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ALGORITHMIC APPROACH TO A CLASS OF GENERALIZED SET-VALUED NONLINEAR VARIATIONAL-LIKE INEQUALITY PROBLEMS IN BANACH SPACES

JAVAD BALOOEE

Abstract. In this paper, the notion of P-η-proximal mapping is used and an iterative algorithm for finding the approximate solution of a new class of generalized set-valued nonlinear variational-like inequality problems is proposed. Under some suitable conditions, the convergence analysis of the sequences generated by our suggested iterative algorithm is studied. The final section is dedicated to the investigation and analysis of the notion of co-proximal operator and related results given in [R. Ahmad, S.S. Irfan, I. Ahmad, M. Rahaman, Co-proximal operators for solving generalized co-variational inclusion problems in q-uniformly smooth Banach spaces, J. Nonlinear Convex Anal. **19**(7)(2018) 1093–1107]. Some facts relating to them are also pointed out.

1. Introduction

Due to their applications to fields like economics, engineering, mechanics, elasticity, fluid mechanics, game theory and optimization [8, 21], variational inequalities and complementarity problems continue to attract the interest of many researchers, see, for example, [1–15, 17, 18, 20, 24, 25, 28, 29, 31–34, 36, 38] and the references therein. The importance of the theory of variational inequalities and its applications have motivated many researchers to extend, generalize and study it in many different directions. For more details and relevant commentaries, we refer the reader to [15]. With the goal of studying and solving of various classes of variational inequalities, in recent decades considerable efforts have been made to develop efficient and implementable numerical methods including projection method and its variant forms, descent method, linear approximation and the method based on the auxiliary principle technique. To find more information, the reader can refer to [2–7,9–14,17–20,24,28,29,31,32,34,38] and the references therein. It is well known that an important and significant generalization of convexity is invexity, which was introduced by Hanson [23] in 1981. In the light of the application of variational inequalities to optimization problems, and the notion of invexity, the introduction of the concept of variational-like inequality or pre-variational inequality was first made by Parida et al. [30] and Yang and Chen [36], independently. In fact, they replaced the linear term $y - x$ appearing in the formulation of variational inequalities by a vector-valued term $\eta(y, x)$, where η is a vector-valued bifunction. But,

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the presence of the vector-valued term $\eta(y, x)$ in the formulation of a variational-like inequality makes limitations for us in using of solution methods to compute approximate solutions of variational-like inequalities. Indeed, the auxiliary principle technique and the proximal method are the most studied methods for solving variational-like inequalities. It is significant to emphasize that we cannot employ the projection method to construct and propose any iterative algorithm for solving variational-like inequalities in the setting of Banach spaces. This is mainly because the standard projection method strictly depend on the inner product property of Hilbert spaces. This drawback motivated the researchers to introduce proximal (resolvent) mappings to overcome and resolve the above-mentioned problem. The concepts of η-subdifferential and η-proximal point mappings of a proper functional were initially introduced by Ding and Luo [18] and Lee et al. [25], independently, and under some suitable conditions, the existence and Lipschitz continuity of η-proximal mapping of a proper functional are proved. At the same time, they developed some perturbed η-proximal point algorithms for computing the approximate solutions of some classes of variational-like inequalities in a Hilbert space context. In order to develop and construct efficient iterative algorithms for solving variational-like inequalities in the framework of Banach spaces, Ding and Xia [20] succeeded to introduce the concept of J-proximal mapping for a nonconvex lower semicontinous and subdifferentiable proper functional on Banach space. Under some appropriate conditions, they proved the existence and Lipschitz continuity of J-proximal mapping of a lower semicontinuous subdifferentiable proper functional. In the meanwhile, they applied the concept of J-proximal mapping and a similar technique of resolvent operator in the context of Hilbert spaces and proposed an iterative algorithm to compute the approximate solution of a new class of variational-like inequalities with nonconvex functional in the framework of reflexive Banach spaces. After that, in 2005, Ahmad et al. [2] and Kazmi and Bhat [24] were, independently, the first to introduce the concept of J^{η}-proximal (also referred to as P-η-proximal) mapping for a nonconvex lower semicontinuous η-subdifferentiable proper functional on Banach space which can be viewed as a generalization of the notion of J-proximal mapping introduced in [20]. They proved the existence and Lipschitz continuity of such mappings under some suitable hypotheses and suggested some iterative algorithms for finding the approximate solutions of some classes of generalized multivalued nonlinear variational-like inequalities in the setting of Banach spaces. They also studied the convergence analysis of the sequences generated by their proposed iterative algorithms under some suitable conditions.

