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AN n-DIMENSIONAL INTERMEDIATE VALUE THEOREM AND ITS APPLICATION TO THE GAME THEORY

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Dedicated to the memory of Professor Wataru Takahashi and Professor Naoki Shioji

Abstract. The Poincaré-Miranda theorem is an n-dimensional zero-point theorem, which was conjectured by Poincaré [4]. Miranda [3] showed that it is equivalent to Brouwer's fixed point theorem. Further it is equivalent to the Hadamard theorem [1], see [2] for simple proofs. Although it is sometimes called an extension of the intermediate value theorem, it does not seem to be explicitly given as an n-dimensional intermediate value theorem. In this paper we explicitly mention an n-dimensional intermediate value theorem, and show that it is equivalent to the Poincaré-Miranda theorem. As an application we deal with a bimatrix game and a three-person strategic game, and show the range of payoffs that can be achieved by mixed strategies.

1. Introduction

Let $[a_i, b_i] \subset \mathbb{R}$ $(i = 1, \ldots, n)$ be closed intervals, and $I = [a, b]$ denote the product $[a_1, b_1] \times \cdots \times [a_n, b_n]$. Let $f = (f_1, \ldots, f_n) : I \to \mathbb{R}^n$ be a continuous function. The classical intermediate value theorem states that when $n = 1$

(1) for any value γ satisfying $\min\{f(a), f(b)\} \leq \gamma \leq \max\{f(a), f(b)\}$ there exists some point $a \leq c \leq b$ such that $f(c) = \gamma$,

(2) in particular when $\min\{f(a), f(b)\} < \gamma < \max\{f(a), f(b)\}$ there exists some point $a < c < b$ such that $f(c) = \gamma$.

Although the intermediate value theorem $(n = 1)$ is usually stated as (2), the first form is equivalent to Brouwer's fixed point theorem for $n = 1$ and the following Poincaré-Miranda theorem for $n = 1$.

Theorem 1.1 (Poincaré-Miranda). *If a continuous function $g = (g_1, \ldots, g_n) : I \to \mathbb{R}^n$ satisfies the following boundary condition:*

$$g_i(x) \leq 0 \ (x \in I,\ x_i = a_i), \quad g_i(x) \geq 0 \ (x \in I,\ x_i = b_i) \tag{1.1}$$

or

$$g_i(x) \geq 0 \ (x \in I,\ x_i = a_i), \quad g_i(x) \leq 0 \ (x \in I,\ x_i = b_i) \tag{1.2}$$

for any $i = 1, \ldots, n$, then g has a zero-point $c \in I$.

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In this paper we present an n-dimensional intermediate value theorem, and show that it is equivalent to the Poincaré-Miranda theorem. Next we apply it to two and three-person strategic games, and show the range of payoffs that can be achieved by mixed strategies.

2. n-dimensional intermediate value theorem

Theorem 2.1. *Let $f = (f_1, \dots, f_n) : I \to \mathbb{R}^n$ be a continuous function. Define $\overline{\alpha}_i$, $\overline{\beta}_i$, $\underline{\alpha}_i$, and $\underline{\beta}_i$ for $i = 1, \dots, n$ by*

$$\overline{\alpha}_i := \max\{f_i(x) \mid x \in I,\ x_i = a_i\},\ \overline{\beta}_i := \max\{f_i(x) \mid x \in I,\ x_i = b_i\}, \tag{2.1}$$

$$\underline{\alpha}_i := \min\{f_i(x) \mid x \in I,\ x_i = a_i\},\ \underline{\beta}_i := \min\{f_i(x) \mid x \in I,\ x_i = b_i\}. \tag{2.2}$$

Then for any $\gamma = (\gamma_1, \dots, \gamma_n)$ satisfying

$$\min\{\overline{\alpha}_i, \overline{\beta}_i\} \le \gamma_i \le \max\{\underline{\alpha}_i, \underline{\beta}_i\} \quad (i = 1, \dots, n), \tag{2.3}$$

there exists some point $c = (c_1, \dots, c_n) \in I$ such that $f(c) = \gamma$. In particular, when γ_i satisfies

$$\min\{\overline{\alpha}_i, \overline{\beta}_i\} < \gamma_i < \max\{\underline{\alpha}_i, \underline{\beta}_i\}, \tag{2.4}$$

it holds that $a_i < c_i < b_i$.

