Linear and Nonlinear Analysis
Volume 8, Number 2, 2022, 147–154
Yokohama Publishers
ISSN 2188-8167

A CHARACTERIZATION OF QUASI-ARITHMETIC SET-VALUED MEANS

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Dedicated to the memory of Professor Wataru Takahashi

Abstract. The notion of set-valued means is investigated. A characterization of quasi-arithmetic set-valued means and a comparison property for such means are presented. Conditions under which the arithmetic mean is the selection of a quasi-arithmetic set-valued mean are given.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $n \geq 2$ be an integer. A function $M : I^n \to I$ is said to be a mean if

$$\min\{x_1, \ldots, x_n\} \leq M(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\}, \tag{1.1}$$

for all $x_1, \ldots, x_n \in I$. The theory of means is an extensive mathematical theory having various applications in mathematics itself as well as in economics, engineering, social and natural science. There are books, survey papers and and numerous articles devoted to it (see e.g. [1, 2, 3, 4, 5, 6, 8] and the references given there).

In [7] the above classical definition of mean has been extended to the case of set-valued means in vector spaces. Since all measurements carried out in reality are made with some errors, sometimes it is better to replace a single-valued function by a set-valued one. Therefore such a novel approach to the idea of means seems to be accurate and may have useful applications.

Let X be a real vector space and D be a convex nonempty subset of X. Denote by $S(D)$ the family of all nonempty subsets of D. A map $M : D^n \to S(D)$ is called the *set-valued mean* if

$$M(x_1, \ldots, x_n) \subset \mathrm{conv}\{x_1, \ldots, x_n\}, \tag{1.2}$$

for all $x_1, \ldots, x_n \in D$.

Clearly, if $X = \mathbb{R}$, $D = I$ and M is single-valued, then condition (1.2) reduces to (1.1). A few natural examples of set-valued means are presented below.

Example 1. Given $x = (x_1, \ldots, x_n) \in D^n$ we put $\overline{x} := \frac{1}{n}(x_1 + \cdots + x_n)$ and, for every $t \in [0, 1]$, define $M_t : D^n \to S(D)$ by

$$M_t(x_1, \ldots, x_n) = t\ \mathrm{conv}\{x_1, \ldots, x_n\} + (1 - t)\ \overline{x}, \quad x_1, \ldots, x_n \in D. \tag{1.3}$$

2020 *Mathematics Subject Classification.* Primary 26E60. Secondary 26A51, 26E25, 39B62.
Key words and phrases. Means, set-valued maps, set-valued means, quasi-arithmetic means, inequalities between means, convex functions.

It is clear that M_t are set-valued means and

$$\overline{x} = M_0(x_1, \ldots, x_n) \subset M_t(x_1, \ldots, x_n) \subset M_1(x_1, \ldots, x_n) = \operatorname{conv}\{x_1, \ldots, x_n\},$$

for every $t \in [0,1]$ (cf. [7]).

Let $m_t : I^n \to I$, $t \in T$, be a family of single-valued means. Then $M : I^n \to S(I)$ defined by

$$M(x_1, \ldots, x_n) = \{m_t(x_1, \ldots, x_n) : t \in T\}, \quad x_1, \ldots, x_n \in I,$$

is a set-valued mean.

Let $m : I^n \to I$ be a single-valued mean and $\varepsilon > 0$. Then $M : I^n \to S(I)$ defined by

$$M(x_1, \ldots, x_n) = \big(m(x_1, \ldots, x_n) + (-\varepsilon, \varepsilon)\big) \cap [\min\{x_1, \ldots, x_n\}, \max\{x_1, \ldots, x_n\}]$$

is a set-valued mean.

