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A NONLINEAR MEAN CONVERGENCE THEOREM FOR TWO MONOTONE NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Dedicated to the memory of Professor Wataru Takahashi with deep gratitude.

Abstract. In this paper, we prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

1. Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T, i.e.,

$$F(T) = \{z \in C : Tz = z\}.$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. The fixed point theory for such mappings is rich and varied. It finds many applications in nonlinear functional analysis. The existence of fixed points for nonexpansive mappings in Banach and metric spaces has been investigated since the early 1960s (For example, see [5, 6, 8, 10, 12]). Among other things, in 1975, Baillon [3] proved the following first nonlinear mean convergence theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$,

$$\{S_n x\} = \left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$$

converges weakly to a fixed point of T (see also [19]).

In recent years, a new direction has been very active essentially after the publication of Ran and Reurings results [16]. They proved an analogue of the classical Banach contraction principle [4] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [11, 15, 20, 21]. Bin Dehaish and Khamsi [7] proved a

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weak convergence theorem of Mann's type [14] for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [14, 17]). Shukla and Wiśnicki [18] obtained a nonlinear mean convergence theorem for monotone nonexpansive mappings in such Banach spaces.

In this paper, we prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

2. Preliminaries and notations

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x + z \preceq y + z,$$
$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow] = \{z \in E : x \preceq z\}$ and $[\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow]$ and $[\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets

$$[a, \rightarrow] = \{x \in X; a \preceq x\} \quad \text{or} \quad [\leftarrow, a] = \{x \in X; x \preceq a\}.$$

for any $a \in E$. As a direct consequence of this, the subset

$$[a, b] = \{x \in X; a \preceq x \preceq b\} = [a, \rightarrow] \cap [\leftarrow, b]$$

is also closed and convex for each $a, b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E. Let C be a nonempty subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \to C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x, y \in C$ such that $x \preceq y$. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the set of all positive integers and the set of all nonnegative integers, respectively. We also denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and the set of all nonnegative real numbers, respectively. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or $\text{w-}\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, $\text{co}A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

A Banach space E is said to be strictly convex if

$$\frac{\|x+y\|}{2} < 1$$

for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon \right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\| \frac{x+y}{2} \right\| \leq r \left(1 - \delta\left(\frac{\varepsilon}{r}\right) \right)$$

for every $x, y \in E$ with $\|x\| \leq r$, $\|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in E : \|x\| = 1\}$ be a unit sphere in a Banach space E.

3. Monotone and approximate fixed point sequences

In this section, we study approximate fixed point sequences and monotone sequences. Let C be a nonempty subset of E and let T be a mapping of C into E. A sequence $\{x_n\}$ in C is said to be an *approximate fixed point sequence* of a mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(see also [13, 19]). A sequence $\{x_n\}$ in E is said to be *monotone* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [7]). The following lemma was obtained by the author and Takahashi [1] (see also [2].

Lemma 3.1 ([1]). *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself with $ST = TS$. Then,*

$$\lim_{n\to\infty} \sup_{x\in C} \left\| \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x - T \left(\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x \right) \right\| = 0$$

and

$$\lim_{n\to\infty} \sup_{x\in C} \left\| \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x - S \left(\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x \right) \right\| = 0.$$

The following theorem was proved by Browder [9].

Theorem 3.2 ([9]). *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element u in C and $\{x_n - Tx_n\}$ converges strongly to 0. Then, u is a fixed point of T.*

Using Theorem 3.2, we can prove the following result which is crucial in this paper.

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself. Let $\{x_n\}$ be a sequence in C which is a monotone, and approximate fixed point sequence of T and S, i.e.,*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Then, then the sequence $\{x_n\}$ converges weakly to a point of $F(S) \cap F(T)$.

Proof. Let w be a common fixed point of T and S. Since E is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. Let z_1, z_2 be two weak cluster-points of $\{x_n\}$. Then, there exists two subsequences of $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$, respectively. By the assumption, we remark

$$\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = \lim_{i\to\infty} \|x_{n_i} - Sx_{n_i}\| = 0$$

and

$$\lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = \lim_{j\to\infty} \|x_{n_j} - Sx_{n_j}\| = 0.$$

It follows from Theorem 3.2 that $z_1, z_2 \in F(T) \cap F(S)$ (see also [1]). Next, we show $z_1 = z_2$ (see also [7]). Fix $k \geq 1$. Since $\{x_n\}$ is monotone and the order interval $[x_k, \rightarrow)$ is weakly closed, we conclude that $z_i \in [x_k, \rightarrow)$ for $i = 1, 2$. So, we see that z_i is an upper bound for $\{x_n\}$ for $i = 1, 2$. Then, we also obtain that $\{x_n\} \subset (\leftarrow, z_i]$ for $i = 1, 2$. It follows from the same reason that $z_j \in (\leftarrow, z_i]$ for $i, j = 1, 2$. So, we have $z_1 = z_2$. Therefore, we obtain that $\{x_n\}$ converges weakly to a point of $F(S) \cap F(T)$. □

4. Nonlinear mean convergence theorems

In this section, we show nonlinear mean convergence theorems for monotone nonexpansive mappings. Using Lemma 3.1, we can prove the following lemma which plays an important role in our results.