Proof. Define a function $g : I \to \mathbb{R}^n$ by $g(x) = \gamma - f(x)$. Taking account of $\underline{\alpha}_i \le \overline{\alpha}_i$ and $\underline{\beta}_i \le \overline{\beta}_i$, we see that (2.3) is equivalent to

$$\overline{\alpha}_i \le \gamma_i \le \underline{\beta}_i \text{ or } \overline{\beta}_i \le \gamma_i \le \underline{\alpha}_i. \tag{2.5}$$

In the former case, it holds that

$$g_i(x) = \gamma_i - f_i(x) \begin{cases} \ge \overline{\alpha}_i - f_i(x) \ge 0 & (x_i = a_i) \\ \le \underline{\beta}_i - f_i(x) \le 0 & (x_i = b_i). \end{cases} \tag{2.6}$$

In the latter case, it holds that

$$g_i(x) = \gamma_i - f_i(x) \begin{cases} \le \underline{\alpha}_i - f_i(x) \le 0 & (x_i = a_i) \\ \ge \overline{\beta}_i - f_i(x) \ge 0 & (x_i = b_i). \end{cases} \tag{2.7}$$

Hence, by the Poincaré-Miranda theorem, there exists $c \in I$ such that $g(c) = 0$, that is, $f(c) = \gamma$.

In particular when γ_i satisfies (2.4), assume that $\overline{\alpha}_i < \gamma_i < \underline{\beta}_i$ and $c_i = a_i$, then $f_i(c) \le \overline{\alpha}_i < \gamma_i = f_i(c)$. Hence $c_i \ne a_i$. Assume that $\overline{\alpha}_i < \gamma_i < \underline{\beta}_i$ and $c_i = b_i$, then $\underline{\beta}_i \le f_i(c) = \gamma_i < \underline{\beta}_i$. Hence $c_i \ne b_i$. Therefore we have $a_i < c_i < b_i$. Similarly we have $a_i < c_i < b_i$ in the case of $\overline{\beta}_i < \gamma_i < \underline{\alpha}_i$. $\square$

Theorem 2.2. *Theorem 2.1 implies the Poincaré-Miranda theorem.*

Proof. If g_i satisfies boundary condition (1.1). Then it holds that

$$\overline{\alpha}_i = \max\{g_i(x) \mid x \in I,\ x_i = a_i\} \le 0 \le \min\{g_i(x) \mid x \in I,\ x_i = b_i\} = \underline{\beta}_i.$$

If g_i satisfies boundary condition (1.2). Then it holds that

$$\overline{\beta}_i = \max\{g_i(x) \mid x \in I,\ x_i = b_i\} \le 0 \le \min\{g_i(x) \mid x \in I,\ x_i = a_i\} = \underline{\alpha}_i.$$

Hence $\gamma_i = 0$ satisfies (2.3). By Theorem 2.1, there exists $c \in I$ such that $g(c) = \mathbf{0}$. $\square$

Example 2.3. This example shows the necessity of assumption (2.3). Let $I = [-1,1]^2$, $f_1(x_1,x_2) = -(x_1+1)^2 - x_2^2$, and $f_1(x_1,x_2) = x_2$. Then

$$\begin{aligned}\overline{\alpha}_1 &= \max\{f_1(x_1,x_2) \mid x_1 = -1\} = \max\{-x_2^2 \mid x_2 \in [-1,1]\} = 0,\\ \underline{\alpha}_1 &= \min\{f_1(x_1,x_2) \mid x_1 = -1\} = \min\{-x_2^2 \mid x_2 \in [-1,1]\} = -1.\end{aligned}$$

