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CONVERGENCE ANALYSIS FOR PRODUCTS OF RESOLVENTS OF CONVEX FUNCTIONS AND MULTIVALUED k-STRICTLY PSEUDONONSPREADING MAPPINGS IN HADAMARD SPACES

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Dedicated to the memory of Professor Wataru Takahashi with deep gratitude.

Abstract. In this paper, using the concept of gate condition for multi-valued mappings, we introduce a modified proximal point algorithm combined with a Halpern iteration process for approximating a common element of the set of minimizers of a finite family of convex functions and common fixed points of a finite family of k-strictly pseudononspreading multi-valued mappings in Hadamard spaces. We prove a strong convergence result without imposing the strict condition of compactness for solving the aforementioned problem. An application to a finite family of convex feasibility and fixed point problems for a finite family of quasi-nonexpansive mappings was discussed. The presented results improve and generalize many recent results in the literature.

1. Introduction

Let (X, d) be a metric space, D be a nonempty subset of X, and $T : D \to 2^D$ be a multi-valued mapping. A point $x \in D$ is called a fixed point of T if $q \in Tq$, and an endpoint (or a strict fixed point) of T if $Tx = \{x\}$. For a single-valued mapping $T : D \to D$, a fixed point of T is any point $q \in D$ such that $Tq = q$. We denote by $F(T)$ and $End(T)$, the sets of all fixed points of T and endpoints of T respectively. Clearly, $End(T) \subseteq F(T)$, but the converse is not generally true. We also denote the family of nonempty closed bounded subsets of D by $CB(D)$, the family of nonempty closed convex subsets of D by $CC(D)$ and the family of nonempty compact subsets of D by $K(D)$. The Pompieu Hausdorff metric on $CB(D)$ is defined by

$$\mathcal{H}(A, B) := \max\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(A, y)\},$$

for all $A, B \in CB(D)$, where $dist(x, B) = \inf\{d(x, b) : b \in B\}$. Suppose for each $x \in X$, there exists $u \in D$, such that $d(x, u) = dist(x, D)$ then D is called proximinal.

Let $T : D \to CB(D)$ be a multi-valued mapping, then for each $x \in D$, we define

$$P_T(x) := \{u \in Tx : d(x, u) = dist(x, Tx)\}. \tag{1.1}$$

Let $x \notin D$, a unique point y is called the gate of x in D, if

$$d(x, z) = d(x, y) + d(y, z),$$

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where $z \in D$. A point u is called a key of T, if for each $x \in F(T)$, x is the gate of u in Tx. We say that T satisfies the gate condition if T has a key in D. It is clear that the endpoint condition implies the gate condition but the converse is not true (see [41]).

Definition 1.1. Let D be a nonempty subset of a CAT(0) space X. A multi-valued mapping $T : D \to CB(D)$ is called

(1) nonexpansive, if
$$\mathcal{H}(Tx, Ty) \leq d(x, y), \text{ for all } x, y \in D;$$
(2) quasi-nonexpansive, if $p \in F(T)$ and
$$\mathcal{H}(Tx, Tp) \leq d(x, p), \text{ for all } x \in D;$$
(3) nonspreading, if
$$2\mathcal{H}(Tx, Ty)^2 \leq dist(y, Ty)^2 + dist(x, Ty)^2 \text{ for all } x, y \in D;$$
(4) k-strictly pseudononspreading if there exists $k \in [0, 1)$ such that
$$\begin{aligned}(2-k)\mathcal{H}(Tx, Ty)^2 \leq\ & kd(x, y)^2 + (1-k)dist(y, Tx)^2 + (1-k)dist(x, Ty)^2 \\ & + kdist(x, Tx)^2 + kdist(y, Ty)^2, \text{ for all } x, y \in D.\end{aligned}$$

Clearly, every nonspreading multi-valued mapping is 0-strictly pseudononspreading. It is also clear that if T is k-strictly pseudononspreading and has a fixed point, then for all $x \in D$ and $p \in F(T)$, we have
$$\mathcal{H}(Tx, Tp)^2 \leq d(x, p)^2 + kdist(x, Tx)^2.$$
Fixed points theory for multi-valued mappings continues to attract a lot of attention due to its numerous real world applications in market economy, differential inclusions, constrained optimization and game theory. They are also preferable in devising critical points in optimal control problems, energy management problems, signal processing, image reconstruction and a host of other problems.

Different iteration processes have been developed to approximate fixed points of multi-valued mappings, some of which require that T has a strict fixed point (endpoint) or that P_T (1.1) satisfies some contractive conditions (see, for example, [16, 23, 33, 38, 44, 43, 53] and the references therein).

It is worth-mentioning that Tits [47] introduced $\mathbb{R}$-trees in 1977, and Kirk [29] studied the fixed point of single-valued mappings in $\mathbb{R}$-trees. Since then, fixed point theorems for various types of single-valued and multi-valued mappings have been rapidly developed in $\mathbb{R}$-trees, (see [4, 5, 12]). Also keep in mind that the fixed point theorems in $\mathbb{R}$-trees finds their applications in graph theory, biology and computer science (see [12, 14, 30]).

An $\mathbb{R}$-tree is an example of a CAT(0) space. More so, a metric space X is a complete $\mathbb{R}$-tree if and only if X is hyperconvex with unique metric segments, see [27].

In 2009, Shahzad and Zegeye [42] employed an Ishikawa iteration for quasi-nonexpansive multi-valued mappings satisfying the endpoint condition in Banach spaces and proved a strong convergence result. Later in 2010, Puttasontiphot [39] obtained a similar result in complete CAT(0) spaces. In 2012, Sumanmit and Panyanak [41] introduced a condition on multi-valued mappings in $\mathbb{R}$-trees which is weaker than the endpoint condition and called it a gate condition. They proved a strong convergence theorem of a modified Ishikawa iteration for quasi-nonexpansive multi-valued mappings.

In 2015, Phuengrattana [37] introduced a new two-step iterative process for two k-strictly pseudononspreading multi-valued mappings in $\mathbb{R}$-trees. They proved the following strong convergence theorems using their proposed iteration to a common fixed points of two k-strictly pseudononspreading multi-valued mappings in $\mathbb{R}$-trees.

Theorem 1.2. *Let D be a nonempty closed convex subset of a complete $\mathbb{R}$-trees X. Let $T_1 : D \to CB(D)$ be a k-strictly pseudononspreading multi-valued mapping and $T_2 : D \to CB(D)$ be a k-strictly pseudononspreading and L-Lipschitzian multi-valued mapping and $\Theta := F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that T_1, T_2 satisfy the gate condition, and u_1, u_2 be keys of T_1 and T_2, respectively. For $x_1 \in D$, the sequence $\{x_n\}$ generated by*

$$y_n = (1 - \alpha_n)x_n \oplus \alpha_n z_n^{(1)},$$

where $z_n^{(1)}$ is the gate of $u_1 \in T_1 x_n$ and

$$x_{n+1} = (1 - \beta_n)y_n \oplus \beta_n z_n^{(2)}, \text{ for all } n \in \mathbb{N},$$

where $z_n^{(2)}$ is the gate of $u_2 \in T_2 y_n$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [0,1] such that $0 < a \leq \alpha_n,\ \beta_n \leq b < 1 - k$. If one of the following is satisfied:

(1) *T_1, T_2 satisfy condition (II) (see [37]),*

(2) *T_1 or T_2 is hemicompact,*

then $\{x_n\}$ converges strongly to an element of Θ.