Lemma 4.1. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Let $x \in C$. For each $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$, let*

$$U_n^{(m)} x = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^{k+m} T^{l+m} x.$$

Then, the sequence $\{U_n^{(m)}x\}_{n=1}^{\infty}$ *in* C *is an approximate fixed point sequence of* S *and* T *uniformly in* $m \in \mathbb{Z}^+$.

Proof. For $x \in C$ and $f \in F(T) \cap F(S)$, put $r = \|x - f\|$ and set $X = \{u \in E : \|u - f\| \leq r\} \cap C$. Then, X is a nonempty bounded closed convex subset of C which is T, S-invariant and contains x. So, without loss of generality, we may assume that C is bounded.

Since $\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^{i+k}T^{j+l}x = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^iT^j(S^kT^lx)$, from Lemma 3.1, we have

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{k,l\in\mathbb{Z}^+}\sup_{x\in C}\left\|\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^{i+k}T^{j+l}x - T\left(\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^{i+k}T^{j+l}x\right)\right\| \\
(4.1)\quad &= \lim_{n\to\infty}\sup_{k,l\in\mathbb{Z}^+}\sup_{x\in C}\left\|\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^iT^j(S^kT^lx) - T\left(\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^iT^j(S^kT^lx)\right)\right\| \\
&= 0
\end{aligned}$$

Since $U_n^{(m)}x = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{i=0}^{n-1} S^{i+m}T^{j+m}x$, from (4.1), we have

$$\lim_{n\to\infty}\sup_{m\in\mathbb{Z}^+}\left\|U_n^{(m)}x - TU_n^{(m)}x\right\| = 0.$$

Similarly, we also have

$$\lim_{n\to\infty}\sup_{m\in\mathbb{Z}^+}\left\|U_n^{(m)}x - SU_n^{(m)}x\right\| = 0.$$

So, we can conclude that $\{U_n^{(m)}x\}_{n=1}^{\infty}$ is an approximate fixed point sequence of T and S uniformly in $m \in \mathbb{Z}^+$. □

Lemma 4.2. *Let* C *be a nonempty closed convex subset of an ordered Banach space* E*. Let* S *and* T *be monotone nonexpansive mappings of* C *into itself such that* $ST = TS$ *and* $F(S) \cap F(T) \neq \emptyset$*. Assume that* $x \preceq Sx$ *and* $x \preceq Tx$ *for each* $x \in C$*. Let* $x \in C$*. For each* $m \in \mathbb{Z}^+$*, let*

$$U_n^{(m)}x = \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^{k+m}T^{l+m}x.$$

Then, for each $m \in \mathbb{Z}^+$*, the sequence* $\{U_n^{(m)}x\}_{n=1}^{\infty}$ *in* C *is monotone.*

Proof. Since a partial order is compatible with the linear structure of X, it is not difficult to see that

$$U_n^{(1)}x \preceq U_n^{(2)}x \preceq U_n^{(3)}x \preceq \cdots$$

for each $n \in \mathbb{N}$ and we also obtain that

$$\begin{aligned}
&U_n^{(m)}x \\
&= \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(n+1)^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x \\
&= \frac{1}{(n+1)^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x + \left(\frac{n}{n^2(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x \\
&\quad + \left(\frac{n}{n^2(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x + \left(\frac{1}{n^2(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x \\
&\preceq \frac{1}{(n+1)^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x + \left(\frac{n}{n^2(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{n+m}T^{j+m}x \\
&\quad + \left(\frac{n}{n^2(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{n+m}x + \left(\frac{1}{n^2(n+1)^2}\right)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{n+m}T^{n+m}x \\
&= \frac{1}{(n+1)^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^{i+m}T^{j+m}x + \frac{n\cdot n}{n^2(n+1)^2}\sum_{j=0}^{n-1} S^{n+m}T^{j+m}x \\
&\quad + \frac{n\cdot n}{n^2(n+1)^2}\sum_{i=0}^{n-1} S^{i+m}T^{n+m}x + \frac{n\cdot n}{n^2(n+1)^2} S^{n+m}T^{n+m}x \\
&= \frac{1}{(n+1)^2}\sum_{i=0}^{n}\sum_{j=0}^{n} S^{i+m}T^{j+m}x = U_{n+1}^{(m)}x
\end{aligned}
$$

for $m \in \mathbb{Z}^+$ and $n \in \mathbb{N}$. So, we can conclude that the sequence $\{U_n^{(m)}x\}_{n=1}^{\infty}$ in C is monotone. □

We can prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings.

Theorem 4.3. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Then,*

$$
\left\{ \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l x \right\}
$$

converges weakly to a point of $F(S) \cap F(T)$.

Proof. It follows from Theorem 3.3 and Lemmas 4.1 and 4.2 that for $m \in \mathbb{Z}^+$, $\left\{ \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^{k+m}T^{l+m}x \right\}_{n=1}^{\infty}$ converges weakly to a point of $F(S) \cap F(T)$. So, especially, we also have $\left\{ \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} S^k T^l x \right\}_{n=1}^{\infty}$ converges weakly to a point of $F(S) \cap F(T)$. □

Using Theorem 4.3, we get some convergence theorems for monotone nonexpansive mappings in ordered uniformly convex Banach spaces (see [18]).

Theorem 4.4 ([18])**.** *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{S_n x\} = \{\frac{1}{n}\sum_{k=0}^{n-1} T^k x\}$ converges weakly to a point of $F(T)$.*

Theorem 4.5 ([18])**.** *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{T^n x\}$ converges weakly a point of $F(T)$.*

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