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STRONG CONVERGENCE ANALYSIS OF A SELF ADAPTIVE METHOD FOR SOLVING SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS

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Abstract. In 2020, Reich et al [34] studied the split feasibility problem with multiple output sets in Hilbert spaces. They proposed two new algorithms for solving this problem which mainly requires estimation of the operator norm and computation of projections onto closed convex sets, which is not easy to do so in general. This paper studies the split feasibility problem with multiple output sets in general Hilbert spaces. For solving the aforementioned problem, we propose a new self-adaptive relaxed CQ algorithm which involves computing of projections onto relaxed sets (half-spaces) instead of computing onto the closed convex sets, and it does not require calculating the operator norm. In addition, we present some newly derived results for solving the split feasibility problem. We establish a strong convergence theorem for our proposed algorithm. Finally, we provide some numerical experiments to illustrate the implementation and applicability of our proposed algorithm compared to some existing results. Our results extend and improve some methods in the literature.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $B : H_1 \to H_2$ be a nonzero bounded linear operator and let $B^* : H_2 \to H_1$ be its adjoint. The *split feasibility problem (SFP)* is formulated to find a point $x^* \in H_1$ satisfying

$$x^* \in C \text{ such that } Bx^* \in Q. \tag{1.1}$$

The SFP was first introduced in 1994 by Censor and Elfving [10] in finite-dimensional Hilbert spaces for modeling certain inverse problems and has received a great attention since then. This is because the SFP can be used to model several inverse problems arising from, for example, phase retrievals and in medical image reconstruction [10, 5], intensity-modulated radiation therapy (IMRT) [11, 9, 13], gene regulatory network inference [45], just to mention but few, for more details one can, see, e.g., [1, 50, 6, 43, 49, 51, 23, 37] and references therein. In the span of the last twenty five years, focusing on real world applications, several iterative methods for solving the SFP (1.1) have been introduced and analyzed. Among them,

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Byrne [5, 6] introduced the first applicable and most celebrated method called the well-known *CQ-algorithm* as follows: for any initial guess $x_0 \in H_1$;

$$x_{n+1} := P_C(x_n - \tau_n B^*(I - P_Q)Bx_n)), \tag{1.2}$$

where P_C and P_Q are the metric projections onto C and Q, respectively and $\tau_n \in \left(0, \frac{2}{\|B\|^2}\right)$ where $\|B\|^2$ is the spectral radius of the matrix B^*B. The CQ algorithm proposed by Byrne [5, 6], requires the computation of metric projection onto the sets C and Q (in some cases, it is impossible or is too expensive to exactly compute the metric projection). In addition, the determination of the stepsize depends on the operator norm in which computation (or at least estimation) of operator norm is not an easy task. In practical applications, the sets C and Q are usually the level sets of convex functions which are given by

$$C := \{x \in H_1 : c(x) \leq 0\} \text{ and } Q = \{y \in H_2 : q(y) \leq 0\}, \tag{1.3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and subdifferentiable functions on H_1 and H_2, respectively, and that (generalized gradients) $\partial c(x)$ and $\partial q(y)$ of c and q, respectively, defined by

$$\partial c(x) := \{\xi \in H_1 : c(z) \geq c(x) + \langle \xi, z - x\rangle, \quad \text{for each } z \in H_1\}$$

and

$$\partial q(y) := \{\eta \in H_2 : q(u) \geq q(y) + \langle \eta, u - y\rangle, \quad \text{for each } u \in H_2\}$$

are bounded operators (i.e., bounded on bounded sets).

Later, in 2004, Yang [51] generalized the CQ method to the so-called *relaxed CQ algorithm*, needing computation of the metric projection onto (relaxed sets) *half-spaces* C_n and Q_n, where

$$C_n := \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\}, \tag{1.4}$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n := \{y \in H_2 : q(Bx_n) \leq \langle \eta_n, Bx_n - y\rangle\}, \tag{1.5}$$

where $\eta_n \in \partial q(Bx_n)$. It is easy to see that $C_n \supseteq C$ and $Q_n \supseteq Q$ for all $n \geq 1$. Moreover, it is known that projections onto half-spaces C_n and Q_n have closed forms. In what follows, define

$$f_n(x_n) := \frac{1}{2}\|(I - P_{Q_n})Bx_n\|^2, \tag{1.6}$$

where Q_n is given as in (1.5) and f_n is a convex and differentiable function with its gradient ∇f_n defined by

$$\nabla f_n(x_n) := B^*(I - P_{Q_n})Bx_n. \tag{1.7}$$

More precisely, Yang [51] introduced the following relaxed CQ algorithm for solving the SFP (1.1) in a finite-dimensional Hilbert space: for any initial guess $x_0 \in H_1$;

$$x_{n+1} := P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{1.8}$$

where $\tau_n \in \left(0, \frac{2}{\|B\|^2}\right)$. Since P_{C_n} and P_{Q_n} are easily calculated, this method appears to be very practical. However, to compute the norm of B turns out to be complicated and costly. To overcome this difficulty, in 2012, López et al. [30] introduced a relaxed CQ algorithm for solving the SFP (1.1) with a new adaptive way of determining the stepsize sequence τ_n defined as follows:

$$\tau_n := \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \tag{1.9}$$

where $\{\rho_n\} \in (0,4), \forall n \geq 1$ such that $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$. It was proved that the sequence $\{x_n\}$ generated by (1.8) with τ_n defined by (1.9) converges weakly to a solution of the SEP (1.1). That is, their algorithm has only weak convergence in the framework of infinite-dimensional Hilbert spaces.

Many authors also proposed algorithms that generate a sequence $\{x_n\}$ converges strongly to a point in the solution set of the SFP (1.1), see, e.g., [30, 19, 25, 52, 40]. In particular, Deepho and Kumam [19] proposed a modified *Halpern's iterative* scheme for solving the SFP (1.1) in the setting of infinite-dimensional Hilbert spaces as follows: for any fixed point $u \in H_1$ and any initial guess $x_0 \in H_1$;

$$x_{n+1} := \beta_n u + \delta_n x_n + \gamma_n P_C\Big(x_n - \tau_n B^*(I-P_Q)Bx_n\Big), \forall n \geq 1, \tag{1.10}$$

where $\tau_n \in \Big(0,\ \frac{2}{\|B\|^2}\Big)$ and $\{\beta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0,1]$ such that $\beta_n+\delta_n+\gamma_n = 1$. Assuming that the SFP (1.1) is consistent, it was proved that, if $\{\beta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the assumptions: (**c1**) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$; (**c2**) $\limsup_{n\to\infty} \delta_n < 1$; (**c3**) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$, and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, then the sequence $\{x_n\}$ generated by (1.10) converges strongly to a solution of the SFP (1.1). However, their algorithm also requires to compute the operator norm, and the projections onto the sets C and Q which is not easy to do so. In 2012, López et al. [30] proposed a Halpern's iterative scheme for solving the SFP (1.1) in the setting of infinite-dimensional Hilbert spaces as follows: for any fixed point $u \in H_1$ and any initial guess $x_0 \in H_1$;

$$x_{n+1} = \beta_n u + (1-\beta_n)P_{C_n}\Big(x_n - \tau_n \nabla f_n(x_n)\Big), \forall n \geq 1, \tag{1.11}$$

and in 2013, He et al. [25] also introduced the following relaxed CQ algorithm for solving the SFP (1.1) such that strong convergence is guaranteed in infinite-dimensional Hilbert space: for any fixed point $u \in H_1$ and any initial guess $x_0 \in H_1$;

$$x_{n+1} := P_{C_n}\Big(\beta_n u + (1-\beta_n)\big(x_n - \tau_n \nabla g_n(x_n)\big)\Big), \tag{1.12}$$

where C_n (half-space) and τ_n (variable step size) are given as in (1.4) and (1.9), respectively, and the sequence $\{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$. Under certain suitable conditions, it was shown that the sequence $\{x_n\}$ generated by (1.11) and (1.12) converges strongly to the point $p = P_\Gamma(u)$. One can see that other related works can be found for example in [16, 22, 30, 17] and references therein.

Some generalizations of the SFP have also been studied by many authors. We mention, for instance, the multiple-sets SFP (MSSFP) [11, 54, 55, 29, 18, 28, 26, 27, 38, 41, 46, 32, 4, 53, 47, 42], the split common fixed point problem (SFPP) [14, 33], the split variational inequality problem (SVIP) [12] and the split common null point problem (SCNPP) [7, 44].

In 2020, Reich and Tuyen [35] first introduced and studied the following generalized split feasibility problem (GSFP).

Let H_i, $i = 1, \dots, N$, be real Hilbert spaces and C_i, $i = 1, \dots, N$, be closed and convex subsets of H_i, respectively. Let $A_i : H_i \to H_{i+1}$, $i = 1, \dots, N-1$, be bounded linear operators such that

$$S := C_1 \cap A_1^{-1}(C_2) \cap \cdots \cap A_1^{-1}\Big(A_2^{-1} \dots \Big(A_{N-1}^{-1}(C_N)\Big)\Big) \neq \emptyset. \tag{1.13}$$

The generalized split feasibility problem (GSFP) is to find an element

$$p^* \in S. \tag{1.14}$$

Reich and Tuyen in [35] proved a strong convergence theorem for a modification of the CQ method which solves the GSFP (1.14). For more details on the GSFP (1.14), one can read the paper [35].

Very recently, Reich et al. [34] considered and studied the following split feasibility problem with multiple output sets in Hilbert spaces: Let H, H_i, $i = 1, \dots, N$, be real Hilbert spaces and let $B_i : H \to H_i$, $i = 1, \dots, N$, be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$, be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$, respectively. Given H, H_i and B_i as above, *the split feasibility problem with multiple output sets (in short, SFPMOS)* is to find an element p^* such that

$$p^* \in \Gamma := C \cap \Big(\cap_{i=1}^{N} B_i^{-1}(Q_i)\Big) \neq \emptyset. \tag{1.15}$$

Reich et al. [34], defined the function $g : H \to \mathbb{R}$ by

$$g(x) := \frac{1}{2} \sum_{i=1}^{N} \|(I - P_{Q_i})B_i x\|^2, \text{ for all } x \in H. \tag{1.16}$$

It is not difficult to see that an element p^* is a solution to the SFPMOS (1.15) if and only if it is a solution to the problem

$$\min_{x \in C} g(x), \tag{1.17}$$

This is equivalent to

$$0 \in \nabla g(p^*) + N_C(p^*), \tag{1.18}$$

where $N_C(x)$ is the normal cone of C at the point x $\Big[$Recall: Let $C \subseteq H$ be a closed convex subset of a real Hilbert space H. The *Normal cone* of C at x denoted by $N_C(x)$ is given by $N_C(x) = \{z \in H : \langle z,\ y - x\rangle \leq 0,\ \forall y \in C\}\Big]$. Which implies

$$p^* = P_C\Big(p^* - \lambda \sum_{i=1}^{N} B_i^*(I - P_{Q_i})B_i p^*\Big),$$

where λ is an arbitrary positive real number. Motivated by these characterizations, Reich et al. [34] introduced the following two methods for solving the SFPMOS (1.15). For any given points $x_0, y_0 \in H$, $\{x_n\}$ and $\{y_n\}$ are sequences generated by

$$x_{n+1} := P_C\Big(x_n - \lambda_n \sum_{i=1}^{N} B_i^*(I - P_{Q_i})B_i x_n\Big), \tag{1.19}$$

$$y_{n+1} := \alpha_n f(y_n) + (1-\alpha_n)P_C\Big(y_n - \lambda_n \sum_{i=1}^{N} B_i^*(I - P_{Q_i})B_i y_n\Big), \tag{1.20}$$

where $f : C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0,1)$, $\lambda_n \subset (0,\infty)$ and $\{\alpha_n\} \subset (0,1)$. It was proved that if the sequence $\{\lambda_n\}$ satisfies the condition:

$$0 < a \le \lambda_n \le b < \frac{2}{N \max_{i=1,2,\ldots,N}\{\|B_i\|^2\}} \text{ for all } n \ge 1,$$

then the sequence $\{x_n\}$ generated by (1.19) converges weakly to a solution point $p^* \in \Gamma$ of the SFPMOS (1.15). Furthermore, if the sequence $\{\alpha_n\}$ satisfies the conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty,$$

then the sequence $\{y_n\}$ generated by (1.20) converges strongly to a solution point $p^* \in \Gamma$ of the SFPMOS (1.15), which is a unique solution of the variational inequality

$$\langle (I-f)p^*, x - p^* \rangle \ge 0 \ \forall x \in \Gamma.$$

An important observation here is that the iterative methods given by Scheme (1.19) and Scheme (1.20) introduced by Reich et al. [34] requires to compute the metric projections on to the sets C and Q_i. Moreover, it needs to compute the operator norm. Due to this reason, the following question naturally arises.