Let X be a CAT(0) space and C be a convex subset of X. A function $f : X \to (-\infty, \infty]$ is said to be lower semi-continuous at a point $x \in X$, if

$$f(x) \leq \liminf_{n \to \infty} f(x_n),$$

for each sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to x. f is said to be lower semi-continuous on C if it is lower semi-continuous at any point in C.

For any $\lambda > 0$, the resolvent of a lower semi-continuous function f in X is defined as

$$J_\lambda^f(x) = \arg\min_{y \in X} \left[f(y) + \frac{1}{2\lambda} d^2(y, x) \right].$$

It is known from [26] that J_λ^f is well defined and nonexpansive for all $\lambda > 0$.

Minimization Problem (MP) has been of great interest in optimization theory, nonlinear analysis and geometry. The problem is defined as: find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{1.2}$$

The solution set of MP is denoted by $\arg\min_{y\in X} f(y)$. It is known that $F(J_\lambda^f)$ coincides with $\arg\min_{y\in X} f(y)$. In 2013, Bačák [11] studied the MP (1.2) in CAT(0) spaces using the following iterative algorithm. Let $x_1 \in X$, then

$$x_{n+1} = \arg\min_{y\in X} \big[f(y) + \frac{1}{2\lambda} d^2(y, x_n)\big],$$

where $\lambda > 0$ for all $n \in \mathbb{N}$. He proved that $\{x_n\}$ is Δ-convergent to the minimizer of f under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Suparatulatorn et al. [45] introduced the following modified Halpern iteration process for nonexpansive mappings in the framework of CAT(0) spaces.

Suppose that $u, x_1 \in X$ are arbitrary chosen and $\{x_n\}$ is generated in the following manner:

$$\begin{cases} y_n = argim_{y\in X}\big[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\big]; \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T y_n; \end{cases}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$;

(4) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

Then $\{x_n\}$ strongly converges to $z \in \Gamma := F(T) \cap argmin_{y\in X} f(y) \neq \emptyset$, which is the nearest point of Γ to u.

In 2015, Cholamjiak et al. [17] introduced the following modified proximal point algorithm involving fixed point iterates for two nonexpansive mappings and prove that the sequence generated by their iterative process converges to a minimizer of a convex function and a fixed point problem of two nonexpansive mappings. Let $\{x_n\}$ be generated in the following manner:

$$\begin{cases} z_n = argmin_{y\in X}\big[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\big]; \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_1 z_n; \\ x_{n+1} = (1-\alpha_n) T_1 x_n \oplus \alpha_n T_2 y_n; \end{cases}$$

for each $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ $\triangle$-converges to an element of Ω, where $\Omega := F(T_1) \cap F(T_2) \cap argmin_{y\in X} f(y)$.

Motivated by the results discussed above and the ongoing research in this direction, we study the notion of gate condition for a finite family of k-strictly pseudononspreading multi-valued mappings. Furthermore, we propose a modified proximal point algorithm combined with the Halpern iteration process for approximating a common element of the set of minimizers of a finite family of convex functions and common fixed points of a finite family of k-strictly pseudononspreading multi-valued mappings in Hadamard spaces. We establish a strong convergence result for approximating the solutions of the aforementioned problem. An application to finite family of convex feasibility problem and fixed point problem for finite family of quasi-nonexpansive multi-valued mappings is discussed, and some numerical examples are displayed to show the implementation of our proposed algorithm. The result presented in this paper extends and complements many related results in literature.

In summary, our contributions in this paper are:

- We use the gate condition for our proposed algorithm, which is a much more weaker condition than the endpoint condition imposed in [24, 3, 39, 42, 53]. Also, our algorithm does not require the compactness condition to converge strongly unlike the methods in [17, 37] where the compactness condition is imposed to prove strong convergence result.
- While the authors in [8, 37, 41] considered a common fixed point of two multi-valued mappings for which the mappings are required to be Lipschitz, we consider a common fixed point of a finite family of k-strictly pseudononspreading multi-valued mappings without the additional Lipschitz conditions on the mappings.
- We consider a class of mappings that includes the classes of nonexpansive, nonspreading and quasi-nonexpansive mappings. Thus, making our results more general than many existing results in literature, (for example, see [24, 17, 41, 45] and the references).
- We solve the problem: Find $x^* \in \cap_{i=1}^{k} F(G_i)$ such that

$$f_j(x^*) = \min_{y \in X} f_j(y),\ \ j = 1, 2, \dots, m. \tag{1.3}$$

Then, we employ the following modified proximal point iterative algorithm to solve (1.3)

$$\begin{cases} w_n = \gamma_n u \oplus (1-\gamma_n) x_n; \\ u_n = y_n^{(0)} = \Delta_{i=1}^{m} J_{\lambda_n}^{(j)} w_n = J_{\lambda_n}^{(1)} \circ J_{\lambda_n}^{(2)} \circ \cdots \circ J_{\lambda_n}^{(m)} w_n; \\ y_n^{(1)} = (1-\beta_n^{(1)}) u_n \oplus \beta_n^{(1)} v_n^{(1)}; \\ y_n^{(2)} = (1-\beta_n^{(2)}) y_n^{(1)} \oplus \beta_n^{(2)} v_n^{(2)}; \\ \qquad \vdots \\ y_n^{(k-1)} = (1-\beta_n^{(k-1)}) y_n^{(k-2)} \oplus \beta_n^{(k-1)} v_n^{(k-1)}; \\ x_{n+1} = y_n^{(k)} = (1-\beta_n^{(k)}) y_n^{(k-1)} \oplus \beta_n^{(k)} v_n^{(k)}; \end{cases} \tag{1.4}$$

where $v_n^{(i)}$ is a gate of $a_i \in G_i y_n^{(i-1)}$ and $\{\beta_n^i\}, \{\gamma_n\} \subset (0,1)$ for $i = 1, 2, \dots, N$.