Recently, Ahmad et al. [1] introduced a proximal operator associated with a proper, lower semicontinuous and subdifferentiable functional called co-proximal operator and claimed that it is a new one. Under some assumptions, they also asserted that such an operator exists and proved its Lipschitz continuity. They considered a generalized co-variational inequality problem in the setting of a q-uniformly smooth Banach space and using co-proximal operator they proposed an iterative algorithm for computing its approximate solution. At the end of their paper, they claimed that under some appropriate conditions, the sequences generated by their suggested iterative algorithm converge strongly to a solution of the problem considered in [1].

The organization of this article is as follows. In Sect. 2, we recall basic definitions and facts about P-η-proximal mappings. In Sect. 3, a variational-like inequality problem involving a proper lower semicontinuous and η-subdifferentiable functional named generalized set-valued nonlinear variational-like inequality problem (in short, (GSNVLIP) is considered and its equivalence with a fixed point problem is proved under some suitable conditions. Applying the concept of P-proximal mapping, an iterative algorithm for approximating the solution of the GSNVLIP is proposed. Section 3 by the study of the convergence analysis of the sequences generated by our proposed iterative algorithm is concluded. Section 4 deals with the investigation and analysis of the concept of co-proximal operator introduced in [1]. We show that under the assumptions considered in [1], contrary to the claim of the authors in [1], the concept co-proximal operator is not a new one. Indeed, it is the same notion of J-proximal mapping introduced by Ding and Xia [20]. We prove that, contrary to the claim in [1], under the conditions mentioned in it, the co-proximal operator associated with a proper, lower semicontinous and subdifferentiable functional is not well defined necessarily. We also show that the proposed iterative algorithm in [1] is not necessarily well defined and point out that the results given in [1] are not correct.

2. Notation, basic definitions and fundamental properties

Throughout the paper, unless otherwise specified, we use the following notations, terminology and assumptions.

For a real Banach space E, we denote by E^* its dual Banach space of bounded linear functionals. As usual, x^* will stand for the weak star topology in E^*, and $\langle ., . \rangle$ will represent the duality pairing of E and E^*. The value of a functional $x^* \in E^*$ at $x \in E$ is denoted by either $\langle x, x^* \rangle$ or $x^*(x)$, as is convenient. For the sake of simplicity, the norms of E and E^* are denoted by the symbol $\|.\|$. We use the symbol 2^E (resp., $CB(E)$) to represent the set of all nonempty (resp., nonempty closed and bounded) subsets of E.

For any given function $f : E \to \mathbb{R} \cup \{\pm\infty\}$, $\operatorname{dom} f = \{x \in E, f(x) < +\infty\}$ is called the *effective domain* of f. Such a function is said to be *proper* if its effective domain is nonempty and it is real-valued on its effective domain, what is equivalent, f is proper if $f(x) > -\infty$ for all $x \in E$ and $f(x) < +\infty$ for at least one $x \in E$.

Definition 2.1. A function $f : E \to \mathbb{R} \cup \{+\infty\}$ is called

(i) convex if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for every $\lambda \in [0, 1]$ and all $x, y \in E$, for which the right-hand side is meaningful;

(ii) lower semicontinuous at $x_0 \in E$, provided that $f(x_0) \leq \liminf_n f(x_n)$, for every sequence $\{x_n\} \subset E$ satisfying $\lim_n x_n = x_0$.

If the property mentioned in Definition 2.1(ii) holds for every point $x_0 \in E$ we say that f is *lower semicontinuous* on E.

Definition 2.2. The function $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ is called lower semicontinuous in the second argument on E if for each $x \in E$, the function $f(x, .) : E \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous on E.

Similarly, one can define the lower semicontinuity of the function f in the first argument.

Definition 2.3 ([38])**.** An extended real-valued functional $f : (x, y) \in E \times E \to f(x, y) \in \mathbb{R} \cup \{\pm\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-DQCV)

(i) in the first argument (or with respect to x), if for any finite subset $\{x_1, x_2, \dots, x_n\}$ of E and any $\hat{x} \in \mathrm{Co}(\{x_1, x_2, \dots, x_n\})$, we have

$$\min_{1 \leq i \leq n} f(x_i, \hat{x}) \leq 0,$$

where for any given set $A \subset E$, $\mathrm{Co}(A)$ denotes the closed convex hull of A consisting of all vectors of the form $\sum_{i=1}^{n} \lambda_i u_i$ with $u_i \in A_i$, $\lambda_i \in \mathbb{R}_+ = [0, +\infty)$ and $\sum_{i=1}^{n} \lambda_i = 1$;

(ii) in the second argument (or with respect to y), if for any finite subset $\{y_1, y_2, \dots, y_n\}$ of E and any $\hat{y} \in \mathrm{Co}(\{y_1, y_2, \dots, y_n\})$, we have

$$\min_{1 \leq i \leq n} f(\hat{y}, y_i) \leq 0.$$

Lemma 2.4 ([19])**.** *Let D be a nonempty convex subset of a topological vector space and let $f : D \times D \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued functional such that*

