geq\beta(m,y),\gamma(n,y)\leq\delta(m,y)$ for $y<1$, we obtain the following two representations by (4.18), (4.19), (4.20) and (4.21).

$$\begin{aligned}
&\frac{(y-1)^2}{y}\min\{\alpha(n,y),\beta(m,y)\}\leq\log y-\frac{y-1}{y} \qquad (4.22)\\
\leq\ & \frac{(y-1)^2}{y}\max\{\alpha(n,y),\beta(m,y)\},
\end{aligned}$$

and

$$(y-1)^2\min\{\gamma(n,y),\delta(m,y)\}\leq y-1-\log y \qquad (4.23)$$

$$\leq \quad (y-1)^2 \max\{\gamma(n,y), \delta(m,y)\}.$$

Let $0 < k < K$. Then we define

$$M(k,K) = \max_{k \leq y \leq K} \frac{(y-1)^2}{y},\ m(k,K) = \min_{k \leq y \leq K} \frac{(y-1)^2}{y},$$

$$N(k,K) = \max_{k \leq y \leq K} (y-1)^2,\ n(k,K) = \min_{k \leq y \leq K} (y-1)^2.$$

Then we have the following two inequalities.

$$m(k,K) \min\{\alpha(n,K), \beta(m,K)\} \leq \log y - \frac{y-1}{y} \leq M(k,K) \max\{\alpha(n,k), \beta(m,k)\},$$

and

$$n(k,K) \min\{\gamma(n,K), \delta(m,K)\} \leq y - 1 - \log y \leq N(k,K) \max\{\gamma(n,k), \delta(m,k)\}.$$

We need the following lemma in order to prove operator inequality.

Lemma 4.2. *Let $0 < k \leq x \leq K$ and $t > 0$, then we have*

$$\begin{aligned} 0 &\leq \min\{\alpha(n,K^t), \beta(m,K^t)\} \left(\frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t} \right) \\ &\leq \log x - \frac{1 - x^{-t}}{t} \\ &\leq \max\{\alpha(n,k^t), \beta(m,k^t)\} \left(\frac{x^t - 1}{t} - \frac{1 - x^{-t}}{t} \right) \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} 0 &\leq \min\{\gamma(n,K^t), \delta(m,K^t)\} t \left(\frac{x^t - 1}{t} \right)^2 \\ &\leq \frac{x^t - 1}{t} - \log x \\ &\leq \max\{\gamma(n,k^t), \delta(m,k^t)\} t \left(\frac{x^t - 1}{t} \right)^2. \end{aligned} \tag{4.25}$$

Proof. We obtain the results by putting $y = x^t$ in (4.22) and (4.23). ☐

Theorem 4.2. *Let A, B be two positive invertible operators and the constants $0 < \ell < L$ with the property that $\ell A \leq B \leq LA$. Then for any $t > 0$ and any $m, n \in \mathbb{N} \cup \{0\}$ we have*

$$\begin{aligned} 0 &\leq \min\{\alpha(n,L^t), \beta(m,L^t)\}(T_t(A|B) - T_{-t}(A|B)) \\ &\leq S(A|B) - T_{-t}(A|B) \\ &\leq \max\{\alpha(n,\ell^t), \beta(m,\ell^t)\}(T_t(A|B) - T_{-t}(A|B)), \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} 0 &\leq \min\{\gamma(n,L^t), \delta(m,L^t)\} t T_t(A|B) A^{-1} T_t(A|B) \\ &\leq T_t(A|B) - S(A|B) \end{aligned} \tag{4.27}$$

$$\leq \max\{\gamma(n,\ell^t),\delta(m,\ell^t)\}tT_t(A|B)A^{-1}T_t(A|B),$$

where

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$
$$S(A|B) = A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2}$$

and

$$T_t(A|B) = A^{1/2}\ln_t(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Proof. Since $\ell A \leq B \leq LA$ and A is invertible, then by multiplying both sides with $A^{-1/2}$ we get $\ell 1_H \leq A^{-1/2}BA^{-1/2} \leq L1_H$. Denote $X = A^{-1/2}BA^{-1/2}$ and by using the functional calculus for X that has its spectrum contained in the interval $[\ell, L]$ and the inequality (4.24), we get