2. Preliminaries

Let X be a metric space and $x, y \in X$. A geodesic from x to y is a map (or a curve) c from the closed interval $[0, d(x,y)] \subset \mathbb{R}$ to X such that $c(0) = x,\ c(d(x,y)) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x,y)]$. The image of c is called a geodesic segment joining from x to y. It is denoted by $[x,y]$ whenever it is unique. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y\in X$. A subset D of a geodesic space X is said to be convex, if for any two points $x,y\in D$, the geodesic joining x and y is contained in D, that is, if $c:[0,d(x,y)]\to X$ is a geodesic such that $x=c(0)$ and $y=c(d(x,y))$, then $c(t)\in D\ \forall\ t\in[0,d(x,y)]$. A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three vertices (points in X) with unparameterized geodesic segments between each pair of vertices. For any geodesic triangle, there is a comparison (Alexandrov) triangle $\bar{\Delta}\subset\mathbb{R}^2$ such that $d(x_i,x_j)=d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j)$, for $i,j\in\{1,2,3\}$. A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in X respectively, then Δ is said to satisfy the CAT(0) inequality for all points x,y of Δ and $\bar{x},\bar{y}$ of $\bar{\Delta}$ if $d(x,y)=d_{\mathbb{R}^2}(\bar{x},\bar{y})$. Let x,y,z be points in X and y_0 be the midpoint of the segment $[y,z]$, then the CAT(0) inequality implies

$$d^2(x,y_0)\leq\frac{1}{2}d^2(x,y)+\frac{1}{2}d^2(x,z)-\frac{1}{4}d(y,z).$$

Berg and Nikolaev [13] introduced the notion of quasi-linearization in a CAT(0) space as follows: Let a pair $(a,b)\in X\times X$ denoted by $\overrightarrow{ab}$, be called a vector. Then, the quasilinearization map $\langle .,.\rangle:(X\times X)\times(X\times X)\to\mathbb{R}$ is defined by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\frac{1}{2}(d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)),\text{ for all }a,b,c,d\in X.$$

It is easy to see that $\langle\overrightarrow{ab},\overrightarrow{ab}\rangle=d^2(a,b)$, $\langle\overrightarrow{ba},\overrightarrow{cd}\rangle=-\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{ae},\overrightarrow{cd}\rangle+\langle\overrightarrow{eb},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{cd},\overrightarrow{ab}\rangle$, for all $a,b,c,d,e\in X$. Furthermore, a geodesic space X is said to satisfy the Cauchy-Schwartz inequality, if

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\leq d(a,b)d(c,d),$$

for all $a,b,c,d\in X$. It is well known that a geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality [20]. Also, it is known that complete CAT(0) spaces are called Hadamard spaces.
Let $\{x_n\}$ be a bounded sequence in X and $r(.,\{x_n\}):X\to[0,\infty)$ be a continuous mapping defined by

$$r(x,\{x_n\})=\limsup_{n\to\infty}d(x,x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}):\inf\{r(x,x_n):x\in X\},$$

while the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\})=\{x\in X:r(x,\{x_n\})=r(\{x_n\})\}.$$

It is well known from [21, 31] that in a complete CAT(0) space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point

$x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$.

Definition 2.1. Let C be a nonempty closed convex subset of a CAT(0) space X. The metric projection $P_C : X \to C$ which assigns to each $x \in X$ the unique point $P_C x$ is defined by

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}.$$

Lemma 2.2. [18] *Let C be a nonempty closed convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \le 0$ for all $y \in C$.*

Lemma 2.3. [19, 20] *Let X be a CAT(0) space. Then for all $w, x, y, z \in X$ and all $t \in [0, 1]$, we have*

(1) $d(tx \oplus (1-t)y, z) \le td(x, z) + (1-t)d(y, z)$,
(2) $d^2(tx \oplus (1-t)y, z) \le td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$,
(3) $d^2(z, tx \oplus (1-t)y) \le t^2 d^2(z, x) + (1-t)^2 d^2(z, y) + 2t(1-t)\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle$,
(4) $d(tx \oplus (1-t)y, tw \oplus (1-t)z) \le td(x, w) + (1-t)d(y, z)$.

Lemma 2.4. [35] *Let X be a CAT(0) space with a convex metric space and U, V be bounded gated subsets of X. Then,*

$$d(P_U(a), P_V(a)) \le \mathcal{H}(U, V),$$

for any $a \in X$, where $P_U(a), P_V(a)$ are respectively the unique nearest point of a in U and V.

Lemma 2.5. [25] *Let X be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semiontinuous function. Then,*

$$d^2(J_\lambda^f x, x) \le d^2(J_\mu^f x, x) \text{ for } 0 < \lambda < \mu \text{ and } x \in X.$$

Lemma 2.6. [32] *Let X be a CAT(0) space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda^f x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda^f x) + f(J_\lambda^f x) \le f(y).$$

Lemma 2.7. [20] *Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergence subsequence.*

Definition 2.8. Let C be a nonempty closed and convex subset of a Hadamard space X. A mapping $T : C \to C$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in X such that Δ-$\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x = Tx$.

Lemma 2.9. [37] *Let C be a nonempty closed and convex subset of a CAT(0) space. Assume that $T : D \to CB(D)$ is a k-strictly pseudononspreading multi-valued mapping and $\{x_n\}$ is a sequence in C such that $x_n \to x$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, then $x \in Tx$.*

Lemma 2.10. [50] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \; n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a real sequence such that:

(1) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup\limits_{n\to\infty} \delta \leq 0$ *or* $\sum\limits_{n=1}^{\infty} |\alpha_n\delta_n| < \infty$.

Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.11. [34] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3. Main results

Lemma 3.1. *Let X be a complete CAT(0) space and $f_j : X \to (-\infty, \infty]$, $j = 1, 2, \cdots, m$ be finite family of proper, convex and lower semi-continuous functions. For $0 < \lambda \leq \mu$, we have that*

$$F\left(\Pi_{j=1}^m J_\mu^{(j)}\right) \subseteq \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right),$$

where, $\Pi_{j=1}^m J_\mu^{(j)} = J_\mu^{(1)} \circ J_\mu^{(2)} \circ \cdots \circ J_\mu^{(m)}$.

Proof. Let $x \in F\left(\Pi_{j=1}^m J_\mu^{(j)}\right)$ and $y \in \left(\cap_{j=1}^m F\left(J_\mu^{(j)}\right)\right)$, then we have that

$$\begin{aligned} d^2(x,y) &= d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, \Pi_{j=1}^m J_\mu^{(j)}y\right) \\ &\leq d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) \end{aligned} \tag{3.1}$$

Also, from Lemma 2.6 we have that

$$\begin{aligned} &\frac{1}{2\mu}d^2\left(\Pi_{j=1}^m J^{(j)}x, y\right) - \frac{1}{2\mu}d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) \\ &\qquad + \frac{1}{2\mu}d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, \Pi_{j=1}^m J_\mu^{(j)}x\right) + f\left(\Pi_{j=1}^m J_\mu^{(j)}x\right) \leq f(y). \end{aligned}$$