(i) *f is lower semicontinuous in the second argument on every nonempty compact subset of D;*

(ii) *f is* 0-DQCV *in the first argument;*

(iii) *there exists a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D \backslash K$, there is an $x \in \mathrm{Co}(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

Definition 2.5. A proper functional $\phi : E \to \mathbb{R} \cup \{+\infty\}$ is said to be subdifferentiable at a point $x \in E$ if there exists a point $x^* \in E^*$ such that

$$\phi(y) - \phi(x) \geq \langle x^*, y - x \rangle, \quad \forall y \in E.$$

Such a point x^* is called subgradient of ϕ at x. The set of all subgradients of ϕ at x is denoted by $\partial\phi(x)$. The mapping $\partial\phi : E \to 2^{E^*}$ defined by

$$\partial\phi(x) = \{x^* \in E^* : \phi(y) - \phi(x) \geq \langle x^*, y - x \rangle, \forall y \in E\}, \quad \forall x \in E,$$

is said to be subdifferential of ϕ at x.

The introduction and study of the notion of η-subdifferential, in a more general setting than that given in [37], was first initiated by Lee et al. [25] and Ding and Luo [18], independently, as follows.

Definition 2.6 ([18,25])**.** Let $\eta : E\times E \to E$ be a vector-valued mapping. A proper functional $\phi : E \to \mathbb{R}\cup\{+\infty\}$ is said to be η-subdifferentiable at a point $x \in E$ if there exists a point $x^* \in E^*$ such that

$$\langle x^*, \eta(y,x)\rangle \leq \phi(y) - \phi(x), \quad \forall y \in E.$$

Such a point x^* is called η-subgradient of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\partial_\eta\phi(x)$. We can associate with each ϕ the η-subdifferential mapping $\partial_\eta\phi$ defined by

$$\partial_\eta\phi(x) = \begin{cases} \{x^* \in E^* : \langle x^*, \eta(y,x)\rangle \leq \phi(y) - \phi(x), \forall y \in E\}, & x \in \operatorname{dom}\phi, \\ \emptyset, & x \notin \operatorname{dom}\phi. \end{cases}$$

For $x \in \operatorname{dom}\phi$, $\partial_\eta\phi(x)$ is called the η-subdifferential of ϕ at x.

Here it is to be noted that in the definition of η-subdifferential in the sense of Yang and Craven [37], the function ϕ needs to be local Lipschitz and cannot take the value $+\infty$. We now present a new example which illustrates that the notion of η-subdifferential introduced in [18, 25] is more general than that given in [37].

Example 2.7. Suppose that E is the set of all real numbers endowed with the Euclidean norm $\|.\| = |.|$ and the mappings $\phi : E \to \mathbb{R}\cup\{+\infty\}$ and $\eta : E\times E \to E$ are defined, respectively, by

$$\phi(x) = \begin{cases} \alpha(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}) + \beta, & x \leq 0, \\ +\infty, & x > 0, \end{cases}$$

and $\eta(x,y) = \varrho(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|})$, for all $x, y \in E$, where k is an arbitrary but fixed odd natural number, and $\alpha, \varrho, \varsigma > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. We now show that for given $x \in \operatorname{dom}\phi$, $\partial_\eta\phi(x) = [\frac{\alpha}{\varrho}, +\infty)$. For this aim, take $x \in \operatorname{dom}\phi$ arbitrarily. Then, we have $\phi(x) = \alpha(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}) + \beta$ and $x \leq 0$.

If $\gamma \in \partial_\eta\phi(x)$, then

$$\begin{aligned} &\gamma(\varrho(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|})) \\ &\leq \phi(y) - \alpha(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}) - \beta, \quad \forall y \in E. \end{aligned}$$

Since $\phi(y) = +\infty$ for all $y > 0$, it follows that

$$\gamma(\varrho(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}))$$
$$\leq \alpha(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}) + \beta - \alpha(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}) - \beta \tag{2.1}$$
$$= \alpha(\sum_{p=1}^{\frac{k+1}{2}} (y^{2p-1}|y| - x^{2p-1}|x|) + \sum_{p=1}^{\frac{k-1}{2}} (\sqrt[2p+1]{y|y|} - \sqrt[2p+1]{x|x|})), \quad \forall y \leq 0.$$

If $x = 0$, then making use of (2.1), we conclude that

$$\gamma\varrho(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}) \leq \alpha(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}), \quad \forall y \leq 0,$$

which implies that $\gamma \geq \frac{\alpha}{\varrho}$.