A large and important class of means are the quasi-arithmetic means $A_f : I^n \to I$ defined by

$$A_f(x_1, \ldots, x_n) = f^{-1}\Big(\frac{f(x_1) + \cdots + f(x_n)}{n}\Big),$$

where the generating function $f : I \to J$ is continuous and strictly monotonic and $I, J \subset \mathbb{R}$ are intervals. In a similar way we can define a set-valued counterpart of the quasi arithmetic means putting, for a given set-valued map $F : I \to S(J)$,

$$A_F(x_1, \ldots, x_n) = F^+\Big(\frac{F(x_1) + \cdots + F(x_n)}{n}\Big),$$

where $F^+(B) := \{x \in I : F(x) \subset B\}$. However, without any additional assumptions, A_F need not be a set-valued mean (see [7, Examples 2 and 3]).

The following result proved in [7] gives conditions under which A_F is a set-valued mean:

Theorem 1.1. *Let $f, g : I \to J$ be strictly increasing functions such that f is concave and g is convex. Assume that $f \le g$ on I and $F(x) = [f(x), g(x)]$ for all $x \in I$. Then, for every $n \ge 2$, the map $A_F : I^n \to S(I)$ given by*

$$A_F(x_1, \ldots, x_n) = F^+\Big(\frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big), \quad x_1, \ldots, x_n \in I,$$

is a set-valued mean.

However, it turns out that the assumptions on f and g in the above theorem are sufficient but not necessary conditions for A_F to be a set-valued mean. For instance, the functions $f(x) = -1/x$, $g(x) = \ln x$, $x > 0$, do not satisfy the above assumptions, but A_F generated by $F(x) = [f(x), g(x)]$, $x > 0$, is a set-valued mean (see Example 2 below).

The aim of this paper is to present a full characterization of set-valued quasi-arithmetic means generated by $F = [f, g]$. Examples and further properties of set-valued quasi-arithmetic means are also presented.

2. A characterization

Let us start by recalling the classical comparison property due to Hardy, Littlewood and Pólya [5, Thm. 92], which will be needed in our investigations.

Lemma 2.1. *Let $f, g : I \to J$ be continuous and strictly monotonic. If g is increasing (decreasing), then $A_f \leq A_g$ on I^n if and only if $g \circ f^{-1}$ is convex (concave).*

The following characterization theorem is the main result of this paper.

Theorem 2.2. *Let $f, g : I \to J$ be continuous and strictly increasing (strictly decreasing) functions. Assume that $f \leq g$ on I and $F(x) = [f(x), g(x)], \quad x \in I$. Then, the following conditions are equivalent:*

1. *For every $n \geq 2$, the map $A_F : I^n \to S(I)$ given by*

$$A_F(x_1, \ldots, x_n) = F^+\Big(\frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big), \quad x_1, \ldots, x_n \in I, \tag{2.1}$$

is a set-valued mean.

2. *2. The map $A_F : I^2 \to S(I)$ given by*

$$A_F(x_1, x_2) = F^+\Big(\frac{F(x_1) + F(x_2)}{2}\Big), \quad x_1, x_2 \in I, \tag{2.2}$$

is a set-valued mean.

3. *The function $g \circ f^{-1}$ is convex.*
4. *For every $n \geq 2$ and all $x_1, \ldots, x_n \in I$*

$$A_F(x_1, \ldots, x_n) = \begin{cases} [A_f(x_1, \ldots, x_n), A_g(x_1, \ldots, x_n)] & \text{if } f, g \text{ are increasing} \\ [A_g(x_1, \ldots, x_n), A_f(x_1, \ldots, x_n)] & \text{if } f, g \text{ are decreasing.} \end{cases} \tag{2.3}$$

Proof. Implication $1 \Rightarrow 2$ is clear.