Since $f(y) \leq f\left(\Pi_{j=1}^m J^{(j)}x\right)$, we obtain from (3.1) that

$$\begin{aligned} d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, \Pi_{j=1}^m J_\mu^{(j)}x\right) &\leq d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) - d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right) \\ &\;\;\vdots \\ &\leq d^2(x,y) - d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right) \end{aligned}$$

$$\leq d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right) - d^2\left(\Pi_{j=1}^m J_\mu^{(j)}x, y\right),$$

which implies that

$$\Pi_{j=1}^m J_\mu^{(j)}x = \Pi_{j=2}^m J_\mu^{(j)}x. \tag{3.2}$$

Similarly, we obtain from Lemma 2.6 and (3.1) that

$$\begin{aligned} d^2\left(\Pi_{j=3}^m J_\mu^{(j)}x, \Pi_{j=2}^m J_\mu^{(j)}x\right) &\leq d^2\left(\Pi_{j=3}^m J_\mu^{(j)}x, y\right) - d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) \\ &\vdots \\ &\leq d^2(x,y) - d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) \\ &\leq d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right) - d^2\left(\Pi_{j=2}^m J_\mu^{(j)}x, y\right), \end{aligned}$$

which also implies that

$$\Pi_{j=2}^m J_\mu^{(j)}x = \Pi_{j=3}^m J_\mu^{(j)}x. \tag{3.3}$$

Continuing in this manner, we obtain that

$$\Pi_{j=3}^m J_\mu^{(j)}x = \Pi_{j=4}^m J_\mu^{(j)}x = \cdots = \Pi_{j=m-1}^m J_\mu^{(j)}x = J_\mu^{(m)}x = x. \tag{3.4}$$

From (3.4), we have

$$x = J_\mu^{(m)}x. \tag{3.5}$$

From (3.4) and (3.5), we obtain that

$$x = \Pi_{j=m-1}^m J_\mu^{(j)}x = J_\mu^{(m-1)}J_\mu^{(m)}x = J_\mu^{(m-1)}x. \tag{3.6}$$

Continuing in this manner, we get from (3.2)and (3.6) that

$$x = J_\mu^{(m-2)}x = \cdots = J_\mu^{(2)}x = J_\mu^{(1)}x. \tag{3.7}$$

That is,

$$J_\mu^{(1)}x = J_\mu^{(2)}x = \cdots = J_\mu^{(m-1)}x = J_\mu^{(m)}x = x. \tag{3.8}$$

Now, since $0 < \lambda \leq \mu$, we obtain from Lemma 2.5 and (3.8) that

$$d^2(x, J_\lambda^{(j)}x) \leq d^2(x, J_\mu^{(j)}x) = 0,\ j = 1, 2, \cdots, m,$$

which implies that $x \in F(J_\lambda^{(j)}),\ j = 1, 2, \cdots, m$. Hence, we conclude that

$$F\left(\Pi_{j=1}^m J_\mu^{(j)}\right) \subseteq \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right).$$

□

Theorem 3.2. *Let D be a nonempty closed and convex subset of an Hadamard space X and $f_j : X \to \mathbb{R}, j = 1, 2, \ldots, m$ be a proper convex and lower semi-continuous function. Let $\{G_i\}_{i=1}^k : X \to CB(X)$ be finite family of ρ_i-strictly pseudononspreading multi-valued mappings. Assume that $\{G_i\}_{i=1}^k$ satisfies the gate condition with $a_i, i = 1, 2, \ldots, k$ being the key of G_i and $\Omega := \cap_{i=1}^k F(G_i) \cap \cap_{j=1}^m argmin_{y\in X} f_j(y) \neq \emptyset$. Let $\{\gamma_n\}$ and $\{\beta_n^i\}$ be sequences in [0,1] such that the following assumptions holds:*

(1) $0 < a \leq \beta_n^i \leq b < 1 - \rho_i$

(2) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Then, for $\lambda > 0$ *and any given* $x_1, u \in X$*, the sequence* $\{x_n\}$ *generated by* (1.4) *converges strongly to a point* $x^* = P_\Omega u$*, where* P_Ω *is the metric projection of* X *onto* Ω*.*

Proof. Let $z \in \Omega$, then from Lemma 2.3, (2), Lemma 2.4 and (1.4), we have that

$$\begin{aligned}
d^2(y_n^{(1)}, z) &\leq (1-\beta_n^{(1)})d^2(u_n, z) + \beta_n^{(1)} d^2(v_n^{(1)}, z) - \beta_n^{(1)}(1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&\leq (1-\beta_n^{(1)})d^2(u_n, z) \\
&\quad + \beta_n^{(1)} d^2(P_{G_1 u_n}(a_1), P_{G_1 z}(a_1)) - \beta_n^{(1)}(1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&\leq (1-\beta_n^{(1)})d^2(u_n, z) \\
&\quad + \beta_n^{(1)} \mathcal{H}^2(G_1 u_n, G_1 z) - \beta_n^{(1)}(1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&\leq (1-\beta_n^{(1)})d^2(u_n, z) \\
&\quad + \beta_n^{(1)} \big(d^2(u_n, z) + \rho_1 dist^2(u_n, G_1 u_n)\big) - \beta_n^{(1)}(1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&= d^2(u_n, z) - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&= d^2(J_{\lambda_n}^{(1)} w_n, z) - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&\leq d^2(J_{\lambda_n}^{(2)} w_n, z) - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \qquad (3.9) \\
&\vdots \\
&\leq d^2(w_n, z). \qquad (3.10)
\end{aligned}$$

Again, by applying (3.9), Lemma 2.3, (2), Lemma 2.4 and (1.4) we have that

$$\begin{aligned}
d^2(y_n^{(2)}, z) &\leq (1-\beta_n^{(2)})d^2(y_n^{(1)}, z) \\
&\quad + \beta_n^{(2)} d^2(v_n^{(2)}, z) - \beta_n^{(2)}(1-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\
&\leq (1-\beta_n^{(2)})d^2(y_n^{(1)}, z) \\
&\quad + \beta_n^{(2)} d^2(P_{G_2 y_n^{(1)}}(a_2), P_{G_2 z}(a_2)) - \beta_n^{(2)}(1-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\
&\leq (1-\beta_n^{(2)})d^2(y_n^{(1)}, z) \\
&\quad + \beta_n^{(2)} \mathcal{H}^2(G_2 y_n^{(1)}, G_2 z) - \beta_n^{(2)}(1-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\
&\leq (1-\beta_n^{(2)})d^2(y_n^{(1)}, z) \\
&\quad + \beta_n^{(2)} \big(d^2(y_n^{(1)}, z) + \rho_2 dist^2(y_n^{(1)}, G_2 y_n^{(1)})\big) - \beta_n^{(2)}(1-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\
&= d^2(y_n^{(1)}, z) - \beta_n^{(2)}(1-\rho_2-\beta_n^{2})d^2(y_n^{(1)}, v_n^{(2)}) \\
&\leq d^2(w_n, z) - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\
&\quad - \beta_n^{(2)}(1-\rho_2-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \qquad (3.11) \\
&\leq d^2(w_n, z).
\end{aligned}$$

Now, following the same argument as in (3.9) and (3.11), we from (1.4) that

$$\begin{aligned} d^2(y_n^{(k-1)}, z) &\leq d^2(w_n, z) - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\ &\quad - \beta_n^{(2)}(1-\rho_2-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\ &\quad - \cdots - \beta_n^{(k-1)}(1-\rho_{(k-1)}-\beta_n^{(k-1)})d^2(y_n^{(k-2)}, v_n^{k-1}) \qquad (3.12) \\ &\leq d^2(w_n, z). \end{aligned}$$