$$\begin{aligned} 0 &\leq \min\{\alpha(n,L^t),\beta(m,L^t)\}\left(\frac{X^t-1_H}{t}-\frac{1_H-X^{-t}}{t}\right) \\ &\leq \log X - \frac{1_H-X^{-t}}{t} \\ &\leq \max\{\alpha(n,\ell^t),\beta(m,\ell^t)\}\left(\frac{X^t-1_H}{t}-\frac{1_H-X^{-t}}{t}\right) \end{aligned} \tag{4.28}$$

for any $t>0$. Now if we multiply both sides of (4.28) by $A^{1/2}$, then we get

$$\begin{aligned} 0 &\leq \min\{\alpha(n,L^t),\beta(m,L^t)\}A^{1/2}\left(\frac{X^t-1_H}{t}-\frac{1_H-X^{-t}}{t}\right)A^{1/2} \\ &\leq A^{1/2}\log X - A^{1/2}\frac{1_H-X^{-t}}{t}A^{1/2} \\ &\leq \max\{\alpha(n,\ell^t),\beta(m,\ell^t)\}A^{1/2}\left(\frac{X^t-1_H}{t}-\frac{1_H-X^{-t}}{t}\right)A^{1/2}. \end{aligned}$$

Since

$$A^{1/2}\frac{1_H-X^{-t}}{t}A^{1/2} = A^{1/2}\frac{X^{-t}-1_H}{-t}A^{1/2} = T_{-t}(A|B),$$

we have (4.26). Similarly by using (4.25) we get

$$\begin{aligned} 0 &\leq \min\{\gamma(n,K^t),\delta(m,K^t)\}t\left(\frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t}\right)^2 \\ &\leq \frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t} - \log(A^{-1/2}BA^{-1/2}) \\ &\leq \max\{\gamma(n,k^t),\delta(m,k^t)\}t\left(\frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t}\right)^2. \end{aligned} \tag{4.29}$$

for any $t>0$. If we multiply both sides of (4.29) by $A^{1/2}$, then we get

$$0 \leq \min\{\gamma(n,L^t),\delta(m,L^t)\}tA^{1/2}\left(\frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t}\right)^2 A^{1/2}$$

$$\begin{aligned} &\leq A^{1/2}\frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t}A^{1/2}-A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2} \\ &\leq \max\{\gamma(n,\ell^t),\delta(m,\ell^t)\}tA^{1/2}\left(\frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t}\right)^2 A^{1/2}. \end{aligned}$$

Since

$$A^{1/2}\left(\frac{(A^{-1/2}BA^{-1/2})^t-1_H}{t}\right)^2 A^{1/2}=T_t(A|B)A^{-1}T_t(A|B),$$

we have (4.27). □

By putting $t=1$ in (4.26) and (4.27), we have the following corollary.

Corollary 4.2. *Let A,B be two positive invertible operators and the constants $0<\ell<L$ with the property that $\ell A\leq B\leq LA$. Then for any $m,n\in\mathbb{N}\cup\{0\}$ we have*

$$\begin{aligned} 0 &\leq \min\{\alpha(n,L),\beta(m,L)\}(B-A)(A^{-1}-B^{-1})A \\ &\leq S(A|B)-(B-A)B^{-1}A \\ &\leq \max\{\alpha(n,\ell),\beta(m,\ell)\}(B-A)(A^{-1}-B^{-1})A, \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \min\{\gamma(n,L),\delta(m,L)\}(B-A)A^{-1}(B-A) \\ &\leq B-A-S(A|B) \\ &\leq \max\{\gamma(n,\ell),\delta(m,\ell)\}(B-A)A^{-1}(B-A). \end{aligned}$$

Proof. When $t=1$, $T_1(A|B)=B-A$ and $T_{-1}(A|B)=(B-A)B^{-1}A$. Since

$$\begin{aligned} &T_1(A|B)-T_{-1}(A|B)=B-A-(B-A)B^{-1}A \\ =\ &(B-A)(I-B^{-1}A)=(B-A)(A^{-1}-B^{-1})A, \end{aligned}$$

we obtain the result. □

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