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A NOTE ON THE TOPOLOGICAL TRANSVERSALITY THEOREM FOR MAPS WITH LOWER SEMICONTINUOUS TYPE SELECTIONS

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Dedicated to Professor's Wataru Takahashi and Naoki Shioji with much admiration.

Abstract. This paper presents a Leray–Schauder alternative and a topological transversality theorem for compact maps which have lower semicontinuous type selections. Basically if we have two compact maps F and G which have lower semicontinuous type selections and $F \cong G$ (defined in an appropriate way) then one map being essential guarantees the essentiality of the other map.

1. Introduction

Consider continuous compact single valued maps F and G with $F \cong G$ in this setting. The topological transversality theorem states that F is essential if and only if G is essential (essential maps were introduced by Granas [3] and extended by O'Regan [5]). The topological transversality theorem (homotopy theorem) for multivalued maps which have lower semicontinuous type selections has not been considered in the literature and this is the first general theory to the best of my knowledge in this direction. The strategy in this paper is to present a simple result which will then generate a Leray–Schauder alternative and a topological transversality theorem in this setting which is new for multivalued maps with lower semicontinuous type selections. It is the new and applicable idea of essentiality that enables one to generate this homotopy theory.

We now describe the class of maps considered in this paper [1, 6]. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 and let F be a multifunction. We say $F \in W(X,Y)$ if $F : X \to 2^Y$ (here 2^Y denotes the family of nonempty subsets of Y) and there exists a $\theta : X \to 2^Y$ which is lower semicontinuous with $\overline{co}\,(\theta(x)) \subseteq F(x)$ for each $x \in X$. The following result was established in [1, 6].

Theorem 1.1. *Let X be a paracompact subset of a Hausdorff topological vector space E_1 and Y a metrizable, complete subset of a Hausdorff locally convex linear topological space E_2. If $F \in W(X,Y)$ then there exists an upper semicontinuous map $G : X \to K(Y)$ with $G(x) \subseteq F(x)$ for $x \in X$; here $K(Y)$ denotes the family of nonempty convex compact subsetes of Y.*

To establish our results recall the following fixed point result of Himmelberg [4].

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Theorem 1.2. *Let Y be a convex subset of a Hausdorff locally convex linear topological space, X a nonempty subset of Y, and let $G : Y \to K(X)$ be an upper semicontinuous compact map. Then there exists an $x \in X$ with $x \in G(x)$.*

2. Topological Transversality Theorem

Let E be a Fréchet space and U an open subset of E. We begin by defining the classes of maps we will consider and then we introduce the idea of an essential map.

Definition 2.1. We say $F \in CW(\overline{U}, E)$ if $F \in W(\overline{U}, E)$ is a compact map; here $\overline{U}$ denotes the closure of U in E.

Definition 2.2. We say $F \in CW_{\partial U}(\overline{U}, E)$ if $F \in CW(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Definition 2.3. We say $G \in D(\overline{U}, E)$ if $G : \overline{U} \to K(E)$ is a upper semicontinuous compact map.

Definition 2.4. We say $G \in D_{\partial U}(\overline{U}, E)$ if $G \in D(\overline{U}, E)$ and $x \notin G(x)$ for $x \in \partial U$.

Now we introduce the notion of an essential map.

Definition 2.5. We say $F \in CW_{\partial U}(\overline{U}, E)$ is essential in $CW_{\partial U}(\overline{U}, E)$ if for any upper semicontinuous (compact) selection $\Phi : \overline{U} \to K(E)$ of F (i.e. $\Phi \in D_{\partial U}(\overline{U}, E)$) and any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = \Phi|_{\partial U}$ there exists a $x \in U$ with $x \in \theta(x)$.

Remark 2.6. Note E is metrizable so $\overline{U}$ is paracompact. Note Theorem 1.1 guarantees that there exists an upper semicontinuous selection $\Phi : \overline{U} \to K(E)$ of F in Definition 2.5.

Remark 2.7. If $F \in CW_{\partial U}(\overline{U}, E)$ is essential in $CW_{\partial U}(\overline{U}, E)$ and if $\Phi \in D(\overline{U}, E)$ is any selection of F then there exists a $x \in U$ with $x \in \Phi(x)$ (take $\theta = \Phi$ in Definition 2.5), so in particular there exists a $x \in U$ with $x \in F(x)$.

Next we present the notion of homotopy.