From (1.4) and (3.12), we obtain that

$$\begin{aligned} d^2(x_{n+1}, z) &\leq (1-\beta_n^k)d^2(y_n^{(k-1)}, z) + \beta_n^k d^2(v_n^k, z) - \beta_n^{(k)}(1-\beta_n^{(k)})d^2(y_n^{(k-1)}, v_n^k) \\ &\leq d^2(w_n, p) - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) \\ &\quad - \beta_n^{(2)}(1-\rho_2-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\ &\quad - \cdots - \beta_n^{(k-1)}(1-\rho_{(k-1)}-\beta_n^{(k-1)})d^2(y_n^{(k-2)}, v_n^{k-1}) \\ &\quad - \beta_n^{(k)}(1-\rho_k-\beta_n^{(k)})d^2(y_n^{(k-1)}, v_n^k) \qquad (3.13) \\ &\leq d^2(w_n, z), \qquad (3.14) \end{aligned}$$

this implies that

$$\begin{aligned} d(x_{n+1}, z) &\leq d(w_n, z) \\ &\gamma_n d(u, z) + (1-\gamma_n)d(x_n, z) \\ &\leq \max\{d(u, z), d(x_n, z)\} \\ &\vdots \\ &\leq \max\{d(u, p), d(x_1, z)\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{w_n\}$ and $\{y_n\}$ are all bounded.

CASE A: Assume that $\{d^2(x_n, z)\}$ is monotone decreasing.Then $\{d^2(x_n, z)\}$ converges and

$$d^2(x_n, z) - d^2(x_{n+1}, z) \to 0 \text{ as } n \to \infty.$$

Hence, we have from (3.13) and Lemma 2.3 (3) that

$$\begin{aligned} d^2(x_{n+1}, z) &\leq \gamma_n^2 d^2(u, z) + (1-\gamma_n)d^2(x_n, z) + 2\gamma_n(1-\gamma_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z}\rangle \\ &\quad - \beta_n^{(1)}(1-\rho_1-\beta_n^{(1)})d^2(u_n, v_n^{(1)}) - \beta_n^{(2)}(1-\rho_2-\beta_n^{(2)})d^2(y_n^{(1)}, v_n^{(2)}) \\ &\quad - \cdots - \beta_n^{(k-1)}(1-\rho_{(k-1)}-\beta_n^{(k-1)})d^2(y_n^{(k-2)}, v_n^{k-1}) \\ &\quad - \beta_n^{(k)}(1-\rho_k-\beta_n^{(k)})d^2(y_n^{(k-1)}, v_n^k), \qquad (3.15) \end{aligned}$$

Hence, using conditions (i) and (ii) of (3.2), we obtain that

$$\lim_{n\to\infty}\left(d^2(u_n, v_n^{(1)}) = d^2(y_n^{(1)}, v_n^{(2)}) = \cdots = d^2(y_n^{(k-2)}, v_n^{(k-1)}) = d^2(y_n^{(k-1)}, v_n^{(k)})\right) = 0. \qquad (3.16)$$

From (1.4), we also have that

$$d(w_n, x_n) \leq \gamma_n d(u, x_n) \to 0, \text{ as } n \to \infty. \qquad (3.17)$$

On applying Lemma 2.6, we have that

$$\begin{aligned}
d^2(u_n, \Delta_{j=2}^m J_{\lambda_n}^{(j)} w_n) &= d^2(\Delta_{j=1}^m J_{\lambda_n}^{(j)} w_n, \Delta_{j=2}^m J_{\lambda_n}^{(j)} w_n) \\
&\le d^2(\Delta_{j=2}^m J_{\lambda_n}^{(j)} w_n, z) - d^2(u_n, z) \\
&\le d^2(w_n, z) - d^2(u_n, z) \\
&\le d^2(w_n, z) - d^2(y_n^{(1)}, z) \\
&\le d^2(w_n, z) - d^2(y_n^{(2)}, z) \\
&\vdots \\
&\le d^2(w_n, z) - d^2(x_{n+1}, z) \\
&\le \gamma_n d^2(u, z) + (1-\gamma_n) d^2(x_n, z) - d^2(x_{n+1}, z) \\
&\quad + 2\gamma_n(1-\gamma_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z}\rangle. \qquad (3.18)
\end{aligned}$$

Therefore, we have that

$$\lim_{n\to\infty} d(u_n, \Delta_{j=2}^m J_{\lambda_n}^{(j)} w_n) = 0. \qquad (3.19)$$

From (3.19), we have that

$$\lim_{n\to\infty} \left(d(u_n, \Delta_{j=2}^m J_{\lambda_n}^{(j)} w_n) = d(\Delta_{j=1}^m J_{\lambda_n}^{(j)} w_n, \Delta_{j=2}^m J_{\lambda_n}^{(j)} w_n) \right) = 0. \qquad (3.20)$$

By following the same approach as in (3.18)-(3.20), we have that

$$\lim_{n\to\infty} \left(d(\Delta_{j=2}^{(j)} w_n, \Delta_{j=3}^m J_{\lambda_n}^{(j)} w_n) = d(\Delta_{j=3}^{(j)} w_n, \Delta_{j=4}^m J_{\lambda_n}^{(j)} w_n) \cdots = d(J_{\lambda_n}^{(m)} w_n, w_n) \right) = 0. \qquad (3.21)$$

Therefore, we have that

$$\begin{aligned}
d(u_n, w_n) &= d(\Delta_{j=1}^m J_{\lambda_n}^{(j)} w_n, w_n) \\
&\le d(\Delta_{j=2}^{(j)} w_n, \Delta_{j=3}^m J_{\lambda_n}^{(j)} w_n) + d(\Delta_{j=3}^{(j)} w_n, \Delta_{j=4}^m J_{\lambda_n}^{(j)} w_n) \\
&\quad + \cdots + d(J_{\lambda_n}^{(m)} w_n, w_n) \to 0 \text{ as } n \to \infty. \qquad (3.22)
\end{aligned}$$

From (3.17) and (3.22), we have that

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \qquad (3.23)$$

Also, from (3.16) and (3.23), we have

$$\lim_{n\to\infty} d(v_n^{(1)}, x_n) = 0. \qquad (3.24)$$

Using (1.4) and (3.16), we obtain that

$$d(y_n^{(1)}, v_n^{(1)}) \le (1-\beta_n^{(1)}) d(u_n, v_n^{(1)}) \to 0, \text{ as } n \to \infty. \qquad (3.25)$$

We have from (3.24) and (3.25) that

$$\lim_{n\to\infty} d(y_n^{(1)}, x_n) = 0. \tag{3.26}$$

Also, from (1.4) and (3.16), we have

$$d(y_n^{(2)}, v_n^{(2)}) \le (1-\beta_n^{(2)}) d(y_n^{(1)}, v_n^{(2)}) \to 0, \text{ as } n \to \infty. \tag{3.27}$$

Following the same argument as in (3.27) and applying (3.16), we have

$$\lim_{n\to\infty} \left(d(y_n^{(3)}, v_n^{(3)}) = d(y_n^{(4)}, v_n^{(4)}) = \cdots = d(y_n^{(k)}, v_n^{(k)}) \right) = 0. \tag{3.28}$$