Question: *Can we have a strongly convergent algorithm for solving the SFPMOS (1.15) which mainly involves a self-adaptive step-size and requires to compute the projections onto half-spaces so that the algorithm is easily implementable?.*

We have a positive answer for the above question which is motivated by the algorithms proposed by Reich et al. [34] for solving the SFPMOS (1.15), the iterative methods given by schemes (1.10)-(1.12) proposed for solving the SFP (1.1), and other similar results in the literature. In this paper, we propose a new self adaptive relaxed CQ algorithm for solving the SFPMOS (1.15) in general Hilbert spaces.

This paper is organized as follows. In the next section, we recall some necessary tools which will be used in establishing our main results. In Section 3, we propose a self-adaptive relaxed CQ algorithm for solving the SFPMOS (1.15), and we establish and analyze a strong convergence theorem for the proposed algorithm. Also, in this section, we present some newly derived results for solving the SFP (1.1). In Section 4, we present the application of our proposed method to solve the GSFP (1.14). Finally, in Section 5, we provide some numerical experiments including an application to signal recovery to illustrate the implementation of our proposed method and we compare with some similar existing results.

2. Preliminaries

In this section, we recall some preliminaries which are needed in the sequel. Let H be a real Hilbert space with the inner product $\langle .,. \rangle$, and induced norm $\|.\|$. Let I stands for the identity operator on H. We denote the fixed point set of an operator $T : H \to H$ (if T has fixed point) by $Fix(T)$, *i.e.*, $Fix(T) = \{x \in H : Tx = x\}$. Let the symbols " $\rightharpoonup$ " and " $\to$ ", denote the weak and strong convergence, respectively. For any sequence $\{x_n\} \subset H$,

$$\omega_\omega(x_n) = \{x \in H : \exists \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\}$$

denotes the weak ω-limit set of $\{x_n\}$.

Definition 2.1. ([3]) Let H be a real Hilbert space with inner product $\langle , \rangle$ and induced norm $\|.\|$. Let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a given operator. Then T is called

(1): *Lipschitz continuous* with constant $\sigma > 0$ on C if

$$\|Tx - Ty\| \leq \sigma \|x - y\|, \forall x, y \in C; \tag{2.1}$$

(2): *nonexpansive* on C if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C; \tag{2.2}$$

(3): *firmly nonexpansive* on C if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in C, \tag{2.3}$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \forall x, y \in C; \tag{2.4}$$

(4): *averaged* if there exist a number $\sigma \in (0, 1)$ and a nonexpansive operator $F : C \to H$ such that

$$T = \sigma F + (1 - \sigma)I, \text{ where } I \text{ is the identity operator.} \tag{2.5}$$

In this case, we say that T is σ-*averaged.*

Definition 2.2. ([3]) Let $C \subseteq H$ be a nonempty, closed and convex set. For every element $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| = \min\{\|x - y\| : y \in C\}. \tag{2.6}$$

The operator $P_C : H \to C$ is called a *metric projection* of H onto C and it has the following well-known properties.

Lemma 2.3. *(*[3, 24]*) Let $C \subseteq H$ be a nonempty, closed and convex set. Then, the following assertions hold for any $x, y \in H$ and $z \in C$:*

(1): $\langle x - P_C(x), z - P_C(x) \rangle \leq 0$;
(2): $\|P_C(x) - P_C(y)\| \leq \|x - y\|$;

(3): $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle$;
(4): $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|x - P_C(x)\|^2$.

Lemma 2.4. *([6, 3]) Let $C \subseteq H$ be a nonempty closed convex subset. Then, $I - P_C$ is firmly nonexpansive and so is nonexpansive.*

Definition 2.5. Let $f : H \to (-\infty, +\infty]$ be a proper function. Then

(1) : f is *convex* if

$$f(\delta x + (1-\delta)y) \leq \delta f(x) + (1-\delta)f(y), \forall \delta \in (0,1) \text{ and } \forall x, y \in H.$$

(2) : f is *strongly convex* with constant σ, where $\sigma > 0$, if

$$f(\delta x+(1-\delta)y)+\frac{\sigma}{2}\delta(1-\delta)\|x-y\|^2 \leq \delta f(x)+(1-\delta)f(y), \forall \delta \in (0,1) \text{ and } \forall x, y \in H.$$

(3) : A vector $w \in H$ is a *subgradient* of f at a point x if

$$f(y) \geq f(x) + \langle w, y - x\rangle, \ \forall y \in H.$$

(4) : The set of all subgradients of a convex function $f : H \to \mathbb{R}$ at $x \in H$, denoted by $\partial f(x)$, is called the *subdifferential* of f, and is defined by

$$\partial f(x) = \{w \in H : f(y) \geq f(x) + \langle w, y - x\rangle, \text{ for each } y \in H\}.$$

(5) : If $\partial f(x) \neq \emptyset$, f is said to be *subdifferentiable* at x. If the function f is continuously differentiable then $\partial f(x) = \{\nabla f(x)\}$.

Definition 2.6. Let $f : H \to (-\infty, +\infty]$ be a proper function.

(1) : f is *lower semicontinuous* (*lsc*) at x if $x_n \to x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(2) : f is *weakly lower semicontinuous* (*w-lsc*) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(3) : f is lower semicontinuous on H if it is lower semicontinuous at every point $x \in H$ and f is weakly lower semicontinuous on H if it is weakly lower semicontinuous at every point $x \in H$.

Lemma 2.7. *([50]) Let C and Q be closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $f : H_1 \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$, where $A : H_1 \to H_2$ be a bounded linear operator. Then for $\delta > 0$ and $x^* \in H_1$, the following statements are equivalent.*

(1): *The point x^* solves the SFP (1.1), i.e, $x^* \in \{x \in C : Ax \in Q\}$.*
(2): *The point x^* is the fixed point of the mapping $P_C(I - \delta\nabla f)$.*
(3): *The point x^* solves the variational inequality problem with respect to the ∇f, that is find a point $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^*\rangle \geq 0, \forall x \in C.$$

Lemma 2.8. *([15]) Let H_1 and H_2 be real Hilbert spaces and $f : H_1 \to \mathbb{R}$ is given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ where Q is closed convex subset of H_2 and $A : H_1 \to H_2$ be a bounded linear operator. Then the following assertions hold:*

(1) : *f is convex and differentiable;*
(2) : *f is weakly lower semicontinuous on H_1;*
(3) : *$\nabla f(x) = A^*(I - P_Q)Ax$, for $x \in H_1$;*
(4) : *∇f is $\|A\|^2$-Lipschitz, i.e., $\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2 \|x - y\|, \ \forall x, y \in H_1$.*

Definition 2.9. Let $\{\Lambda_n\}$ be a real sequence. Then, $\{\Lambda_n\}$ decrease at infinity if there exists $n_0 \in \mathbb{N}$ such that $\Lambda_{n+1} \leq \Lambda_n$, for $n \geq n_0$. In other words, the sequence $\{\Lambda_n\}$ does not decrease at infinity, if there exists a subsequence $\{\Lambda_{n_t}\}_{t \geq 1}$ of $\{\Lambda_n\}$ such that $\Lambda_{n_t} < \Lambda_{n_t+1}$, for all $t \geq 1$.

Lemma 2.10. *([31]) Let $\{\Lambda_n\}$ be a sequence of real numbers that does not decrease at infinity. Also consider the sequence of integers $\{\varphi(n)\}_{n \geq n_0}$ defined by*

$$\varphi(n) = \max\{m \in \mathbb{N} : m \leq n, \Lambda_m \leq \Lambda_{m+1}\}.$$

Then $\{\varphi(n)\}_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \to \infty} \varphi(n) = \infty$, and for all $n \geq n_0$, the following two estimates hold:

$$\Lambda_{\varphi(n)} \leq \Lambda_{\varphi(n)+1} \text{ and } \Lambda_n \leq \Lambda_{\varphi(n)+1}.$$

Lemma 2.11. *([48]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$s_{n+1} \leq (1 - \sigma_n)s_n + \sigma_n \mu_n + \beta_n, n \geq 1,$$

*where $\{\sigma_n\}$, $\{\mu_n\}$ and $\{\beta_n\}$ satisfying the conditions: **(1)** $\{\sigma_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$; **(2)** $\limsup_{n \to \infty} \mu_n \leq 0$; **(3)** $\beta_n \geq 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$.*
Then, $\lim_{n \to \infty} s_n = 0$.

3. Main results

In this section, we propose a new self adaptive relaxed CQ-method for solving the SFPMOS (1.15), and we prove a strong convergence theorem of the proposed algorithm. We consider a general case of the SFPMOS (1.15), where the nonempty, closed and convex sets C and $Q_i (i = 1, \ldots, N)$ are given by level sets of convex functions. Throughout this section, we assume that $c : H \to \mathbb{R}$ and $q_i : H_i \to \mathbb{R}$ are lower semicontinuous convex functions and the sets C and Q_i are given by

$$C := \{x \in H : c(x) \leq 0\} \text{ and } Q_i := \{y \in H_i : q_i(y) \leq 0\}. \tag{3.1}$$

We assume that c and each q_i are subdifferentiable on H and H_i, respectively, with subdifferential ∂c and ∂q_i, respectively. Moreover, we suppose that for any $x \in H$ a subgradient $\xi_n \in \partial c(x)$ can be calculated, and for any $y \in H_i$ and for each $i \in \{1, \ldots, N\}$, a subgradient $\eta_i^n \in \partial q_i(y)$ can be calculated. Again, we assume that both ∂c and $\partial q_i (i = 1, \ldots, N)$ are bounded operators (i.e., bounded on bounded set).