Similarly computing $\overline{\beta}_1$, $\underline{\beta}_2$, etc., we have

$$\begin{aligned}\min\{\overline{\alpha}_1, \overline{\beta}_1\} &= \min\{0,-4\} = -4, \qquad \max\{\underline{\alpha}_1, \underline{\beta}_1\} = \max\{-1,-5\} = -1\\ \min\{\overline{\alpha}_2, \overline{\beta}_2\} &= \min\{-1,1\} = -1, \qquad \max\{\underline{\alpha}_2, \underline{\beta}_2\} = \max\{-1,1\} = 1.\end{aligned}$$

By Theorem 2.1, for any $\gamma \in [-4,-1]\times[-1,1]$, there exists a point $c \in [-1,1]^2$ such that $f(c) = \gamma$. Further, for $\gamma \in (-4,-1)\times(-1,1)$, c is in $(-1,1)^2$. On the other

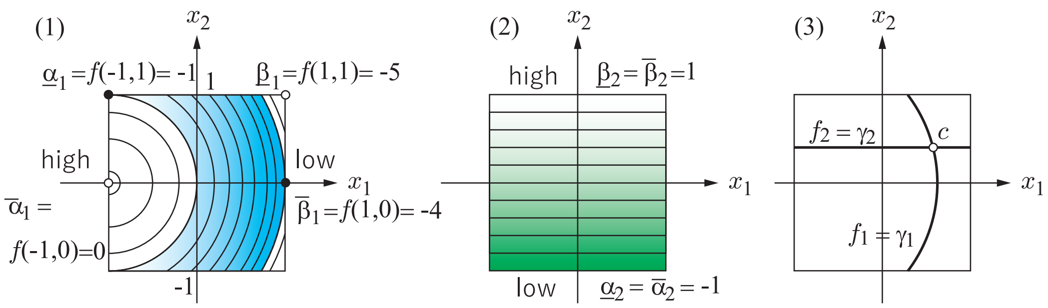

FIGURE 1. (1) Level sets of f_1. (2) Level sets of f_2. (3) $f(c) = \gamma$.

hand, it holds that for $\gamma := (-\varepsilon^2, 1-\varepsilon) \notin [-4,-1] \times [-1,1]$

$$\begin{aligned} f(x_1,x_2)=\gamma \;&\Leftrightarrow\; (x_1+1)^2+x_2^2=\varepsilon^2,\ x_2=1-\varepsilon \\ &\Rightarrow\; (x_1+1)^2=\varepsilon^2-(1-\varepsilon)^2=2\varepsilon-1<0, \end{aligned}$$

where $\varepsilon > 0$ is sufficiently small. Hence there is no $x \in I$ such that $f(x)=\gamma$.

3. Application to the game theory

First, we apply Theorem 2.1 to the following bimatrix game:

- Each player has two actions called pure strategies.
- The payoff of player i is given by

$$f_i(x_1,y_1)=(x_1,1-x_1)\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} y_1 \\ 1-y_1 \end{pmatrix} \quad (i=1,2) \tag{3.1}$$

  where $x_1, y_1 \in [0,1]$. Player i wants to maximize f_i, predicting opponent's action.

In the following Δ denotes the set of 1-dimensional probability vectors. Probability vectors $\boldsymbol{x}=(x_1,x_2)\in\Delta$ and $\boldsymbol{y}=(y_1,y_2)\in\Delta$ are called mixed strategies.