Using (3.16) and (3.25), we have that

$$d(v_n^{(1)}, v_n^{(2)}) \le d(y_n^{(1)}, v_n^{(2)}) + d(y_n^{(1)}, v_n^{(1)}) \to 0, \text{ as } n \to \infty. \tag{3.29}$$

Continuing the same approach as in (3.29) and applying (3.16) and (3.27), we have that

$$\lim_{n\to\infty} \left(d(v_n^{(2)}, v_n^{(3)}) = d(v_n^{(3)}, v_n^{(4)}) = \cdots = d(v_n^{(k-1)}, v_n^{(k)}) \right) = 0. \tag{3.30}$$

From (3.24) and (3.29), we obtain that

$$\lim_{n\to\infty} d(v_n^{(2)}, x_n) = 0. \tag{3.31}$$

Also, on applying (3.16) and (3.31), we have

$$\lim_{n\to\infty} d(y_n^{(2)}, x_n) = 0. \tag{3.32}$$

Also, from (3.30) and (3.31), we obtain

$$\lim_{n\to\infty} d(v_n^{(3)}, x_n) = 0. \tag{3.33}$$

Using (3.28) and (3.33), we have

$$\lim_{n\to\infty} d(y_n^{(3)}, x_n) = 0. \tag{3.34}$$

Following the same process from (3.28)-(3.34), we obtain that

$$\lim_{n\to\infty} \left(d(y_n^{(4)}, x_n) = d(y_n^{(5)}, x_n) = \cdots = d(y_n^{(k)}, x_n) \right) = 0. \tag{3.35}$$

We can now conclude from (3.28) and (3.35) that

$$\lim_{n\to\infty} d(v_n^{(k)}, x_n) = 0. \tag{3.36}$$

Now, suppose $\{x_n\}$ is bounded, it follows from Lemma 2.7 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $\Delta - \lim_{k\to\infty} x_{n_k} = x^*$. It follows from (3.17), (3.23), (3.35) and Lemma 2.7 that there exists subsequences $\{w_{n_k}\}$ of $\{w_n\}, \{u_{n_k}\}$ of $\{u_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$, such that $\Delta - \lim w_{n_k} = x^*, \Delta - \lim u_{n_k} = x^*$ and $\Delta - \lim y_{n_k} = x^*$, respectively. Also, using (3.28) and Lemma 2.9, we have that $x^* \in \bigcap_{i=1}^k F(G_i)$. Since $J_{\lambda_n}^{f_j}$, $j = 1, 2, \ldots, m$ is nonexpansive, we obtain from (3.22) and Lemma 2.9 that $x^* \in \bigcap_{j=1}^m F(J_{\lambda_n}^{f_j})$. Therefore, we conclude that $x^* \in \Omega$.

Next, we show that $\limsup_{n\to\infty}\langle\overrightarrow{uz}, \overrightarrow{x_nz}\rangle \leq 0$. Now, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\overrightarrow{uz}, \overrightarrow{x_nz}\rangle = \lim_{k\to\infty}\langle\overrightarrow{uz}, \overrightarrow{x_{n_k}z}\rangle.$$

Since $x_{n_k} \rightharpoonup x^*$, it follows from Lemma 2.2 that

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{uz}, \overrightarrow{x_nz}\rangle &= \lim_{k\to\infty}\langle\overrightarrow{uz}, \overrightarrow{x_{n_k}z}\rangle \\ &= \langle\overrightarrow{uz}, \overrightarrow{x^*z}\rangle \leq 0. \end{aligned} \tag{3.37}$$

We now prove that $\{x_n\}$ converges strongly to x^*. From (3.15), we obtain that

$$\begin{aligned} d^2(x_{n+1}, x^*) &\leq \gamma_n^2 d^2(u, x^*) + (1-\gamma_n)d^2(x_n, x^*) + 2\gamma_n(1-\gamma_n)\langle\overrightarrow{ux^*}, \overrightarrow{x_nx^*}\rangle \\ &= (1-\gamma_n)d^2(x_n, x^*) + \gamma_n\bigg(2(1-\gamma_n)\langle\overrightarrow{ux^*}, \overrightarrow{x_nx^*}\rangle + \gamma_n d(u, x^*)\bigg). \end{aligned} \tag{3.38}$$

Using Lemma 2.10 and (3.37) in (3.38), we conclude that $d(x_n, x^*) \to 0$ as $n \to \infty$. Hence, we have that $\{x_n\}$ converges strongly to $x^* = P_\Omega u$.
CASE 2: Assume that $\{d^2(x_n, x^*)\}$ is not monotone decreasing. Then, there exists a subsequence $\{d^2(x^*, x_{n_j})\}$ of $\{d^2(x^*, x_n)\}$ such that $d^2(x^*, x_{n_j}) < d^2(x^*, x_{n_j+1})$ for all $j \in \mathbb{N}$. Hence, by Lemma 2.11, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and

$$d^2(x^*, x_{m_k}) \leq d^2(x^*, x_{m_k+1}) \text{ and } d^2(x^*, x_k) \leq d^2(x^*, x_{m_k+1}) \text{ for all } k \in \mathbb{N}. \tag{3.39}$$

Thus, from (3.14), (3.39) and Lemma 2.3, we have

$$\begin{aligned} 0 &\leq \lim_{k\to\infty}\bigg(d^2(x^*, x_{m_k+1}) - d^2(x^*, x_{m_k})\bigg) \\ &\leq \limsup_{n\to\infty}\bigg(d^2(x^*, x_{n+1}) - d^2(x^*, x_n)\bigg) \\ &\leq \limsup_{n\to\infty}\bigg(d^2(x^*, w_n) - d^2(x^*, x_n)\bigg) \\ &\leq \limsup_{n\to\infty}\bigg((1-\gamma_n)d^2(x^*, x_n) + \gamma_n d^2(x^*, u) - d^2(x^*, x_n)\bigg) \\ &= \limsup_{n\to\infty}\bigg(\gamma_n(d^2(x^*, u) - d^2(x^*, x_n)\bigg) = 0, \end{aligned} \tag{3.40}$$

hence, we have that

$$\lim_{k\to\infty}\bigg(d^2(x^*, x_{m_k+1}) - d^2(x^*, x_{m_k})\bigg) = 0. \tag{3.41}$$

Following the same approach as in (3.37), we can prove that

$$\lim_{k\to\infty}\bigg(\gamma_{m_k}d^2(x^*, u) + 2(1-\gamma_{m_k})\langle\overrightarrow{ux^*}, \overrightarrow{x_{m_k}x^*}\rangle\bigg). \tag{3.42}$$

More so, using (3.38), we have

$$d^2(x^*, x_{m_k+1}) \leq (1-\gamma_{m_k})d^2(x^*, x_{m_k})$$

$$+\gamma_{m_k}\bigg(\gamma_{m_k}d(x^*,u)+2(1-\gamma_{m_k})\langle\overrightarrow{ux^*},\overrightarrow{x_{m_k}x^*}\rangle\bigg).$$