In this situation, the projections onto C and Q_i are not easily implemented in general. To avoid this difficulty, we introduce a relaxed projection gradient methods, in which the projections onto the half-spaces are adopted in stead of the projections onto C and Q_i. In particular for $n \in \mathbb{N}$, we define the relaxed sets (half-spaces) C_n and $Q_i^n (i = 1, \dots, N)$ of C and Q_i, respectively at x_n as follows:

$$C_n := \{x \in H : c(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \tag{3.2}$$

where $\xi_n \in \partial c(x_n)$ is subgradient of c at x_n and

$$Q_i^n := \{y \in H_i : q_i(B_i x_n) \leq \langle \eta_i^n, B_i x_n - y \rangle\}, \tag{3.3}$$

where $\eta_i^n \in \partial q_i(B_i x_n)$. By the definition of subgradient, it follows that $C \subseteq C_n$ and $Q_i \subseteq Q_i^n$ (see [21]) hold for every $n \geq 0$. Moreover, in order to remove the requirement of estimating the value of operator norm, in which finding operator norm is not easy, we now introduce a new way of selecting the step sizes for solving the SFPMOS (1.15). Now, we define the following (relaxed) proximity function: for $x \in H$,

$$g_n(x) := \frac{1}{2} \sum_{i=1}^{N} \|(I - P_{Q_i^n}) B_i x\|^2. \tag{3.4}$$

We note that $g_n(.)$ is differentiable with its gradient given by

$$\nabla g_n(x) := \sum_{i=1}^{N} B_i^* (I - P_{Q_i^n}) B_i x, \tag{3.5}$$

where each Q_i^n are half-spaces given in (3.3). We note that g_n is weakly lower semi-continuous, convex and differentiable function [2] and ∇g_n is Lipschitz continuous.

Next, we present a self-adaptive relaxed CQ algorithm, that we wish to propose for solving the SFPMOS (1.15).

Algorithm 1: Strongly convergent self-adaptive relaxed CQ algorithm for the SFPMOS (1.15)

Initialization: Choose positive sequences $\{\rho_n\} \subset (0, 4)$, $\{\beta_n\} \subset (0, 1)$, $\{\delta_n\} \subset [0, 1)$ and $\{\varepsilon_n\} \subset (0, 1)$ such that $\beta_n + \delta_n + \varepsilon_n = 1$. Let $u \in H$ be a fixed point. Select an arbitrary starting point $x_0 \in H$, and set $n = 0$.

Step 1: Given the current iterate $x_n \in H$. If $\nabla g_n(x_n) = 0$ for some $n \in \mathbb{N}$, then stop. Otherwise, continue and calculate

$$\tau_n := \frac{\rho_n g_n(x_n)}{\|\nabla g_n(x_n)\|^2}. \tag{3.6}$$

Step 2: Compute the next iterate as

$$x_{n+1} := P_{C_n}\Big(\beta_n u + \delta_n x_n + \varepsilon_n \big(x_n - \tau_n \nabla g_n(x_n)\big)\Big), \tag{3.7}$$

where C_n is the half-space given as in (3.2).

We can see that Algorithm 1 terminates at some iterate (say n) when $\nabla g_n(x_n) = 0$; otherwise, if Algorithm 1 does not stop, then we have the following strong convergence theorem for approximating the solution of the SFPMOS (1.15).

Theorem 3.1. *Let H, H_i, $i = 1, \dots, N$, be real Hilbert spaces and let $B_i : H \to H_i$, $i = 1, \dots, N$, be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$, be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$, respectively. Assume that the SFPMOS (1.15) is consistent. Suppose the sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{\varepsilon_n\}$ in Algorithm 1 satisfy the following conditions:*

- **(A1):** $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = +\infty$;
- **(A2):** $0 < \liminf_{n\to\infty} \varepsilon_n \le \limsup_{n\to\infty} \varepsilon_n < 1$;
- **(A3):** $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$.

Then, the sequence $\{x_n\}$ generated by Algorithm 1 strongly converges to the solution $p \in \Gamma$, where $p = P_\Gamma(u)$.

Proof. We may assume that the sequence $\{x_n\}$ is infinite, that is, Algorithm 1 does not terminate in a finite number of iterations. Thus $\nabla g_n(x_n) \neq 0$ for all $n \ge 0$. Recall that Γ is the solution set of the problem (1.15). In the consistent case of the problem (1.15), Γ is nonempty, closed and convex. Thus, the metric projection P_Γ is well-defined.

Let $p \in \Gamma$ and set $z_n = x_n - \tau_n \nabla g_n(x_n)$. Note that $I - P_{Q_i^n}$ for each $i = 1, \dots, N$ is firmly nonexpansive and $\nabla g_n(p) = 0$. Hence, we have from Lemma 2.3 that

$$\begin{aligned} \langle \nabla g_n(x_n), x_n - p\rangle &= \left\langle \sum_{i=1}^{N} B_i^*(I - P_{Q_i^n})B_i x_n, x_n - p \right\rangle \\ &= \sum_{i=1}^{N} \left\langle B_i^*(I - P_{Q_i^n})B_i x_n, x_n - p \right\rangle \\ &= \sum_{i=1}^{N} \left\langle (I - P_{Q_i^n})B_i x_n, B_i x_n - B_i p \right\rangle \\ &\ge \sum_{i=1}^{N} \left\| (I - P_{Q_i^n})B_i x_n \right\|^2 = 2g_n(x_n), \end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned} \|z_n - p\|^2 &= \|(x_n - p) - \tau_n \nabla g_n(x_n)\|^2 \\ &= \|x_n - p\|^2 + \tau_n^2 \|\nabla g_n(x_n)\|^2 - 2\tau_n \langle \nabla g_n(x_n), x_n - p\rangle \\ &\le \|x_n - p\|^2 + \frac{\rho_n^2 g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} - \frac{2\rho_n g_n(x_n)}{\|\nabla g_n(x_n)\|^2}(2g_n(x_n)) \\ &= \|x_n - p\|^2 + \frac{\rho_n^2 g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} - \frac{4\rho_n g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} \\ &= \|x_n - p\|^2 - \rho_n(4 - \rho_n)\frac{g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2}. \end{aligned} \tag{3.9}$$

Using the condition (A3), we have

$$\|z_n - p\|^2 \le \|x_n - p\|^2, \ \forall n \ge 0. \tag{3.10}$$

Next, we show $\{x_n\}$ is bounded. Since $p \in \Gamma \subseteq C_n$ and the projection operator P_{C_n} is nonexpansive, we obtain from (3.7) and (3.10) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_{C_n}(\beta_n u + \delta_n x_n + \varepsilon_n z_n) - p\|^2 \\
&\le \|(\beta_n u + \delta_n x_n + \varepsilon_n z_n) - p\|^2 \\
&\le \beta_n\|u-p\|^2 + \delta_n\|x_n-p\|^2 + \varepsilon_n\|z_n-p\|^2 \\
&\le \beta_n\|u-p\|^2 + \delta_n\|x_n-p\|^2 + \varepsilon_n\|x_n-p\|^2 \\
&= \beta_n\|u-p\|^2 + (1-\beta_n)\|x_n-p\|^2 \\
&\le \max\left\{\|u-p\|^2, \|x_n-p\|^2\right\} \\
&\ \vdots \\
&\le \max\left\{\|u-p\|^2, \|x_0-p\|^2\right\}. \qquad (3.11)
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{z_n\}$ and $\{B_i x_n\}_{i=1}^N$ are also bounded. The rest of the proof will be divided in to two parts.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|^2\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\|x_n-p\|^2\}_{n=1}^{\infty}$ converges and $\|x_n-p\|^2 - \|x_{n+1}-p\|^2 \to 0$, as $n \to \infty$. Then from (3.9), we obtain

$$\rho_n(4-\rho_n)\frac{g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} \le \|x_n-p\|^2 - \|z_n-p\|^2. \qquad (3.12)$$

Since $\beta_n + \delta_n + \varepsilon_n = 1$, also, from (3.11), we have the following estimation

$$\begin{aligned}
\|x_n-p\|^2 - \|z_n-p\|^2 &\le \frac{\beta_n}{\varepsilon_n}\|u-p\|^2 + \frac{1-\beta_n}{\varepsilon_n}\|x_n-p\|^2 - \frac{1}{\varepsilon_n}\|x_{n+1}-p\|^2 \\
&= \frac{\beta_n}{\varepsilon_n}\|u-p\|^2 - \frac{\beta_n}{\varepsilon_n}\|x_n-p\|^2 \\
&\quad + \frac{1}{\varepsilon_n}\Big[\|x_n-p\|^2 - \|x_{n+1}-p\|^2\Big] \\
&\le \frac{\beta_n}{\varepsilon_n}\|u-p\|^2 + \frac{1}{\varepsilon_n}\Big[\|x_n-p\|^2 - \|x_{n+1}-p\|^2\Big] \\
&= \frac{1}{\varepsilon_n}\Big[\beta_n\|u-p\|^2 + \Big[\|x_n-p\|^2 - \|x_{n+1}-p\|^2\Big]\Big]. \qquad (3.13)
\end{aligned}$$

Combining (3.12) and (3.13) together, we obtain

$$\begin{aligned}
\rho_n(4-\rho_n)\frac{g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} &\le \|x_n-p\|^2 - \|z_n-p\|^2 \\
&\le \frac{1}{\varepsilon_n}\Big[\beta_n\|u-p\|^2 + \Big[\|x_n-p\|^2 - \|x_{n+1}-p\|^2\Big]\Big]. \qquad (3.14)
\end{aligned}$$

By conditions (A2) and (A3) and (3.14), we have as $n \to \infty$

$$0 < \rho_n(4-\rho_n)\frac{g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} \le \frac{1}{\varepsilon_n}\Big[\beta_n\|u-p\|^2 + \Big[\|x_n-p\|^2 - \|x_{n+1}-p\|^2\Big]\Big] \to 0,$$

which implies that

$$\lim_{n\to\infty} \frac{g_n^2(x_n)}{\|\nabla g_n(x_n)\|^2} = 0. \tag{3.15}$$

We note that for each $i = 1, \dots, N$, $B_i^*(I - P_{Q_i^n})B_i(.)$ is Lipschitz continuous. Since the sequence $\{x_n\}$ is bounded and

$$\left\|B_i^*(I-P_{Q_i^n})B_i x_n\right\| = \left\|B_i^*(I-P_{Q_i^n})B_i x_n - B_i^*(I-P_{Q_i^n})B_i p\right\| \leq \left(\max_{1\leq i\leq N} \|B_i\|^2\right)\|x_n - p\|,$$

for all $i = 1, \dots, N$, we have the sequence $\{\|B_i^*(I - P_{Q_i^n})B_i x_n\|\}_{n=1}^{\infty}$ is bounded. Hence, $\{\|\nabla g_n(x_n)\|\}_{n=1}^{\infty}$ is bounded. Consequently, we have from (3.15) that

$$\lim_{n\to\infty} \|(I - P_{Q_i^n})B_i x_n\| = 0 \tag{3.16}$$

for each $i = 1, \dots, N$. Since $z_n = x_n - \tau_n \nabla g_n(x_n)$, then we have from (3.16) that

$$\|z_n - x_n\| \leq \tau_n \|\nabla g_n(x_n)\| \to 0, \text{ as } n \to \infty.$$

That is

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.17}$$

$$\lim_{n\to\infty} \|B_i^*(I - P_{Q_i^n})B_i x_n\| = 0 \tag{3.18}$$

for each $i = 1, \dots, N$. Furthermore, we have the following estimations.