$2 \Rightarrow 3$. Since A_F defined by (2.2) is a set-valued mean, $A_F(x_1, x_2) \neq \emptyset$ for all $x_1, x_2 \in I$. Fix $x_1, x_2 \in I$ and take any $z \in A_F(x_1, x_2)$. Then

$$[f(z), g(z)] = F(z) \subset \frac{F(x_1) + F(x_2)}{2} = \Big[\frac{f(x_1) + f(x_2)}{2}, \frac{g(x_1) + g(x_2)}{2}\Big],$$

which implies

$$\frac{f(x_1) + f(x_2)}{2} \leq f(z) \quad \text{and} \quad g(z) \leq \frac{g(x_1) + g(x_2)}{2}. \tag{2.4}$$

Denote $y_1 = f(x_1), y_2 = f(x_2)$. Since f^{-1} and g are both strictly increasing (or strictly decreasing), by (2.4) we get

$$g\Big(f^{-1}\Big(\frac{y_1 + y_2}{2}\Big)\Big) \leq g(z) \leq \frac{g(f^{-1}(y_1)) + g(f^{-1}(y_2))}{2},$$

which means that the function $g \circ f^{-1}$ is midconvex. Since it is continuous (because f and g are continuous), it is also convex.

To prove that $3 \Rightarrow 4$ assume first that f, g are strictly increasing. Since $g \circ f^{-1}$ is convex, by the comparison property (Lemma 2.1) $A_f \leq A_g$ on I^n. Let $z \in [A_f(x_1, \ldots, x_n), A_g(x_1, \ldots, x_n)]$. Then

$$f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big) \leq z \leq g^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\Big)$$

and, by the monotonicity of f, g

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \leq f(z) \quad \text{and} \quad g(z) \leq \frac{1}{n}\sum_{i=1}^{n} g(x_i).$$

Hence

$$F(z) = [f(z), g(z)] \subset \Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i), \frac{1}{n}\sum_{i=1}^{n} g(x_i)\Big] = \frac{1}{n}\sum_{i=1}^{n} F(x_i),$$

which implies

$$z \in F^{+}\Big(\frac{1}{n}\sum_{i=1}^{n} F(x_i)\Big) = A_F(x_1, \ldots, x_n),$$

and shows that

$$[A_f(x_1, \ldots, x_n), A_g(x_1, \ldots, x_n)] \subset A_F(x_1, \ldots, x_n). \tag{2.5}$$

To prove the reverse inclusion take any $z \in A_F(x_1, \ldots, x_n)$ ($A_F(x_1, \ldots, x_n) \neq \emptyset$ by the previous step). Then

$$F(z) \subset \frac{1}{n}\sum_{i=1}^{n} F(x_i) = \Big[\frac{1}{n}\sum_{i=1}^{n} f(x_i), \frac{1}{n}\sum_{i=1}^{n} g(x_i)\Big],$$

which implies

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \leq f(z) \quad \text{and} \quad g(z) \leq \frac{1}{n}\sum_{i=1}^{n} g(x_i).$$

and hence

$$f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big) \leq z \leq g^{-1}\Big(\frac{1}{n}\sum_{i=1}^{n} g(x_i)\Big).$$

This show that

$$A_F(x_1, \ldots, x_n) \subset [A_f(x_1, \ldots, x_n), A_g(x_1, \ldots, x_n)],$$

which together with (2.5) proves $A_F(x_1, \ldots, x_n) = [A_f(x_1, \ldots, x_n), A_g(x_1, \ldots, x_n)]$.

Now, assume that f, g are strictly decreasing. Then $g \circ f^{-1}$ is strictly increasing. Since, by the assumption, $g \circ f^{-1}$ is convex, $f \circ g^{-1} = \big(g \circ f^{-1}\big)^{-1}$ is concave. Hence, by Lemma 2.1, $A_g \leq A_f$ on I^n. The proof that $A_F(x_1, \ldots, x_n) = [A_g(x_1, \ldots, x_n), A_f(x_1, \ldots, x_n)]$ is analogous as previously.