Since $d^2(x^*,x_{m_k})<d^2(x^*,x_{m_k+1})$, we obtain that

$$d^2(x^*,x_{m_k})\leq\bigg(\gamma_{m_k}d^2(x^*,u)+2(1-\gamma_{m_k})\langle\overrightarrow{ux^*},\overrightarrow{x_{m_k}x^*}\rangle\bigg).$$

Hence, by (3.42), we obtain

$$d^2(x^*,x_{m_k})=0. \tag{3.43}$$

Therefore, we have that $\lim\limits_{k\to\infty}d^2(x^*,x_k)=0$ which implies that $\{x_n\}$ converges strongly to $x^*\in\Omega$. □

We state the consequences of our result as follows:

Corollary 3.3. *Let D be a nonempty closed and convex subset of an Hadamard space X and $f_j:X\to\mathbb{R}, j=1,2,\ldots,m$ be a proper convex and lower semi-continuous function. Let $\{G_i\}_{i=1}^k:X\to CB(X)$ be finite family of quasi-nonexpansive multi-valued mappings. Assume that $\{G_i\}_{i=1}^k$ satisfies the gate condition with $a_i, i=1,2,\ldots,k$ being the key of G_i and $\Omega\neq\emptyset$. Let $\{\gamma_n\}$ and $\{\beta_n^i\}$ be sequences in [0,1] such that the following assumptions holds:*

(1) $0<a\leq\beta_n^i\leq b<1$;

(2) $\lim\limits_{n\to\infty}\gamma_n=0$ *and* $\sum\limits_{n=1}^{\infty}\gamma_n=\infty$.

Then, for $\lambda>0$ and any given $x_1,u\in X$, the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point $x^*=P_\Omega u$, where P_Ω is the metric projection of X onto Ω.

Corollary 3.4. *Let D be a nonempty closed and convex subset of a real Hilbert space H, and $f_j:X\to\mathbb{R}, j=1,2,\ldots,m$ be a proper convex and lower semi-continuous function. Let $\{G_i\}_{i=1}^k:X\to CB(X)$ be finite family of ρ_i- strictly pseudononspreading multi-valued mappings. Assume that $\{G_i\}_{i=1}^k$ satisfies the type-one (see [1]) condition and $\Omega\neq\emptyset$. Let $\{\gamma_n\}$ and $\{\beta_n^i\}$ be sequences in [0,1] such that the following assumptions holds:*

(1) $0<a\leq\beta_n^i\leq b<1-\rho_i$;

(2) $\lim_{n\to\infty}\gamma_n=0$ *and* $\sum_{n=1}^{\infty}\gamma_n=\infty$.

Then, for $\lambda > 0$ and any given $x_1, u \in X$, the sequence $\{x_n\}$ generated iteratively by

$$\begin{cases} w_n = \gamma_n u + (1-\gamma_n)x_n; \\ u_n = y_n^{(0)} = \Delta_{i=1}^m J_{\lambda_n}^{(j)} w_n = J_{\lambda_n}^{(1)} \circ J_{\lambda_n}^{(2)} \circ \cdots \circ J_{\lambda_n}^{(m)} w_n; \\ y_n^{(1)} = (1-\beta_n^{(1)})u_n + \beta_n^{(1)} v_n^{(1)}; \\ y_n^{(2)} = (1-\beta_n^{(2)})y_n^{(1)} + \beta_n^{(2)} v_n^{(2)}; \\ \vdots \\ y_n^{(k-1)} = (1-\beta_n^{(k-1)})y_n^{(k-2)} + \beta_n^{(k-1)} v_n^{(k-1)}; \\ x_{n+1} = y_n^{(k)} = (1-\beta_n^{(k)})y_n^{(k-1)} + \beta_n^{(k)} v_n^{(k)}; \end{cases} \tag{3.44}$$

where $v_n^{(i)} \in G_i y_n^{(i-1)}$, then $\{x_n\}$ converges strongly to a point $x^* = P_\Omega u$, where P_Ω is the metric projection of X onto Ω.

The result discussed in Corollary 3.4 generalizes the result in [38].

4. Application and Numerical Example

4.1. **Convex Feasibility Problem.** Let $\{D_j\}_{j=1}^m$ be a finite family of nonempty, closed and convex subsets of an Hadamard space X such that $\cap_{j=1}^m D_j \neq \emptyset$. The Convex Feasibility Problem (CFP) is to find $x^* \in \cap_{j=1}^m D_j$.

For a nonempty, closed and convex subset D of an Hadamard space X, the indicator function

$$i_D(x) = \begin{cases} 0, \ x \in D, \\ \infty, \ x \in X \backslash D, \end{cases}$$

is proper convex and lower semi-continuous and $J_\lambda^{i_D} = P_D$. Therefore, by letting $f_j = i_D, (j = 1, 2, \dots, m)$, we have the following theorem

Theorem 4.1. *Let $\{D_j\}_{j=1}^\infty$ be a finite family of nonempty closed and convex subsets of an Hadamard space X. Let $\{G_i\}_{i=1}^k : X \to CB(X)$ be finite family of ρ_i- strictly pseudononspreading multi-valued mappings. Assume that $\{G_i\}_{i=1}^k$ satisfies the gate condition and $\Omega \neq \emptyset$. Let $\{\gamma_n\}$ and $\{\beta_n^i\}$ be sequences in [0,1] such that the following assumptions holds:*

(1) $0 < a \leq \beta_n^i \leq b < 1 - \rho_i$;

(2) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^\infty \gamma_n = \infty$.

Then, for $\lambda > 0$ and any given $x_1, u \in X$, the sequence $\{x_n\}$ generated iteratively by

$$(4.1)\qquad \begin{cases} w_n = \gamma_n u \oplus (1-\gamma_n)x_n; \\ u_n = y_n^{(0)} = \Delta_{i=1}^m P_D w_n = P_D^{(1)} \circ P_D^{(2)} \circ \cdots \circ P_D^{(m)} w_n; \\ y_n^{(1)} = (1-\beta_n^{(1)})u_n \oplus \beta_n^{(1)} v_n^{(1)}; \\ y_n^{(2)} = (1-\beta_n^{(2)})y_n^{(1)} \oplus \beta_n^{(2)} v_n^{(2)}; \\ \qquad\qquad\qquad\qquad \vdots \\ y_n^{(k-1)} = (1-\beta_n^{(k-1)})y_n^{(k-2)} \oplus \beta_n^{(k-1)} v_n^{(k-1)}; \\ x_{n+1} = y_n^{(k)} = (1-\beta_n^{(k)})y_n^{(k-1)} \oplus \beta_n^{(k)} v_n^{(k)}; \end{cases}$$

where $v_n^{(i)} \in G_i y_n^{(i-1)}$, then $\{x_n\}$ converges strongly to a point $x^* = P_\Omega u$, where P_Ω is the metric projection of X onto Ω.