Let

$$y_n = \beta_n u + \delta_n x_n + \varepsilon_n z_n = \beta_n u + (1 - \beta_n)v_n, \tag{3.19}$$

where

$$v_n = \frac{\delta_n}{1-\beta_n} x_n + \frac{\varepsilon_n}{1-\beta_n} z_n.$$

This gives,

$$\|v_n - x_n\| \leq \frac{\varepsilon_n}{1-\beta_n}\|z_n - x_n\| \to 0, \;\; n \to \infty,$$

$$\|y_n - v_n\| \leq \beta_n \|u - v_n\| \to 0, \;\; n \to \infty.$$

and

$$\|y_n - x_n\| \leq \|y_n - v_n\| + \|v_n - x_n\| \to 0, \;\; n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \|y_n - v_n\| = \lim_{n\to\infty} \|v_n - x_n\| = 0. \tag{3.20}$$

Since the P_{C_n} is firmly nonexpansive, by Lemma 2.3 (4), we have the following estimation

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_{C_n}(y_n)-P_{C_n}(p)\|^2 \\
&= \|P_{C_n}(y_n)-P_{C_n}(x_n)+P_{C_n}(x_n)-P_{C_n}(p)\|^2 \\
&\leq \|P_{C_n}(x_n)-P_{C_n}(p)\|^2+2\langle P_{C_n}(y_n)-P_{C_n}(x_n), x_{n+1}-p\rangle \\
&\leq \|P_{C_n}(x_n)-P_{C_n}(p)\|^2+2\|P_{C_n}(y_n)-P_{C_n}(x_n)\|\|x_{n+1}-p\| \\
&\leq \|P_{C_n}(x_n)-P_{C_n}(p)\|^2+2\|y_n-x_n\|\|x_{n+1}-p\| \\
&\leq \|x_n-p\|^2-\|(I-P_{C_n})x_n\|^2+2\|y_n-x_n\|\|x_{n+1}-p\|. \qquad (3.21)
\end{aligned}$$

Noting that $\{x_n\}$ is bounded, we have from (3.21) that

$$\|(I-P_{C_n})x_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\|y_n-x_n\|M, \qquad (3.22)$$

where M is some positive constant. Due to the fact that $(\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\|y_n-x_n\|M)\to 0$, as $n\to\infty$, we get from (3.22) that

$$\lim_{n\to\infty}\|(I-P_{C_n})x_n\|^2=0. \qquad (3.23)$$

We claim here that $\omega_\omega(x_n)\subseteq\Gamma$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}\rightharpoonup p^*\in\omega_\omega(x_n)$. Next we show that $p^*\in\Gamma$. That is, we need to show $p^*\in C$ and $B_ip^*\in Q_i$ for each $i=1,\dots,N$. Since $\{x_n\}$ is bounded and from the boundedness assumption of the subdifferential operator ∂c, the sequence $\{\xi_{n_k}\}_{k=1}^{\infty}$ is bounded. Indeed, by (3.2) and (3.23), we obtain

$$c(x_{n_k})\leq\langle\xi_{n_k}, x_{n_k}-P_{C_{n_k}}(x_{n_k})\rangle\leq\|\xi_{n_k}\|\|x_{n_k}-P_{C_{n_k}}(x_{n_k})\|\to 0 \qquad (3.24)$$

The weak lower semi-continuity of $c(.)$ and (3.24) implies that

$$c(p^*)\leq\liminf_{k\to\infty}c(x_{n_k})=0.$$

Consequently, $p^*\in C$.

Since $\{x_n\}$ is bounded and from the boundedness assumption of the subdifferential operator ∂q_i, the sequence $\{\eta_i^{n_k}\}_{k=1}^{\infty}$ is bounded. This together with (3.16) gives

$$\begin{aligned}
q_i\big(B_ix_{n_k}\big) &\leq \Big\langle\eta_i^{n_k}, B_ix_{n_k}-P_{Q_i^{n_k}}\big(B_ix_{n_k}\big)\Big\rangle \\
&\leq \Big\|\eta_i^{n_k}\Big\|\Big\|(I-P_{Q_i^{n_k}})B_ix_{n_k}\Big\|\to 0, \;\; k\to\infty, \qquad (3.25)
\end{aligned}$$

for all $i=1,\dots,N$. Since $x_{n_k}\rightharpoonup p^*$, we have $B_ix_{n_k}\rightharpoonup B_ip^*$. The weak lower semi-continuity of $q_i(.)$ and (3.25) implies that

$$q_i(B_ip^*)\leq\liminf_{k\to\infty}q_i(B_ix_{n_k})\leq\limsup_{k\to\infty}q_i(B_ix_{n_k})\leq 0,$$

for all $i=1,\dots,N$. That is, $B_ip^*\in Q_i$ for all $i=1,\dots,N$. Hence, $p^*\in\Gamma$.

Moreover, for $p=P_\Gamma u$, we can see that

$$\begin{aligned}
\limsup_{n\to\infty}\langle y_n-p, u-p\rangle &= \lim_{k\to\infty}\langle y_{n_k}-p, u-p\rangle \\
&\leq \langle p^*-p, u-p\rangle\leq 0. \qquad (3.26)
\end{aligned}$$

By Lemma 2.3 and (3.19), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_C(y_n)-p\|^2 \\
&\leq \|y_n-p\|^2 \\
&= \langle \beta_n u+(1-\beta_n)v_n-p, y_n-p\rangle \\
&= \beta_n\langle u-p, y_n-p\rangle+(1-\beta_n)\langle v_n-p, y_n-p\rangle \\
&\leq \beta_n\langle u-p, y_n-p\rangle+(1-\beta_n)\|v_n-p\|\|y_n-p\| \\
&\leq (1-\beta_n)\|x_n-p\|^2+\beta_n\langle u-p, y_n-p\rangle. \qquad (3.27)
\end{aligned}$$

Using (3.26) in (3.27) and applying Lemma 2.11, we obtain

$$\lim_{n\to\infty}\|x_n-p\|^2=0.$$

Therefore, as $n\to\infty$, $x_n\to p=P_\Gamma u$.

Case 2: Set $\Lambda_n=\|x_n-p\|^2$. Assume that $\{\Lambda_n\}$ is not decreasing at infinity. Let $\phi:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\geq n_0$ (for some n_0 large enough) defined by

$$\phi(n)=\max\{t\in\mathbb{N}: t\leq n, \Lambda_t\leq\Lambda_{t+1}\}.$$

By Lemma 2.10, $\{\phi(n)\}_{n=n_0}^{\infty}$ is a nondecreasing sequence such that $\phi(n)\to\infty$ as $n\to\infty$ and

$$\max\{\Lambda_{\phi(n)},\ \Lambda_n\}\leq\Lambda_{\phi(n)+1},\ \ \forall n\geq n_0. \qquad (3.28)$$

After a similar conclusion from (3.16), it is easy to see that

$$\lim_{n\to\infty}\|(I-P_{Q_i^{\phi(n)}})B_i x_{\phi(n)}\|=0. \qquad (3.29)$$

By the similar argument as above in Case 1, we conclude immediately that

$$\lim_{n\to\infty}\|B_i^*(I-P_{Q_i^{\phi(n)}})B_i x_{\phi(n)}\|=0$$

and

$$\limsup_{n\to\infty}\langle y_{\phi(n)}-p, u-p\rangle\leq 0.$$

Since $\{x_{\phi(n)}\}$ is bounded, there exists a subsequence of $\{x_{\phi(n)}\}$, still denoted by $\{x_{\phi(n)}\}$ which converges weakly to p^*. By similar argument as above in Case 1, we conclude immediately that $p^*\in C$ and $B_i p^*\in Q_i\Rightarrow p^*\in\Gamma$.

From (3.27) we have that

$$\|x_{\phi(n)+1}-p\|^2 \leq (1-\beta_{\phi(n)})\|x_{\phi(n)}-p\|^2+\beta_{\phi(n)}\langle u-p, y_{\phi(n)}-p\rangle$$

which implies by Lemma (2.11)

$$\lim_{n\to\infty}\|x_{\phi(n)}-p\|^2=0.$$

and

$$\lim_{n\to\infty}\|x_{\phi(n)+1}-p\|^2=0.$$

Moreover, for $n \geq n_0$, it is easy to see that $\Lambda_{\phi(n)} - \Lambda_{\phi(n)+1} \leq 0$ if $n \neq \phi(n)$ (that is $n > \phi(n)$), because for $\phi(n) + 1 \leq m \leq n$, $\Lambda_m > \Lambda_{m+1}$. As a consequence, we obtain,

$$0 \leq \Lambda_n \leq \max\{\Lambda_{\phi(n)},\ \Lambda_{\phi(n)+1}\} = \Lambda_{\phi(n)+1},\ \ \forall n \geq n_0.$$

Therefore, we obtain $\lim_{n\to\infty} \Lambda_n = 0$, that is, $\{x_n\}$ converges strongly to p. This completes the proof. □

Remark 3.2. For the special case, where $N = 1$, the SFPMOS (1.15) becomes the SFP (1.1). Thus, it is worth to mention that, we have the following corollary for solving the SFP (1.1), an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces and let $B : H_1 \to H_2$ be bounded linear operator. Let C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Assume that $\Omega = C \cap B^{-1}(Q) \neq \emptyset$. Let $u \in H_1$ be a fixed point. For any starting point $x_0 \in H_1$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} := P_{C_n}\Big(\beta_n u + \delta_n x_n + \varepsilon_n\big(x_n - \tau_n \nabla f_n(x_n)\big)\Big) \tag{3.30}$$

where C_n, τ_n, and ∇f_n are given by (1.4), (1.9) and (1.7), respectively. Suppose the sequences $\{\beta_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\}$ satisfy the conditions in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to the solution $p \in \Omega$, where $p = P_\Omega(u)$.

Remark 3.4. The iterative scheme (27) in [38] was reduced to the iterative scheme (43) in [42]. Here we note that our iterative scheme (3.30) extends the mentioned iterative scheme to self-adaptive relaxed method.

We note also the following results regarding to the SFPMOS (1.15).

Corollary 3.5. *Let H, H_i, $i = 1, \dots, N$, be real Hilbert spaces and let $B_i : H \to H_i$, $i = 1, \dots, N$, be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$, be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$, respectively. Assume that the problem (1.15) is consistent. For any initial guess $x_0 \in H$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} := P_{C_n}\Big(\beta_n x_0 + \delta_n x_n + \varepsilon_n\big(x_n - \tau_n \nabla g_n(x_n)\big)\Big) \tag{3.31}$$

where C_n, τ_n, and ∇g_n are given by (3.2), (3.6) and (3.5)), respectively. Suppose the sequences $\{\beta_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\}$ satisfy the conditions in Theorem 3.1. Then, the sequence $\{x_n\}$ generated by (3.31) strongly converges to the solution $p = P_\Gamma(x_0) \in \Gamma$.

Remark 3.6. We note that by letting $\delta_n \equiv 0$ of in Algorithm 1, we obtain the following result regarding the SFPMOS (1.15).

Corollary 3.7. *Let H, H_i, $i = 1, \dots, N$, be real Hilbert spaces and let $B_i : H \to H_i$, $i = 1, \dots, N$, be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$, be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$, respectively. Assume that the problem (1.15) is consistent. For a fixed point $u \in H$ and any initial guess $x_0 \in H$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} := P_{C_n}\Big(\beta_n u + (1 - \beta_n)\big(x_n - \tau_n \nabla g_n(x_n)\big)\Big) \tag{3.32}$$

where C_n, τ_n, and ∇g_n are given by (3.2), (3.6) and (3.5)), respectively. Suppose the sequence $\{\beta_n\}$ is as in Algorithm 1 and satisfies the conditions (A1) in Theorem 3.1. Then, the sequence $\{x_n\}$ generated by (3.32) strongly converges to the solution $p = P_\Gamma(u) \in \Gamma$.

Corollary 3.8. *Let H, H_i, $i = 1, \dots, N$, be real Hilbert spaces and let $B_i : H \to H_i$, $i = 1, \dots, N$, be bounded linear operators. Let C and Q_i, $i = 1, \dots, N$, be nonempty, closed and convex subsets of H and H_i, $i = 1, \dots, N$, respectively. Assume that the problem (1.15) is consistent. For any initial guess $x_0 \in H$, let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} := P_{C_n}\Big(\beta_n x_0 + (1-\beta_n)\big(x_n - \tau_n \nabla g_n(x_n)\big)\Big) \tag{3.33}$$

where C_n, τ_n, and ∇g_n are given by (3.2), (3.6) and (3.5)), respectively. Suppose the sequence $\{\beta_n\}$ satisfies the condition (A1) in in Theorem 3.1. Then, the sequence $\{x_n\}$ generated by (3.33) strongly converges to the solution $p = P_\Gamma(u) \in \Gamma$.