The last implication $4 \Rightarrow 1$ is clear. Indeed, if A_F is of the form (2.3), then $A_F(x_1,\dots,x_n) \neq \emptyset$ for all $x_1,\dots,x_n \in I$, and

$$A_F(x_1,\dots,x_n) \subset [\min\{x_1,\dots,x_n\}, \max\{x_1,\dots,x_n\}] = \operatorname{conv}\{x_1,\dots,x_n\},$$

because, being means,

$$A_f(x_1,\dots,x_n), A_g(x_1,\dots,x_n) \in [\min\{x_1,\dots,x_n\}, \max\{x_1,\dots,x_n\}]$$

for all $x_1,\dots,x_n \in I$. This shows that A_F is a set-valued mean and finishes the proof.

□

Remark 2.3. The functions f and g in the above theorem must be both strictly increasing or both strictly decreasing. If this condition is not satisfied, then the map A_F generated by $F = [f, g]$ need not be a set-valued mean. For example, let $f(x) = -x$ and $g(x) = x$, $x \in [0,\infty)$. Then f is strictly decreasing, g is strictly increasing and $g \circ f^{-1}$ is convex. However, A_F generated by $F(x) = [-x, x]$, $x \geq 0$, is not a set-valued mean. Indeed, for any $x > 0$ we have

$$A_F(x,x) = F^+\Big(\frac{F(x)+F(x)}{2}\Big) = F^+([-x,x]) = [0,x] \not\subseteq \{x\} = \operatorname{conv}\{x,x\}.$$

To present some examples of set-valued quasi-arithmetic means consider the classical harmonic, geometric, arithmetic and quadratic means defined for $x_1,\dots,x_n > 0$ by

$$H(x_1,\dots,x_n) = \frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\ , \quad G(x_1,\dots,x_n) = \sqrt[n]{x_1\cdots x_n}\ ,$$

$$A(x_1,\dots,x_n) = \frac{x_1+\cdots+x_n}{n}\ , \quad K(x_1,\dots,x_n) = \sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}.$$

All these means are quasi-arithmetic means generated by the functions $f(x) = -1/x$, $f(x) = \ln x$, $f(x) = x$ and $f(x) = x^2$, respectively. It is known that

$$H(x_1,\dots,x_n) \leq G(x_1,\dots,x_n) \leq A(x_1,\dots,x_n) \leq K(x_1,\dots,x_n), \tag{2.6}$$

for all $x_1,\dots,x_n > 0$.

Given two means $M, N : I^n \to I$ with $M \leq N$ we denote by MN the set-valued mean defined by

$$MN(x_1,\dots,x_n) = \big[M(x_1,\dots,x_n), N(x_1,\dots,x_n)\big],\ x_1,\dots,x_n \in I.$$

Example 2. Let $f(x) = -1/x$, $g(x) = \ln x$, $x > 0$ and $F_1(x) = [f(x), g(x)], x > 0$. Then f, g are strictly increasing, $f < g$ and $g \circ f^{-1}(x) = -\ln(-x)$ is convex on $(-\infty, 0)$. Therefore A_{F_1} is an quasi-arithmetic set-valued mean and

$$A_{F_1}(x_1,\dots,x_n) = [A_f(x_1,\dots,x_n), A_g(x_1,\dots,x_n)] = HG(x_1,\dots,x_n),\ x_1,\dots,x_n > 0.$$

Similarly, taking one by one $F_2(x) = [\ln x, x^2]$, $F_3(x) = \big[-\frac{1}{x}, x\big]$, $F_4(x) = [\ln x, x]$, $F_5(x) = [x, x^2+1]$ and $F_6(x) = [-\frac{1}{x}, x^2]$, $x > 0$, we obtain

$$A_{F_2}(x_1,\dots,x_n)) = GK(x_1,\dots,x_n), \quad A_{F_3}(x_1,\dots,x_n)) = HA(x_1,\dots,x_n),$$

$$A_{F_4}(x_1,\dots,x_n)) = GA(x_1,\dots,x_n), \quad A_{F_5}(x_1,\dots,x_n)) = AK(x_1,\dots,x_n),$$

and

$$A_{F_6}(x_1,\dots,x_n)) = HK(x_1,\dots,x_n), \quad \text{for all} \quad x_1,\dots,x_n > 0.$$

3. Further properties

In this section we discuss some more results on quasi-arithmetic set-valued means. In particular we present a comparison property for such means and give conditions under which the arithmetic mean is a selection of a given set-valued quasi-arithmetic mean. As a consequence we obtain an alternative proof of certain known inequalities between classical means.