Definition 2.8. Let Φ, $\Psi \in D_{\partial U}(\overline{U}, E)$. We say $\Phi \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous, compact map $H : \overline{U} \times [0, 1] \to K(E)$ with $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0, 1)$ (here $H_t(x) = H(x, t)$), $H_0 = \Phi$ and $H_1 = \Psi$.

Remark 2.9. A standard argument guarantees that $\cong$ in $D_{\partial U}(\overline{U}, E)$ is an equivalence relation.

Definition 2.10. Let F, $G \in CW_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $CW_{\partial U}(\overline{U}, E)$ if for any selection $\Phi \in D_{\partial U}(\overline{U}, E)$ (respectively, $\Psi \in D_{\partial U}(\overline{U}, E)$) of F (respectively, of G) we have $\Phi \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$.

Next we present a simple result which will then generate a Leray–Schauder alternative and a topological transversality theorem.

Theorem 2.11. *Let E be a Fréchet space, U an open subset of E and $F \in WC_{\partial U}(\overline{U}, E)$. Assume $G \in WC_{\partial U}(\overline{U}, E)$ is essential in $WC_{\partial U}(\overline{U}, E)$ and suppose the following holds:*

$$\text{(2.1)} \qquad \begin{cases} \text{for any selection } \Phi \in D_{\partial U}(\overline{U}, E) \ \ (\text{respectively, } \Psi \in D_{\partial U}(\overline{U}, E)) \\ \text{of } F \ \ (\text{respectively, of } G) \text{ and any map } \theta \in D_{\partial U}(\overline{U}, E) \\ \text{with } \theta|_{\partial U} = \Phi|_{\partial U} \ \text{ we have } \Psi \cong \theta \ \text{ in } D_{\partial U}(\overline{U}, E). \end{cases}$$

Then F is essential in $WC_{\partial U}(\overline{U}, E)$.

Proof. Let $\Phi \in D_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = \Phi|_{\partial U}$. We must show there exists a $x \in U$ with $x \in \theta(x)$. Let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of G. Now (2.1) guarantees that there exists a upper semicontinuous, compact map $H : \overline{U} \times [0,1] \to K(E)$ with $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $H_0 = \Psi$ and $H_1 = \theta$. Let

$$\Omega = \left\{ x \in \overline{U} : \ x \in H(x,t) \ \text{ for some } \ t \in [0,1] \right\}.$$

Now $\Omega \neq \emptyset$ since G is essential in $CW_{\partial U}(\overline{U}, E)$ (see Remark 2.7). Also Ω is closed since H is upper semicontinuous (in fact Ω is compact since H is compact). Also note $\Omega \cap \partial U = \emptyset$ since $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$. Then there exists a Urysohn continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H(x, \mu(x)) = H \circ g(x)$ where $g : \overline{U} \to \overline{U} \times [0,1]$ is given by $g(x) = (x, \mu(x))$. Note R is a upper semicontinuous compact map with $R|_{\partial U} = \Psi|_{\partial U}$ since if $x \in \partial U$ then $R(x) = H(x,0) = \Psi(x)$ and thus $R \in D_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = \Psi|_{\partial U}$. Now since G is essential in $CW_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $x \in R(x)$ i.e. $x \in H_{\mu(x)}(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $x \in H_1(x) = \theta(x)$. □

The above result then generates a very general Leray–Schauder type result.

Theorem 2.12. *Let E be a Fréchet space, U an open subset of E and $F \in WC_{\partial U}(\overline{U}, E)$. Assume $G \in WC_{\partial U}(\overline{U}, E)$ is essential in $WC_{\partial U}(\overline{U}, E)$ and $x \notin t\,F(x) + (1-t)\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $WC_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Proof. Let $\Phi \in D_{\partial U}(\overline{U}, E)$ (respectively, $\Psi \in D_{\partial U}(\overline{U}, E)$) be any selection of F (respectively, of G). Now consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = \Phi|_{\partial U}$. Let $H(x,t) = t\,\theta(x) + (1-t)\,\Psi(x)$. Note $H : \overline{U} \times [0,1] \to K(E)$ is a upper semicontinuous compact map (see [2, Theorem 4.18]) and $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (note if $x \in \partial U$ and $t \in (0,1)$ then since $\theta|_{\partial U} = \Phi|_{\partial U}$ we have $H_t(x) = t\,\Phi(x) + (1-t)\,\Psi(x)$ and note $x \notin t\,\Phi(x) + (1-t)\,\Psi(x)$ since $x \notin t\,F(x) + (1-t)\,G(x)$), $H_0 = \Psi$ and $H_1 = \theta$. Thus (2.1) holds so our result follows from Theorem 2.11. □

Next we present an applicable example of an essential map.