Remark 3.9. In Corollary 3.7 above, for the particular case, where $N = 1$, the iterative scheme (3.32) reduced exactly to iterative scheme (1.12) proposed by He et al. [25, Theorem 3.2].

4. Application to the generalized split feasibility problem

In this section, we present an application of Theorem 3.1 for solving the GSFP (1.14) in Hilbert spaces. We first recall the GSFP.

Let H_i, $i = 1, \dots, N$, be the real Hilbert spaces and C_i, $i = 1, \dots, N$, be closed and convex subsets of H_i, respectively. Let $A_i : H_i \to H_{i+1}$, $i = 1, \dots, N-1$, be bounded linear operators such that

$$S := C_1 \cap A_1^{-1}(C_2) \cap \cdots \cap A_1^{-1}\Big(A_2^{-1} \dots \Big(A_{N-1}^{-1}(C_N)\Big)\Big) \neq \emptyset.$$

The GSFP [35] is to find an element

$$p^* \in S,$$

that is $p^* \in C_1$, $A_1 p^* \in C_2, \dots, A_{N-1}A_{N-2}\dots A_1 p^* \in C_N$.

Remark 4.1. ([34, Remark 1.1]) Letting $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $B_1 = A_1, B_2 = A_2A_1, \dots,$ and $B_{N-1} = A_{N-1}A_{N-2}A_{N-3}\dots A_2A_1$, then the SFPMOS (1.15) becomes (1.14).

Using Theorem 3.1 and Remark 4.1, we note the following theorem for solving the GSFP (1.14).

Theorem 4.2. *Let $H = H_1, C = C_1, Q_i = C_{i+1}, 1 \leq i \leq N-1$, $B_1 = A_1, B_2 = A_2A_1, \dots,$ and $B_{N-1} = A_{N-1}A_{N-2}A_{N-3}\dots A_2A_1$. Let $u, x_0 \in C_1$, and let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} := P_{C_1^n}\Big(\beta_n u + \delta_n x_n + \varepsilon_n\big(x_n - \tau_n \sum_{i=1}^{N-1} B_i^*(I - P_{Q_i^n})B_i x_n\big)\Big), \tag{4.1}$$

where C_1^n and C_{i+1}^n are half-spaces of C_1 and C_{i+1} (at the nth iterate), respectively, and can be defined by (3.2) and (3.3), respectively, and

$$\tau_n := \frac{\rho_n \sum_{i=1}^{N-1} \left\|(I - P_{C_{i+1}^n})B_i x_n\right\|^2}{2\left\|\sum_{i=1}^{N-1} B_i^*(I - P_{C_{i+1}^n})B_i x_n\right\|^2}, \forall n \geq 0.$$

Suppose the sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\}$ satisfy the following conditions:

(C1): $\{\beta_n\} \subset (0, 1)$ *with* $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = +\infty$*;*

(C2): $\{\varepsilon_n\} \subset (0, 1)$ *with* $0 < \liminf_{n\to\infty} \varepsilon_n \leq \limsup_{n\to\infty} \varepsilon_n < 1$*;*

(C3): $\{\delta_n\} \subset [0, 1)$ *with* $\beta_n + \delta_n + \varepsilon_n = 1$*;*

(C4): $\{\rho_n\} \subset (0, +\infty)$ *with* $\liminf_{n\to\infty} \rho_n(4 - \rho_n) > 0$.

Then, the sequence $\{x_n\}$ generated by iterative scheme (4.1) converges strongly to the solution $p \in S$, where $p = P_S(u)$.

5. Numerical experiments

In this section, we provide some numerical experiments to illustrate the implementation of our proposed methods compared to many existing results by solving three problems. In Example 5.1, we study the behavior and implementation of Algorithm 1 and compare it with the Scheme (1.19), Scheme (1.20), and Scheme (5.4) by solving a problem adopted from [34]. In Example 5.2, we compare Scheme (3.30) with that of Scheme (1.11) and Scheme (1.12) by solving a problem in infinite-dimensional real Hilbert spaces. In Example 5.3, we apply Scheme (3.30) for signal recovery and we compare its performance with Scheme (1.11) and Scheme (1.12). The numerical results are completed on a standard TOSHIBA laptop with Intel(R) Core(TM) i5-2450M CPU@2.5GHz with memory 4GB. The code is implemented in MATLAB R2020a.

In these numerical experiments, Iter. (n) stands for the number of iterations and CPU(s) for the Elapsed time-run in seconds. For the sake of convenience, we denote $e_1 = (1, 1, \ldots, 1)^T \in \mathbb{R}^{10}$.

Example 5.1. ([34]) Consider $H = \mathbb{R}^{10}$, $H_1 = \mathbb{R}^{20}$, $H_2 = \mathbb{R}^{30}$, and $H_3 = \mathbb{R}^{40}$. Find a point $p^* \in \mathbb{R}^{10}$ such that

$$p^* \in \Gamma := C \cap B_1^{-1}(Q_1) \cap B_2^{-1}(Q_2) \cap B_3^{-1}(Q_3) \neq \emptyset, \tag{5.1}$$

where the closed convex sets C and Q_i $(i = 1, 2, 3)$ are defined by

$$\begin{aligned} C &= \{x \in \mathbb{R}^{10} : \|x - \mathbf{c}\|^2 \leq \mathbf{r}^2\}, \\ Q_1 &= \{B_1 x \in \mathbb{R}^{20} : \|B_1 x - \mathbf{c}_1\|^2 \leq \mathbf{r}_1^2\}, \\ Q_2 &= \{B_2 x \in \mathbb{R}^{30} : \|B_2 x - \mathbf{c}_2\|^2 \leq \mathbf{r}_2^2\}, \\ Q_3 &= \{B_3 x \in \mathbb{R}^{40} : \|B_3 x - \mathbf{c}_3\|^2 \leq \mathbf{r}_3^2\}. \end{aligned} \tag{5.2}$$

where $\mathbf{c} \in \mathbb{R}^{10}$, $\mathbf{c}_1 \in \mathbb{R}^{20}$, $\mathbf{c}_2 \in \mathbb{R}^{30}$, $\mathbf{c}_3 \in \mathbb{R}^{40}$, $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathbb{R}$, and the linear bounded operators B_i are defined by $B_1 : \mathbb{R}^{10} \to \mathbb{R}^{20}$, $B_2 : \mathbb{R}^{10} \to \mathbb{R}^{30}$, $B_3 : \mathbb{R}^{10} \to \mathbb{R}^{40}$. In this case, for any $x \in \mathbb{R}^{10}$, we have $c(x) = \|x - \mathbf{c}\|^2 - \mathbf{r}^2$ and $q_i(B_i x) = \|B_i x - \mathbf{c}_i\|^2 - \mathbf{r}_i^2$

for $i = 1, 2, 3$. According to (3.2) and (3.3), the half-spaces C_n and Q_i^n $(i = 1, 2, 3)$, respectively of the sets C and Q_i are determined at a point x_n and $B_i x_n$, respectively as follows:

$$
\begin{aligned}
C_n &= \{x \in \mathbb{R}^{10} : \|x_n - \mathbf{c}\|^2 - \mathbf{r}^2 \leq 2\langle x_n - \mathbf{c}, x_n - x\rangle\},\\
Q_1^n &= \{y \in \mathbb{R}^{20} : \|B_1 x_n - \mathbf{c}_1\|^2 - \mathbf{r}_1^2 \leq 2\langle B_1 x_n - \mathbf{c}_1, B_1 x_n - y\rangle\},\\
Q_2^n &= \{y \in \mathbb{R}^{30} : \|B_2 x_n - \mathbf{c}_2\|^2 - \mathbf{r}_2^2 \leq 2\langle B_2 x_n - \mathbf{c}_2, B_2 x_n - y\rangle\},\\
Q_3^n &= \{y \in \mathbb{R}^{40} : \|B_3 x_n - \mathbf{c}_3\|^2 - \mathbf{r}_3^2 \leq 2\langle B_3 x_n - \mathbf{c}_3, B_3 x_n - y\rangle\}.
\end{aligned}
\tag{5.3}
$$

Then, the metric projections onto the half-spaces C_n and Q_i^n $(i = 1, 2, 3)$, can be easily calculated. The elements of the representing matrices B_i are randomly generated in the closed interval $[-5, 5]$. The coordinates of the centers $\mathbf{c}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are randomly generated in the closed interval $[-1, 1]$. The radii $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are randomly generated in the closed intervals $[10, 20]$, $[20, 40]$, $[30, 60]$ and $[40, 80]$, respectively.

In this example, we examine the convergence of the sequence $\{x_n\}$ which is defined by Algorithm 1 and compare it with that of Scheme (1.19), Scheme (1.20), and with the following viscosity approximation an optimization approach method proposed by Reich et al. [36] for solving the SFPMOS (1.15). For any given point $x_0 \in H$, $\{x_n\}$ is a sequence generated by the iterative method

$$
x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) P_C\Big(x_n - \lambda_n \sum_{i \in I(x_n)} \gamma_{i,n} B_i^*(I - P_{Q_i}) B_i x_n\Big),
\tag{5.4}
$$

where $f : C \to C$ is a strict contraction mapping of H into itself with the contraction constant $\theta \in [0, 1)$, $\{\alpha_n\} \subset (0, 1)$, $I(x_n) = \big\{i : \|B_i x_n - P_{Q_i} B_i x_n\| = \max_{i=1,2,\ldots,N} \|B_i x_n - P_{Q_i} B_i x_n\|\big\}$, $\gamma_{i,n} \geq 0$ for all $i \in I(x_n)$ with $\sum_{i \in I(x_n)} \gamma_{i,n} = 1$, and for $\{\rho_n\} \subset [\underline{a}, \bar{a}] \subset (0, 2)$ $\{\lambda_n\} \subset [0, \infty)$ such that

$$
\lambda_n = \begin{cases} \rho_n \frac{(\max_{i=1,2,\ldots,N} \|B_i x_n - P_{Q_i} B_i x_n\|)^2}{\|\sum_{i \in I(x_n)} \gamma_{i,n} B_i^*(I - P_{Q_i}) B_i x_n\|^2}, & \text{if } \|\sum_{i \in I(x_n)} \gamma_{i,n} B_i^*(I - P_{Q_i}) B_i x_n\| > 0,\\ 0, & \text{otherwise}, \end{cases}
\tag{5.5}
$$

by solving problem (5.1).

First, we examine the convergence of the sequence $\{x_n\}$ generated by Algorithm 1. Here, we take $u = 10e_1$ and $x_0 = 100e_1$, $\beta_n = \frac{1}{2n+1}$, $\delta_n = \varepsilon_n = \frac{n}{2n+1}$, and we consider different choices of the sequence ρ_n. We use $E_n = \|x_{n+1} - x_n\|^2 < \epsilon$ as stopping criteria, where ϵ is a small enough positive number (note that if at the nth step, $E_n = 0$, then $x_n \in \Gamma$). The numerical outcomes of the experiments are reported in Table 1.