4.2. **Numerical Example.** In this section, we give a numerical example of an algorithm to illustrate its performance. Let $X = \mathbb{R}^2$ be endowed with the Euclidean norm. For $i = 1, 2$, let $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_i x = \left[0, \frac{x}{10i}\right], \quad x \in X.$$

Then T_i is k-strictly pseudononspreading multivalued mapping with $k = 0$. Now, define $f_1 : \mathbb{R}^2 \to \mathbb{R}^2 by f_1(x_1, x_2) = 100((x_2+1) - (x_1+1))^2 + x_1^2$. Then f_1 is a proper convex and lower semicontinuous function in $(\mathbb{R}^2, d)$ (see [22]). We also define $f_j : \mathbb{R}^2 \to \mathbb{R}^2$ by $f_j(x_1, x_2) = 70jx_1^2, j = 2, 3$. Then f_j is a proper convex and lower semicontinuous function for each $j = 2, 3$. Let $\gamma_n = \frac{1}{n}, \beta_n^1 = \frac{4n+1}{100n+9}, \beta_n^2 = \frac{n}{2n+3}$ and $\lambda_n = \frac{2n}{n+1}$. Hence, for arbitrary $x_0, x_1, u \in \mathbb{R}^2$, we have the following algorithm;

$$\begin{cases} w_n = \frac{u}{n} \circ \frac{n-1}{n} x_n; \\ u_n = y_n^{(0)} = J_{\lambda_n}^1 \circ J_{\lambda_n}^2; \\ y_n^{(1)} = \frac{96n+8}{100n+9} u_n \circ \frac{4n+1}{100n+9} v_n^1; \\ y_n^2 = \frac{n+3}{2n+3} \circ \frac{n}{2n+3} v_n^2. \end{cases}$$

We test the iterative scheme for the following values of u and x_0 : Case I: $u = (0.2, 3)^T, x_0 = (5, 7)^T$,
Case II: $u = (-3, 4)^T, x_0 = (10, 20)^T$,
Case III: $u = (5, 5)^T, x_0 = (1, 3)^T$,
Case IV: $u = (-1, 1)^T, x_0 = (20, 5)^T$.
Using $\|x_{n+1} - x_n\| < 10^{-4}$ as stopping criterion, we show that the change in the initial values does not have significant effect on the performance of the iterative scheme.

4.3. **Conclusion.** The concept of end point for the class of multi-valued mappings have been used extensively both in linear and nonlinear spaces to approximate solutions of fixed point problems. The end point condition has a very strong (strict) condition which can be dispense with using the gate condition. In this paper, using

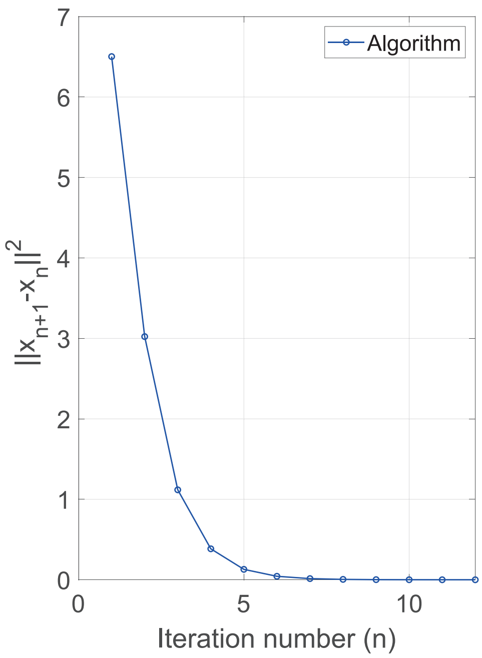

FIGURE 1. Case I, Time: 0.0087secs.

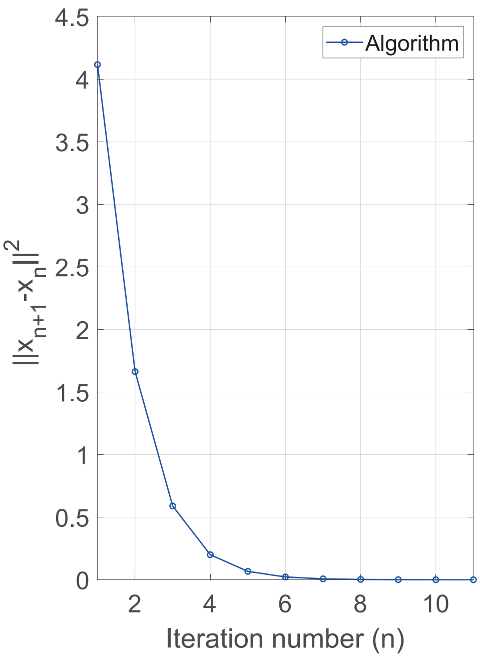

FIGURE 2. Case II: Time : 0.0032secs.

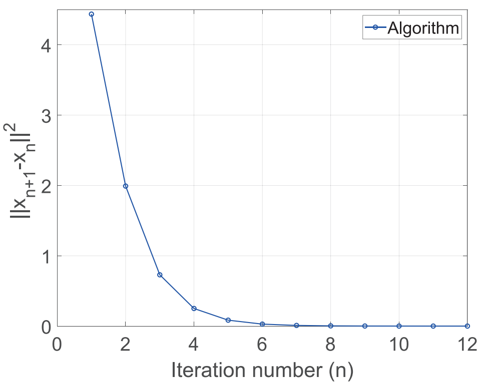

FIGURE 3. Case III, Time: 0.0033secs.

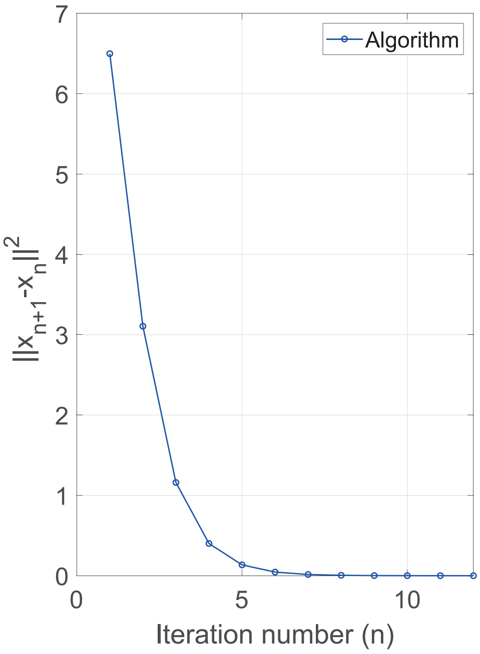

FIGURE 4. Case IV, Time: 0.0045secs.

the concept of gate condition for multi-valued mappings, we introduce a modified proximal point algorithm combined with a Halpern iteration process for approximating a common element of the set of minimizers of a finite family of convex functions and common fixed points of a finite family of k-strictly pseudononspreading multi-valued mappings in Hadamard spaces. We prove a strong convergence result without imposing the strict condition of compactness for solving the aforementioned problem. An application to a finite family of convex feasibility and fixed point problems for a finite family of quasi-nonexpansive mappings was discussed.

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