Theorem 3.1. *In the bimatrix game above, for any (γ_1,γ_2) satisfying*

$$\min\{\max\{a_1,b_1\},\max\{c_1,d_1\}\} \le \gamma_1 \le \max\{\min\{a_1,b_1\},\min\{c_1,d_1\}\} \tag{3.2}$$

$$\min\{\max\{a_2,c_2\},\max\{b_2,d_2\}\} \le \gamma_2 \le \max\{\min\{a_2,c_2\},\min\{b_2,d_2\}\}, \tag{3.3}$$

the payoffs (γ_1, γ_2) are achieved by some mixed strategies $(\boldsymbol{x},\boldsymbol{y})\in\Delta^2$. Further, if γ_1 satisfies (3.2) with strict inequalities, then $x_1\in(0,1)$. If γ_2 satisfies (3.3) with strict inequalities, then $y_1\in(0,1)$.

Proof. Taking $I=[0,1]^2$ in Theorem 2.1, we easily see that

$$\begin{aligned} \underline{\alpha}_1 &= \min\{f_1(0,y_1)\mid y_1\in[0,1]\}=\min\{c_1,d_1\}, \\ \overline{\alpha}_1 &= \max\{f_1(0,y_1)\mid y_1\in[0,1]\}=\max\{c_1,d_1\}, \\ \underline{\beta}_1 &= \min\{f_1(1,y_1)\mid y_1\in[0,1]\}=\min\{a_1,b_1\}, \\ \overline{\beta}_1 &= \max\{f_1(1,y_1)\mid y_1\in[0,1]\}=\max\{a_1,b_1\}, \end{aligned}$$

$$\begin{aligned} \underline{\alpha}_2 &= \min\{f_2(x_1,0)\mid x_1\in[0,1]\}=\min\{b_2,d_2\}, \\ \overline{\alpha}_2 &= \max\{f_2(x_1,0)\mid x_1\in[0,1]\}=\max\{b_2,d_2\}, \\ \underline{\beta}_2 &= \min\{f_2(x_1,1)\mid x_1\in[0,1]\}=\min\{a_2,c_2\}, \\ \overline{\beta}_2 &= \max\{f_2(x_1,1)\mid x_1\in[0,1]\}=\max\{a_2,c_2\}. \end{aligned}$$

Therefore (2.1) reduces to (3.2) and (3.3). □

Example 3.2. Let

$$\begin{aligned} f_1(x_1,y_1) &= (x_1,1-x_1)\begin{pmatrix} 2 & 3 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ 1-y_1 \end{pmatrix}, \\ f_2(x_1,y_1) &= (x_1,1-x_1)\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ 1-y_1 \end{pmatrix}. \end{aligned}$$

Then LHS and RHS of (3.2) are

$$LHS = \min\{\max\{2, 3\}, \max\{-2, 0\}\} = 0,$$

$$RHS = \max\{\min\{2, 3\}, \min\{-2, 0\}\} = 2,$$

respectively. Similarly LHS and RHS of (3.3) are

$$LHS = \min\{\max\{2, 4\}, \max\{-1, 1\}\} = 1,$$

$$RHS = \max\{\min\{2, 4\}, \min\{-1, 1\}\} = 2.$$

respectively. Therefore any $(\gamma_1, \gamma_2) \in [0, 2] \times [1, 2]$ can be achieved by some mixed strategies.

Next, we consider the following three-person game:

- It is a simultaneous game by three players in which each player has two actions.
- The payoffs of player $i = 1, 2, 3$ are given by

$$\begin{aligned} f_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= \textstyle\sum_{i,j,k\in\{1,2\}} a_{ijk} x_i y_j z_k, \\ f_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= \textstyle\sum_{i,j,k\in\{1,2\}} b_{ijk} x_i y_j z_k, \\ f_3(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) &= \textstyle\sum_{i,j,k\in\{1,2\}} c_{ijk} x_i y_j z_k, \end{aligned}$$

  where $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$, and $\boldsymbol{z} = (z_1, z_2)$ are probability vectors. Player i wants to maximize f_i, predicting individual players' actions.