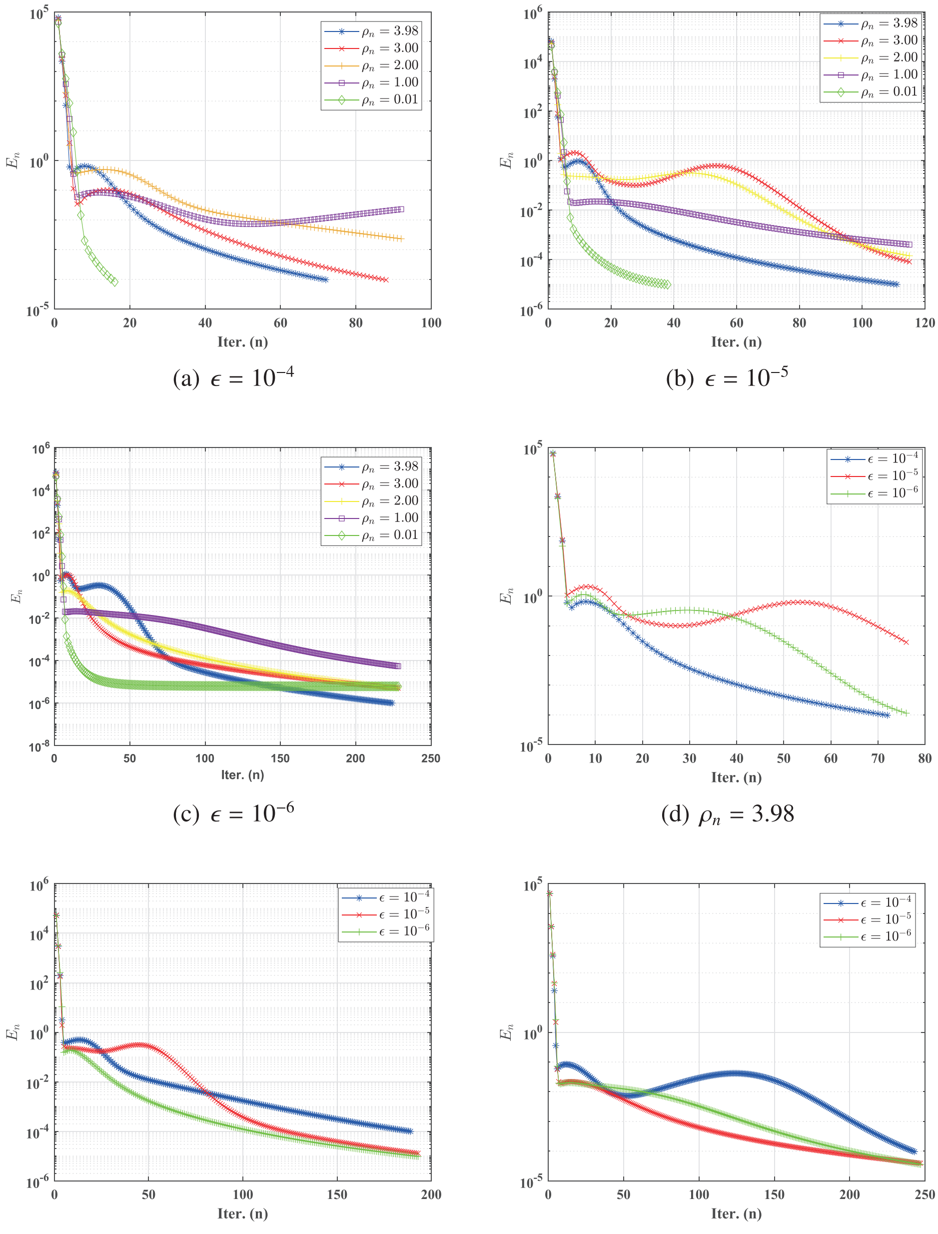
TABLE 1. Numerical results of Algorithm 1 for different choices of ρ_n and ϵ

	$\epsilon = 10^{-4}$		$\epsilon = 10^{-5}$		$\epsilon = 10^{-6}$	
	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)
$\rho_n = 3.98$	72	0.022767	111	0.024639	224	0.035905
$\rho_n = 3.00$	88	0.022052	172	0.02908	356	0.029861
$\rho_n = 2.00$	189	0.026134	206	0.022539	345	0.026904
$\rho_n = 1.00$	243	0.007429	384	0.025907	550	0.031092
$\rho_n = 0.01$	16	0.019556	38	0.023237	6802	0.120587

The behavior of the function E_n in Table 1 is described in Figure 1.

TABLE 2. Comparison of Algorithm 1 with Scheme (1.19), Scheme (1.20), and Scheme (5.4)

	$x_0 = 10e_1$					
	$\epsilon = 10^{-6}$		$\epsilon = 10^{-8}$		$\epsilon = 10^{-10}$	
	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)
Algorithm 1	20	0.019655	39	0.026011	83	0.00455
Scheme (1.19)	69	0.042288	53	0.045127	206	0.007512
Scheme (1.20)	35	0.062453	407	0.060721	3363	0.033598
Scheme (5.4)	34	0.024152	378	0.026378	3044	0.049601
	$\epsilon = 10^{-8}$					
	$x_0 = e_1$		$x_0 = 100e_1$		$x_0 = 30e_1$	
	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)	Iter. (n)	CPU(s)
Algorithm 1	89	0.006857	108	0.004559	65	0.00224
Scheme (1.19)	147	0.010925	112	0.006441	91	0.003845
Scheme (1.20)	298	0.016471	451	0.014648	409	0.008438
Scheme (5.4)	278	0.009539	419	0.01584	379	0.007924



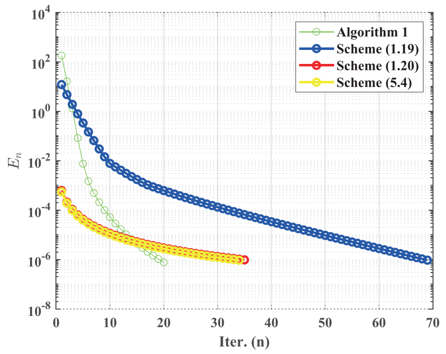
(a) $\epsilon = 10^{-4}$ (b) $\epsilon = 10^{-5}$

(c) $\epsilon = 10^{-6}$ (d) $\rho_n = 3.98$

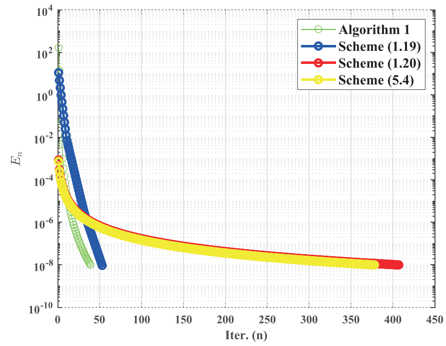
(e) $\rho_n = 3.00$ (f) $\rho_n = 1.00$

FIGURE 1. Numerical results of Algorithm 1 for different choices of ρ_n and ϵ

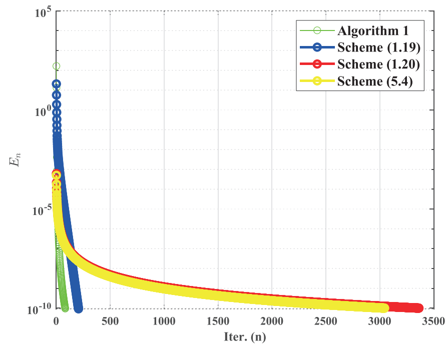
Next, we compare Algorithm 1 with that of Scheme (1.19), Scheme (1.20), and Scheme (5.4) by solving the same problem (5.1). For Algorithm 1, we choose $u = 10e_1$, $\beta_n = \frac{1}{2n+1}$, $\delta_n = \varepsilon_n = \frac{n}{2n+1}$. For Scheme (1.19) and Scheme (1.20), we take $\lambda_n = 0.0005$. For Scheme (1.20) and Scheme (5.4), we take $f(x) = 0.975x$ and $\alpha_n = \frac{1}{2n+1}$. For Algorithm 1 and Scheme (5.4), we take $\rho_n = \frac{n}{10^{10}n+1}$. Moreover, we take $\gamma_{1,n} = \frac{1}{6}$, $\gamma_{2,n} = \frac{1}{3}$, $\gamma_{3,n} = \frac{1}{2}$ for Scheme (5.4). Using $E_n = \|x_{n+1} - x_n\|^2 < \epsilon$ as stopping criteria, for different choices of the initial point x_0 and different values of ϵ, the results of numerical experiments are reported in Table 2 and Figure 2.



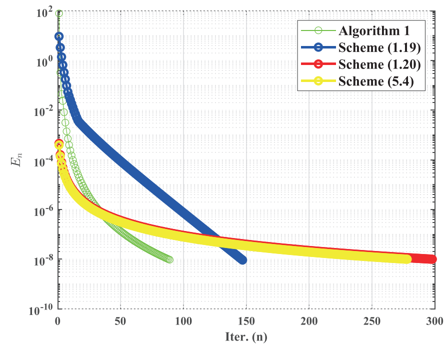
(a) $x_0 = 10e_1$, $\epsilon = 10^{-6}$



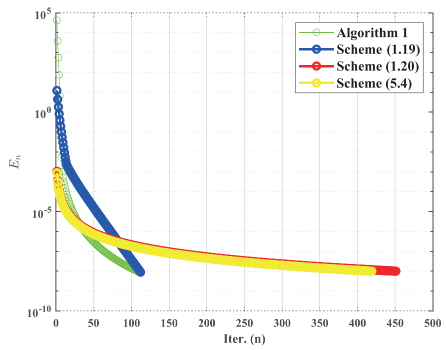
(b) $x_0 = 10e_1$, $\epsilon = 10^{-8}$



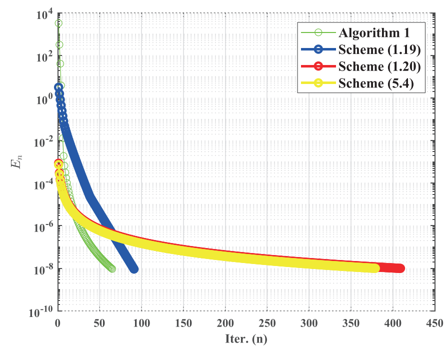
(c) $x_0 = 10e_1$, $\epsilon = 10^{-10}$



(d) $\epsilon = 10^{-8}$, $x_0 = e_1$



(e) $\epsilon = 10^{-8}$, $x_0 = 100e_1$



(f) $\epsilon = 10^{-8}$, $x_0 = 30e_1$

FIGURE 2. Comparison of Algorithm 1 with Scheme (1.19), Scheme (1.20), and Scheme (5.4)

It is readily seen from Table 2 and Figure 2 that, for each choices of the initial point x_0 and each values of ϵ, Algorithm 1 has better performance interms of less number of iterations and small CPU-run time in seconds than the compared methods.