Theorem 3.1. *Let $f_1, f_2, g_1, g_2 : I \to J$ be continuous and strictly increasing (strictly decreasing). Assume that $f_1 \le f_2$ and $g_1 \le g_2$ on I. Put $F(x) = [f_1(x), f_2(x)], G(x) = [g_1(x), g_2(x)], x \in I$ and assume that $A_F, A_G : I^n \to S(I)$ are set-valued means. Then*

$$A_F(x_1,\dots,x_n) \subset A_G(x_1,\dots,x_n), \ x_1,\dots,x_n \in I$$

if and only if $f_1 \circ g_1^{-1}$ and $g_2 \circ f_2^{-1}$ are convex.

Proof. Assume first that f_1, f_2, g_1, g_2 are strictly increasing. Since A_F and A_G are set-valued means, by Theorem 2.2 we obtain $A_F = [A_{f_1}, A_{f_2}]$ and $A_G = [A_{g_1}, A_{g_2}]$. The condition $A_F \subset A_G$ implies that $A_{f_1} \ge A_{g_1}$ and $A_{f_2} \le A_{g_2}$, which, by Lemma 2.1, means that $f_1 \circ g_1^{-1}$ and $g_2 \circ f_2^{-1}$ are convex. Conversely, if $f_1 \circ g_1^{-1}$ and $g_2 \circ f_2^{-1}$ are convex, then, by Lemma 2.1, $A_{f_1} \ge A_{g_1}$ and $A_{f_2} \le A_{g_2}$. Consequently,

$$\begin{aligned} A_F(x_1,\dots,x_n) &= [A_{f_1}(x_1,\dots,x_n), A_{f_2}(x_1,\dots,x_n)] \\ &\subset [A_{g_1}(x_1,\dots,x_n), A_{g_2}(x_1,\dots,x_n)] = A_G(x_1,\dots,x_n), \end{aligned}$$

for all $x_1,\dots,x_n \in I$.

Now assume that f_1, f_2, g_1, g_2 are strictly decreasing. Then, by Theorem 2.2, $A_F = [A_{f_2}, A_{f_1}]$ and $A_G = [A_{g_2}, A_{g_1}]$. Since $A_F \subset A_G$, we have $A_{f_2} \ge A_{g_2}$ and $A_{f_1} \le A_{g_1}$. Hence, by Lemma 2.1, $f_2 \circ g_2^{-1}$ and $g_1 \circ f_1^{-1}$ are concave (and strictly increasing). Therefore $g_2 \circ f_2^{-1} = \left(f_2 \circ g_2^{-1}\right)^{-1}$ and $f_1 \circ g_1^{-1} = \left(g_1 \circ f_1^{-1}\right)^{-1}$ are convex.

Conversely, if $f_1 \circ g_1^{-1}$ and $g_2 \circ f_2^{-1}$ are convex, then $g_1 \circ f_1^{-1}$ and $f_2 \circ g_2^{-1}$ are concave. Therefore by Lemma 2.1, $A_{f_1} \le A_{g_1}$ and $A_{g_2} \le A_{f_2}$, and hence

$$\begin{aligned} A_F(x_1,\dots,x_n) &= [A_{f_2}(x_1,\dots,x_n), A_{f_1}(x_1,\dots,x_n)] \\ &\subset [A_{g_2}(x_1,\dots,x_n), A_{g_1}(x_1,\dots,x_n)] = A_G(x_1,\dots,x_n), \end{aligned}$$

for all $x_1,\dots,x_n \in I$. This completes the proof. □

As an immediate consequence of the above theorem we get the following corollary.