Theorem 2.13. *Let E be a Fréchet space, U an open subset of E and $0 \in U$. Then the zero map is essential in $WC_{\partial U}(\overline{U}, E)$.*

Proof. Let $F(x) = \{0\}$ for $x \in \overline{U}$ (i.e. F is the zero map) and let $\Phi \in D_{\partial U}(\overline{U}, E)$ be any selection of F. Note $\Phi(x) = \{0\}$ for $x \in \overline{U}$. Consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = \Phi|_{\partial U} = \{0\}$. We must show there exists a $x \in U$ with $x \in \theta(x)$. Let

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Note $J : E \to K(E)$ is a upper semicontinuous, compact map so Theorem 1.2 guarantees that there exists a $x \in E$ with $x \in J(x)$. If $x \in E \backslash U$ then $J(x) = \{0\}$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \theta(x)$. □

Theorem 2.14. *Let E be a Fréchet space, U an open subset of E and $0 \in U$. Suppose $F \in WC_{\partial U}(\overline{U}, E)$ with $x \notin t\,F(x)$ for $x \in \partial U$ and $t \in (0, 1)$. Then F is essential in $WC_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Proof. The result follows from Theorem 2.12 and Theorem 2.13 with G being the zero map. □

To establish the topological transversality theorem first note the following:

$$\text{(2.2)} \qquad \begin{cases} \text{if } \ \Phi,\, \Psi \in D_{\partial U}(\overline{U}, E) \ \text{ with } \ \Phi|_{\partial U} = \Psi|_{\partial U} \\ \text{then } \ \Phi \cong \Psi \ \text{ in } \ D_{\partial U}(\overline{U}, E). \end{cases}$$

To see this let $H(x, t) = (1 - t)\,\Phi(x) + t\,\Psi(x)$ and note $H : \overline{U} \times [0, 1] \to K(E)$ is a upper semicontinuous compact map with $x \notin H_t(x)$ for any $x \in \partial\, U$ and $t \in (0, 1)$ (note if $x \in \partial U$ and $t \in (0, 1)$ then $H_t(x) = (1 - t)\,\Phi(x) + t\,\Psi(x) = \Phi(x)$ since $\Phi|_{\partial U} = \Psi|_{\partial U}$ and note $x \notin \Phi(x)$ since $\Phi \in D_{\partial U}(\overline{U}, E)$).

Remark 2.15. From (2.2) note in (2.1) that since $\theta \in D_{\partial U}(\overline{U}, E)$ and $\theta|_{\partial U} = \Phi|_{\partial U}$ then $\theta \cong \Phi$ in $D_{\partial U}(\overline{U}, E)$.

Theorem 2.16. *Let E be a Fréchet space and U an open subset of E. Suppose F and G are two maps in $CW_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $CW_{\partial U}(\overline{U}, E)$. Now F is essential in $CW_{\partial U}(\overline{U}, E)$ if and only if G is essential in $CW_{\partial U}(\overline{U}, E)$.*

Proof. Assume G is essential in $CW_{\partial U}(\overline{U}, E)$. We will use Theorem 2.11 to show F is essential in $CW_{\partial U}(\overline{U}, E)$. Let $\Phi \in D_{\partial U}(\overline{U}, E)$ be any selection of F and let $\Psi \in D_{\partial U}(\overline{U}, E)$ be any selection of G. Now consider any map $\theta \in D_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = \Phi|_{\partial U}$. Now (2.2) guarantees that $\Phi \cong \theta$ in $D_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $CW_{\partial U}(\overline{U}, E)$ (so $\Phi \cong \Psi$ in $D_{\partial U}(\overline{U}, E)$) and Remark 2.9 (i.e. $\cong$ in $D_{\partial U}(\overline{U}, E)$ is an equivalence relation) guarantees that $\Psi \cong \theta$ in $D_{\partial U}(\overline{U}, E)$. Thus (2.1) holds. Now Theorem 2.11 guarantees that F is essential in $CW_{\partial U}(\overline{U}, E)$. A similar argument shows if F is essential in $CW_{\partial U}(\overline{U}, E)$ then G is essential in $CW_{\partial U}(\overline{U}, E)$. □

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