Example 5.2. Let $H_1 = H_2 = L_2([0, 2\pi])$ with the inner product $\langle . \rangle$ defined by

$$\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt, \ \forall x, y \in L_2([0, 2\pi])$$

and with the norm $\|.\|$ defined by

$$\|x\|_2 := \sqrt{\int_0^{2\pi} |x(t)|^2 dt}, \ \forall x, y \in L^2([0, 2\pi]).$$

Further, we consider the following half-spaces

$$C := \left\{ x \in L_2([0, 2\pi]) : \int_0^{2\pi} x(t)dt \leq 1 \right\}$$

and

$$Q := \left\{ y \in L_2([0, 2\pi]) : \int_0^{2\pi} |y(t) - sin(t)|^2 dt \leq 16 \right\}.$$

In addition, we consider a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Ax)(t) = x(t)$. Then, $(A^*x)(t) = x(t)$ and $\|A\| = 1$. That is, A is an identity operator. The metric projection onto C and Q have an explicit formula (see [8]). We can also write the projections onto C and the projections onto Q as follows:

$$P_C(x(t)) = \begin{cases} x(t) + \frac{1 - \int_0^{2\pi} x(t)dt}{4\pi^2}, & \text{if } \int_0^{2\pi} x(t)dt > 1, \\ x(t), & \text{if } \int_0^{2\pi} x(t)dt \leq 1. \end{cases}$$

$$P_Q(y(t)) = \begin{cases} sin(t) + \frac{4(y(t) - sin(t))}{\sqrt{\int_0^{2\pi} |y(t) - sin(t)|^2 dt}}, & \text{if } \int_0^{2\pi} |y(t) - sin(t)|^2 dt > 16, \\ y(t), & \text{if } \int_0^{2\pi} |y(t) - sin(t)|^2 dt \leq 16. \end{cases}$$

Now, we solve the following problem

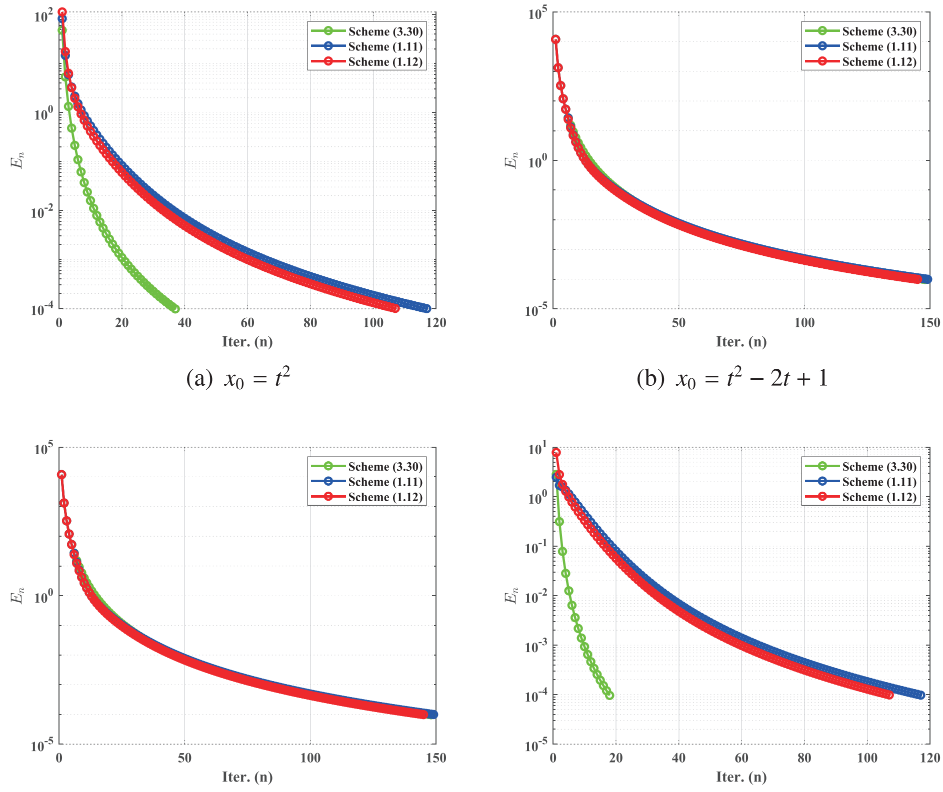
$$\text{find } p^* \in C \text{ such that } Ap^* \in Q. \tag{5.6}$$

We see here that our iterative method can be implemented to solve the problem (5.6) considered in this example. In this example, we compare Scheme (3.30) with the strong convergence results given by Scheme (1.11) and Scheme (1.12). For all methods, we choose $u = \frac{2^t}{2}, \alpha_n = \beta_n = \frac{1}{n+1}$, $\varepsilon_n = \frac{n}{2(n+1)}$, $\delta_n = 1 - \beta_n - \varepsilon_n$, $\rho_n = \frac{n}{10^{10}n+1}$ for different choices of $x_0 \in H_1$.

The error of the iterative algorithms is denoted by $E_n = \|x_{n+1} - x_n\|^2$. We use $E_n < 10^{-4}$ as stopping criteria for all methods and the outcomes of the numerical experiment of the compared methods are reported in Table 3 and Figure 3.

TABLE 3. Comparison of Scheme (3.30) with Scheme (1.11) and Scheme (1.12) for different choices of x_0

		Scheme (3.30)	Scheme (1.11)	Scheme (1.12)
$x_0 = t^2$	Iter. (n)	37	117	107
	CPU(s)	315.75445	878.604041	734.067342
	E_n	9.69942E-05	9.81776E-05	9.92371E-05
$x_0 = t^2 - 2t + 1$	Iter. (n)	25	117	107
	CPU(s)	154.638795	814.225934	758.915085
	E_n	8.9232E-05	9.79585E-05	9.89244E-05
$x_0 = sin(t)e^t$	Iter. (n)	142	149	145
	CPU(s)	909.102986	1050.792024	1021.03839
	E_n	9.78966E-05	9.96792E-05	9.89159E-05
$x_0 = \frac{2^t}{2} - \frac{3^t}{200}$	Iter. (n)	18	117	107
	CPU(s)	118.03319	815.592689	764.397045
	E_n	9.64645E-05	9.78326E-05	9.87446E-05



(a) $x_0 = t^2$ (b) $x_0 = t^2 - 2t + 1$

(c) $x_0 = sin(t)e^t$ (d) $x_0 = \frac{2^t}{2} - \frac{3^t}{200}$

FIGURE 3. Comparison of Scheme (3.30) with Scheme (1.11) and Scheme (1.12) for different choices of x_0

It is observed from the numerical experiment outcomes reported in Table 3 and Figure 3 that Scheme (3.30) is easily implementable and has better performance than Scheme (1.11) and Scheme (1.12) in terms of less number of iterations and small CPU-time run in seconds.

Example 5.3. Compressed Sensing: In this section, we consider numerical experiments to illustrate the application of the proposed algorithm to inverse problems arising from signal processing. Compressed sensing is a very active domain of research and applications, based on the fact that an N-sample signal x with exactly L nonzero components can be recovered from $L \ll M < N$ measurements as long as the number of measurements is smaller than the number of signal samples and at the same time much larger than the sparsity level of x. Likewise, the measurements are required to be incoherent, which means that the information contained in the signal is spread out in the domain. Since $M < N$, the problem of recovering x from M measurements is ill conditioned because we encounter an underdeterminated system of linear equations. But, using a sparsity prior, it turns out that reconstructing x from b is possible as long as the number of nonzero elements is small enough (see [39]). More specifically, compressed sensing can be formulated as inverting the equation system

$$b = Ax + \Theta, \tag{5.7}$$

where $x \in \mathbb{R}^N$ is a vector with L nonzero components to be recovered, $b \in \mathbb{R}^M$ is the vector of noisy observations or measurements (the measured data) with noisy Θ (when $\Theta = 0$, it means that there is no noise to the observed data), and $A : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded linear observation operator, often ill-conditioned because it models a process with loss of information. A powerful approach for problem (5.7) consists in considering a solution x represented by a sparse expansion, that is, represented by a series expansion with respect to an orthonormal basis that has only a small number of large coefficients. When attempting to find sparse solutions to linear inverse problems of type (5.7), successful model is the convex unconstrained minimization problem

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax - b\|_2^2 + \varpi\|x\|_1, \tag{5.8}$$

where ϖ is positive parameter and $\|.\|_1$ is the ℓ_1 norm. Problem (5.8) consists in minimizing an objective function, which includes a quadratic error term combined with a sparseness-including ℓ_1 regularization term, which is to make small component of x to become zero. Problem (5.7) can be seen as the following least absolute shrinkage and selection operator (LASSO), which is commonly used in the theory of signal processing (see [22])

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Ax - b\|_2^2 \text{ subject to } \|x\|_1 \le t, \tag{5.9}$$

where $t > 0$ is a given constant. By the theory of convex analysis, one is able to show that a solution to the LASSO problem (5.9), for appropriate choices $t > 0$, is a minimizer of (5.8) (see [20]). It can be observed that (5.9) indicates the potential of finding a sparse solution of the SFP (1.1) due to the ℓ_1 constraint. More precisely, it is readily seen that problem (5.9) is a particular case of the SFP (1.1) with $C := \{x : \|x\|_1 \leq t\}$ and $Q = \{b\}$, and thus can be solved by Scheme (3.30) and the iterative methods given by Scheme (1.11) and Scheme (1.12). We define the convex function $c(x) = \|x\|_1 - t$, and according (1.4), the level set C_n is defined by

$$C_n = \{x \in \mathbb{R}^N : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\},$$

where $\xi_n \in \partial\omega(x_n)$. Observe that the metric projection onto C_n can be computed by the following manner,

$$P_{C_n}(z)) = \begin{cases} z, & \text{if} \quad c(x_n) + \langle \xi_n, z - x_n \rangle \leq 0, \\ z - \frac{\langle c(x_n) + \langle \xi_n, z - x_n \rangle}{\|\xi_n\|^2_{L^2}} \xi_n, & \text{otherwise .} \end{cases}$$

We choose a subgradient $\xi_n \in \partial c(x_n)$ as

$$(\xi_n)_i = \begin{cases} 1 & \text{if} \quad (\xi_n)_i > 0, \\ 0 & \text{if} \quad (\xi_n)_i = 0, \\ -1 & \text{if} \quad (\xi_n)_i < 0. \end{cases}$$

In a special case where $Q = Q_n = \{b\}$, Scheme (3.30) converges to the solution of (5.9). Moreover, Scheme (3.30) can be implemented easily, because the projection onto the level set has an explicit formula. In order to show the efficacy of Scheme (3.30), a comparative sparse signal recovery experiments were carried-out with Scheme (1.11) and Scheme (1.12).