Corollary 3.2. *Let $f_1, f_2, g_1, g_2 : I \to J$ be continuous and strictly increasing (strictly decreasing). Assume that $f_1 \le f_2$ and $g_1 \le g_2$ on I. Put $F(x) =$*

$[f_1(x), f_2(x)],\ G(x) = [g_1(x), g_2(x)],\ x \in I$, and assume that $A_F, A_G : I^n \to S(I)$ are set-valued means. Then

$$A_F(x_1, \dots, x_n) = A_G(x_1, \dots, x_n),\ x_1, \dots, x_n \in I$$

if and only if $f_1 \circ g_1^{-1}$ and $g_2 \circ f_2^{-1}$ are affine.

We say that a mean $m : I^n \to I$ is the selection of a set-valued mean $M : I^n \to S(I)$ if

$$m(x_1, \dots, x_n) \in M(x_1, \dots, x_n),\ x_1, \dots, x_n \in I.$$

Theorem 3.3. *Let $f, g : I \to J$ be continuous and strictly increasing (strictly decreasing). Assume that $f \le g$ on I and put $F(x) = [f(x), g(x)],\ x \in I$. Then $A_F : I^n \to I$ is a set-valued mean and the arithmetic mean A is a selection of A_F if and only if f is concave and g is convex.*

Proof. Assume first that f and g are strictly increasing. If A_F is a set-valued mean and A is its selection, then, by Theorem 2.2, $A_F = [A_f, A_g]$ and

$$f^{-1}\Big(\frac{f(x_1) + \dots + f(x_n)}{n}\Big) \le \frac{x_1 + \dots + x_n}{n} \le g^{-1}\Big(\frac{g(x_1) + \dots + g(x_n)}{n}\Big).$$

From here

$$\frac{f(x_1) + \dots + f(x_n)}{n} \le f\Big(\frac{x_1 + \dots + x_n}{n}\Big)$$

and

$$g\Big(\frac{x_1 + \dots + x_n}{n}\Big) \le \frac{g(x_1) + \dots + g(x_n)}{n}.$$

Since f and g are continuous, this imply the concavity of f and the convexity of g. Conversely, if f is concave and g is convex, then $g \circ f^{-1}$ is convex. Thus from Theorem 2.2 we get that A_F is a set-valued mean and $A_F = [A_f, A_g]$. Let $h(x) = x,\ x \in I$. Then $h \circ f^{-1}$ and $g \circ h^{-1}$ are convex. Therefore, using Lemma 2.1 we obtain $A_f \le A_h = A$ and $A = A_h \le A_g$. Consequently,

$$A(x_1, \dots, x_n) \in [A_f(x_1, \dots, x_n), A_g(x_1, \dots, x_n)] = A_F(x_1, \dots, x_n), \quad x_1, \dots, x_n \in I,$$

which proves that A is a selection of A_F.
The proof in the case where f and g are decreasing is similar (note only that $A_F = [A_g, A_f]$ in this case) and we omit it. □

As an immediate consequence of the above theorem we can obtain in an alternative way certain known inequalities between classical means.

Example 3. Let $f(x) = -1/x$, $g(x) = x^2$ and $F(x) = [f(x), g(x)],\ x > 0$. Then $A_F = HK$ and, by Theorem 3.3, A is a selection of A_F. Hence

$$\frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \le \frac{x_1 + \dots + x_n}{n} \le \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}}, \quad x_1, \dots, x_n > 0.$$

Let $f(x) = \ln x$, $g(x) = x^2$ and $F(x) = [f(x), g(x)]$, $x > 0$. Then $A_F = GK$ and, by Theorem 3.3, A is a selection of A_F. Hence

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}, \quad x_1, \ldots, x_n > 0.$$

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Manuscript received 4 March 2022
revised 17 December 2022

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