Theorem 3.3. *In the three-person game above, for any $(\gamma_1, \gamma_2, \gamma_3)$ satisfying*

$$\min\{\max_{j,k\in\{1,2\}} a_{1jk}, \max_{j,k\in\{1,2\}} a_{2jk}\} \le \gamma_1 \le \max\{\min_{j,k\in\{1,2\}} a_{1jk}, \min_{j,k\in\{1,2\}} a_{2jk}\}, \tag{3.4}$$

$$\min\{\max_{i,k\in\{1,2\}} b_{i1k}, \max_{i,k\in\{1,2\}} b_{i2k}\} \le \gamma_2 \le \max\{\min_{i,k\in\{1,2\}} b_{i1k}, \min_{i,k\in\{1,2\}} b_{i2k}\}, \tag{3.5}$$

$$\min\{\max_{i,j\in\{1,2\}} c_{ij1}, \max_{i,j\in\{1,2\}} c_{ij2}\} \le \gamma_3 \le \max\{\min_{i,j\in\{1,2\}} c_{ij1}, \min_{i,j\in\{1,2\}} c_{ij2}\}, \tag{3.6}$$

the payoffs $(\gamma_1, \gamma_2, \gamma_3)$ are achieved by some mixed strategies $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \Delta^3$. Namely $(f_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), f_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), f_3(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})) = (\gamma_1, \gamma_2, \gamma_3)$.

Further, if γ_1 satisfies (3.4) with strict inequality, then $x_1 \in (0, 1)$. If γ_2 satisfies (3.5) with strict inequality, then $y_1 \in (0, 1)$. If γ_3 satisfies (3.6) with strict inequality, then $z_1 \in (0, 1)$.

Proof. Since $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are 1-dimensional probability vectors, payoff f_i can be regarded as a function with variables $(x_1, y_1, z_1) \in [0,1]^3$, so that we simply denote it by $f_i(x_1, y_1, z_1)$. For example, $f_1(x_1, y_1, z_1)$ is equal to

$$\begin{aligned}
&\sum_{i,j,k\in\{1,2\}} a_{ijk}x_iy_jz_k \\
= \;& a_{111}x_1y_1z_1 + a_{112}x_1y_1(1-z_1) + a_{121}x_1(1-y_1)z_1 \\
&+a_{122}x_1(1-y_1)(1-z_1) + a_{211}(1-x_1)y_1z_1 + a_{212}(1-x_1)y_1(1-z_1) \qquad (3.7)\\
&+a_{221}(1-x_1)(1-y_1)z_1 + a_{222}(1-x_1)(1-y_1)(1-z_1).
\end{aligned}$$

Hence $f_1(0, y_1, z_1)$ is equal to

$$(a_{211} - a_{212} - a_{221} + a_{222})y_1z_1 + (a_{212} - a_{222})y_1 + (a_{221} - a_{222})z_1 + a_{222},$$

and $\overline{\alpha}_1$ in (2.1) is

$$\overline{\alpha}_1 = \max\left\{f_1(0, y_1, z_1) \mid (y_1, z_1) \in [0,1]^2\right\}.$$

When $a_{211} - a_{212} - a_{221} + a_{222} \neq 0$, the stationary point of (3.7) is a saddle point. If the stationary point is in $(0,1)^2$, it is neither a maximum point nor a minimum point of (3.7). When $a_{211} - a_{212} - a_{221} + a_{222} = 0$, (3.7) is linear. In either case a maximum point of (3.7) can be found in the boundary of $[0,1]^2$. Furthermore since (3.7) is linear on each edge of the boundary we may assume that the maximum of (3.7) is attained by a corner of $[0,1]^2$. The minimum of (3.7) is also attained by a corner. Therefore

$$\underline{\alpha}_1 = \min_{j,k\in\{1,2\}} a_{2jk}, \quad \overline{\alpha}_1 = \max_{j,k\in\{1,2\}} a_{2jk}.$$

Similarly, we have

$$\underline{\beta}_1 = \min_{j,k\in\{1,2\}} a_{1jk}, \quad \overline{\beta}_1 = \max_{j,k\in\{1,2\}} a_{1jk}.$$

□

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