The vector x is a L sparse signal with non-zero L elements that are generated from uniform distribution within an interval of $[-2, 2]$, A is a matrix generated from normal distribution with mean zero and variance of one and b is an observation generated by white Gaussian noise with signal-to-noise ratio $SNR = 40$. The process of sparse signal recovery start by randomly generating $t = L$ and u, x_0 are $N \times 1$ vectors. The main target is then to recover the L sparse signal by solving (5.9) for x. The restoration accuracy is then measured by mean squared error (MSE) as follows:

$$MSE = \frac{\|x_{n+1} - x\|}{N} \leq \epsilon, \tag{5.10}$$

where x_n is an estimated signal of x, and $\epsilon > 0$ is a given small constant. We choose the parameters $\alpha_n = \beta_n = \frac{1}{100n+1}$, $\delta_n = \frac{1}{2} - \frac{1}{100n}$, $\varepsilon_n = \frac{1}{2} + \frac{1}{100n(100n+1)}$, $\rho_n = 3.5$. In our numerical experiments, for $u =$ ones $([N, 1])$ and $x_0 =$ ones $([N, 1])$, we consider four choices: **Choice 1:** $L = 20, N = 2^{12}, M = 2^{10}$; **Choice 2:** $L = 40, N = 2^{12}, M = 2^{10}$; **Choice 3:** $L = 20, N = 2^{14}, M = 2^{12}$; **Choice 4:** $L = 40, N = 2^{14}, M = 2^{12}$. *We use $MSE < \epsilon = 10^{-4}$ as stopping criterion for all methods. The results of the numerical experiments interms of number of iterations (Iter. (n)) and the CPU-run time in seconds (CPU(s)) are reported in Table 4 and Figures 4-8.*

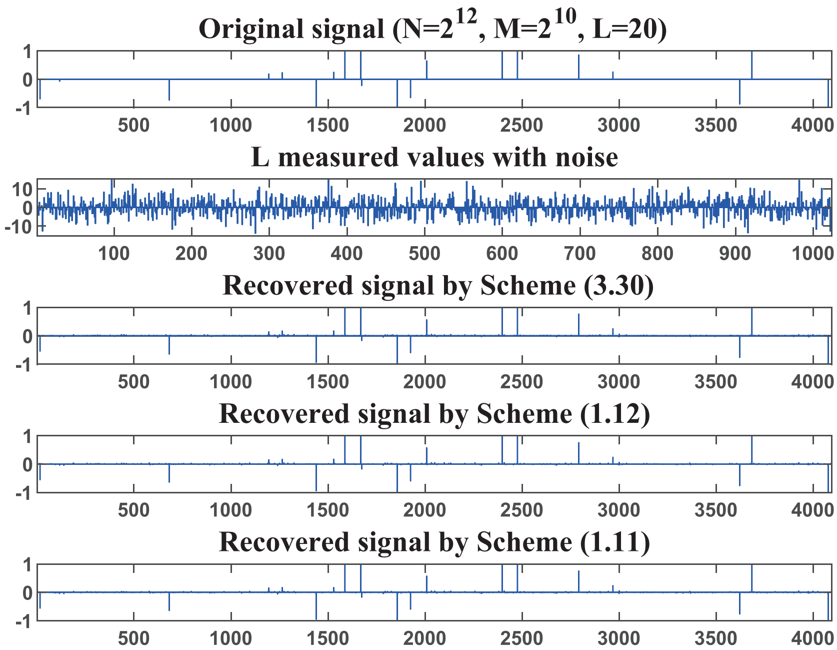


FIGURE 4. Original L-sparse signal versus recovered sparse signals by compared methods for **Choice 1**

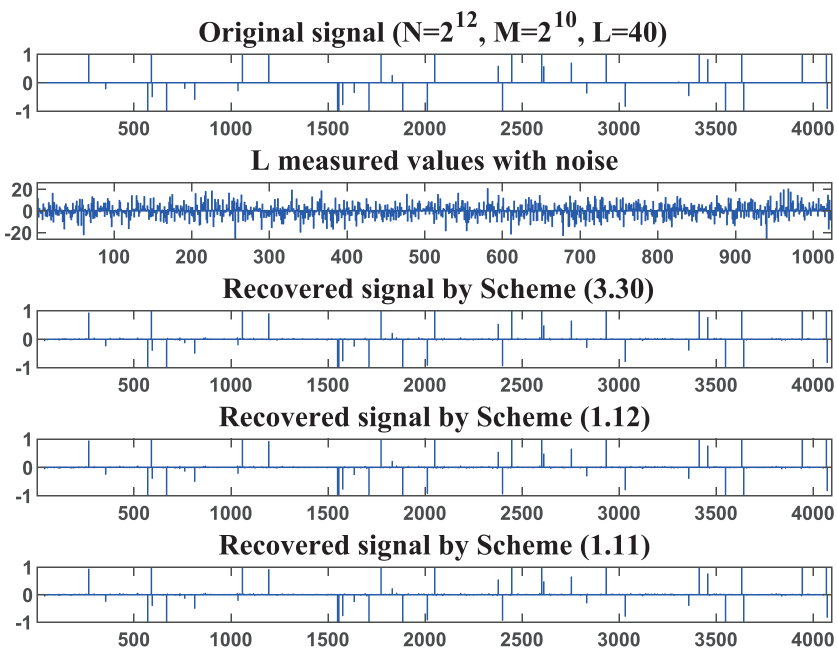


FIGURE 5. Original L-sparse signal versus recovered sparse signals by compared methods for **Choice 2**

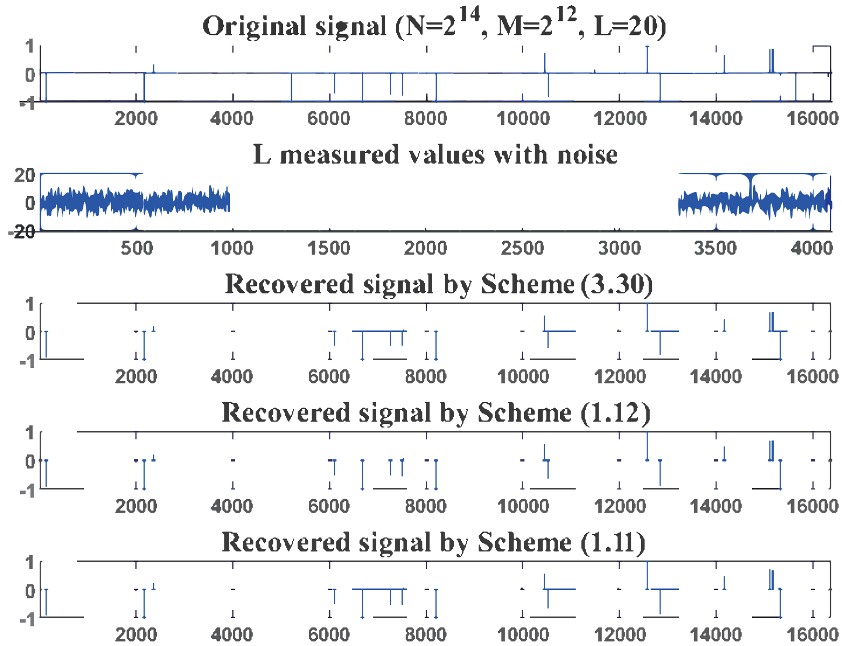


FIGURE 6. Original L-sparse signal versus recovered sparse signals by compared methods for **Choice 3**

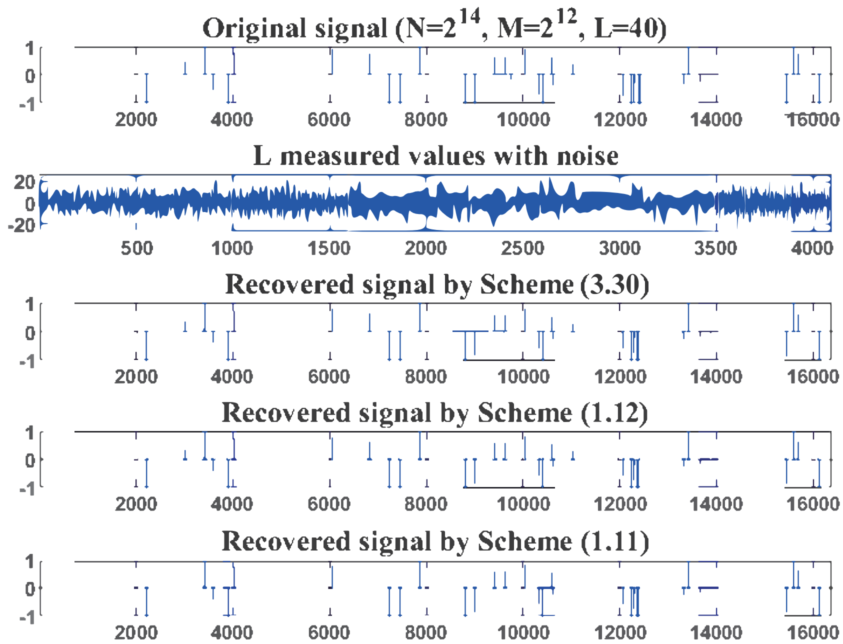
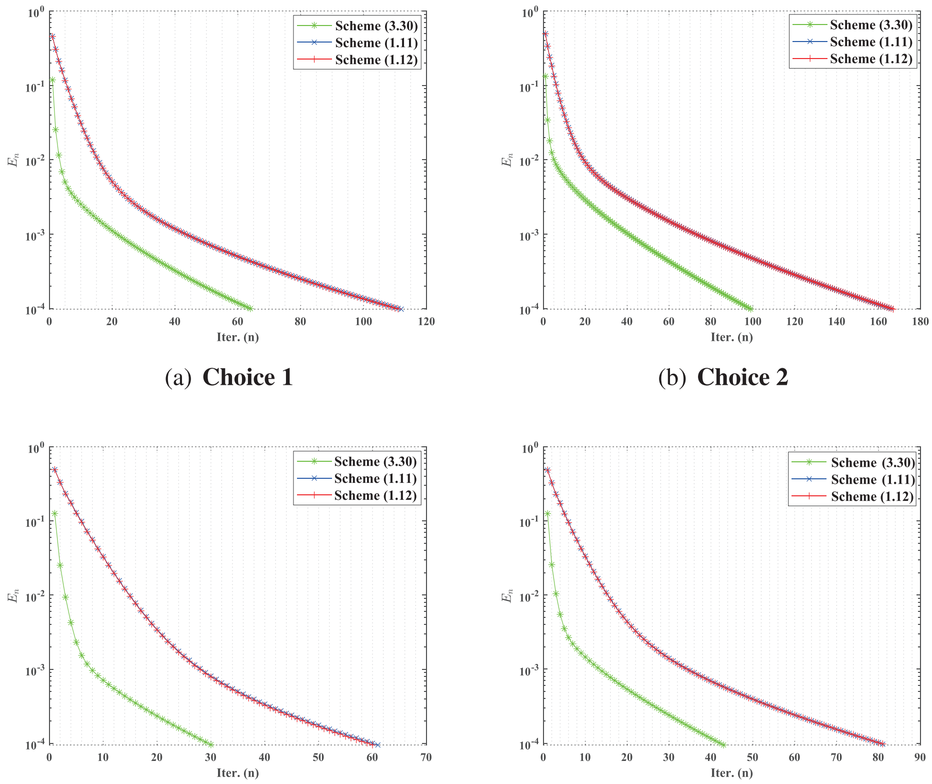


FIGURE 7. Original L-sparse signal versus recovered sparse signals by compared methods for **Choice 4**



(a) **Choice 1** (b) **Choice 2**

(c) **Choice 3** (d) **Choice 4**

FIGURE 8. MSE against Iter. (n) for comparison of Scheme (3.30) with Scheme (1.11) and Scheme (1.12)

It can be observed from Table 4 and Figures 4-8 that the recovered signal by the proposed method has less number of iterations and small CPU(s) time to converge than by the compared methods.

TABLE 4. The experiments of compressed sensing via Scheme (3.30), Scheme (1.11), and Scheme (1.12)

	Scheme (3.30)			*Scheme (1.11)*			*Scheme (1.12)*		
	Iter. (n)	*CPU(s)*	*MSE*	*Iter. (n)*	*CPU(s)*	*MSE*	*Iter. (n)*	*CPU(s)*	*MSE*
Choice 1	*64*	*0.9348*	*9.95E-05*	*112*	*1.6343*	*9.79E-05*	*111*	*1.6464*	*9.88E-05*
Choice 2	*99*	*1.5069*	*9.83E-05*	*166*	*2.6089*	*9.94E-05*	*167*	*2.5307*	*9.79E-05*
Choice 3	*30*	*6.3471*	*9.60E-05*	*61*	*13.0252*	*9.52E-05*	*60*	*12.5858*	*9.54E-05*
Choice 4	*43*	*9.3767*	*9.49E-05*	*81*	*17.245*	*9.73E-05*	*81*	*17.8173*	*9.83E-05*

COMPETING INTERESTS

The authors declare that they have no competing interests.

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Author's contributions

The work presented here was carried out in collaboration between all authors. All authors have contributed equally to the manuscript, checked, read and approved the final manuscript.

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