For the case when $x < 0$, due to the fact that

$$\varrho(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|}) < 0,$$

recalling (2.1), for all $y \leq 0$, we yield

$$\gamma \geq \frac{\alpha(\sum_{p=1}^{\frac{k+1}{2}} (y^{2p-1}|y| - x^{2p-1}|x|) + \sum_{p=1}^{\frac{k-1}{2}} (\sqrt[2p+1]{y|y|} - \sqrt[2p+1]{x|x|}))}{\varrho(\sum_{p=1}^{\frac{k+1}{2}} y^{2p-1}|y| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y|y|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x^{2p-1}|x| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x|x|})}. \tag{2.2}$$

Passing to the limit in (2.2) as $y \to -\infty$, we deduce that $\gamma \geq \frac{\alpha}{\varrho}$. Hence, in any case, we infer that $\gamma \geq \frac{\alpha}{\varrho}$, and so, $\partial_\eta \phi(x) \subseteq [\frac{\alpha}{\varrho}, +\infty)$ for all $x \leq 0$. To prove $\partial_\eta \phi(x) = [\frac{\alpha}{\varrho}, +\infty)$, it is sufficient to show that $[\frac{\alpha}{\varrho}, +\infty) \subseteq \partial_\eta \phi(x)$ for all $x \leq 0$. Take $\gamma \in [\frac{\alpha}{\varrho}, +\infty)$ arbitrarily and on the contrary, suppose that $\gamma \notin \partial_\eta \phi(x_0)$ for some $x_0 \leq 0$. Then there exists $y_0 \leq 0$ such that

$$\gamma(\varrho(\sum_{p=1}^{\frac{k+1}{2}} y_0^{2p-1}|y_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y_0|y_0|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x_0^{2p-1}|x_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x_0|x_0|}))$$
$$> \alpha(\sum_{p=1}^{\frac{k+1}{2}} (y_0^{2p-1}|y_0| - x_0^{2p-1}|x_0|) + \sum_{p=1}^{\frac{k-1}{2}} (\sqrt[2p+1]{y_0|y_0|} - \sqrt[2p+1]{x_0|x_0|})). \tag{2.3}$$

It is obvious that the case where $x_0 = y_0 = 0$ cannot happen. If $x_0, y_0 < 0$, then taking into account that

$$\varrho(\sum_{p=1}^{\frac{k+1}{2}} y_0^{2p-1}|y_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y_0|y_0|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x_0^{2p-1}|x_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x_0|x_0|}) < 0,$$

by using (2.3), yields

$$\frac{\alpha}{\varrho} \leq \gamma$$
$$< \frac{\alpha(\sum_{p=1}^{\frac{k+1}{2}}(y_0^{2p-1}|y_0| - x_0^{2p-1}|x_0|) + \sum_{p=1}^{\frac{k-1}{2}}(\sqrt[2p+1]{y_0|y_0|} - \sqrt[2p+1]{x_0|x_0|}))}{\varrho(\sum_{p=1}^{\frac{k+1}{2}} y_0^{2p-1}|y_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{y_0|y_0|}) + \varsigma(\sum_{p=1}^{\frac{k+1}{2}} x_0^{2p-1}|x_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x_0|x_0|})},$$

for which it follows that

$$\alpha(\varsigma + \varrho)(\sum_{p=1}^{\frac{k+1}{2}} x_0^{2p-1}|x_0| + \sum_{p=1}^{\frac{k-1}{2}} \sqrt[2p+1]{x_0|x_0|}) > 0. \tag{2.4}$$

Taking into consideration the fact that $\alpha, \varsigma, \varrho > 0$ and k is an odd natural number, using (2.4) we conclude that $x_0 > 0$, which is a contradiction. If $x_0 < 0$ and $y_0 = 0$, then making use of (2.3), we deduce that $\gamma < -\frac{\alpha}{\varsigma}$, which leads to a contradiction. Finally, for the case when $x_0 = 0$ and $y_0 < 0$, then employing (2.3), we deduce that $\gamma < \frac{\alpha}{\varrho}$, which is also a contradiction. Thanks to these facts, it follows that $[\frac{\alpha}{\varrho}, +\infty) \subseteq \partial_\eta \phi(x)$, for all $x \leq 0$. Consequently, $\partial_\eta \phi(x) = [\frac{\alpha}{\varrho}, +\infty)$ for all $x \leq 0$.

Definition 2.8 ([2, 24])**.** Let $\eta : E \times E \to E$ be a vector-valued mapping, $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper η-subdifferentiable (may not be convex) functional and $P : E \to E^*$ be a mapping. If for any given $x^* \in E^*$ and $\rho > 0$, there exists a unique point $x \in E$ satisfying

$$\langle P(x) - x^*, \eta(y, x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \qquad \forall y \in E,$$

then the mapping $x^* \to x$, denoted by $J_{\rho,P}^{\partial_\eta \phi}$, is called P-η-proximal mapping of ϕ. Evidently, in the light of Definition 2.6, we have $x^* - P(x) \in \rho\partial_\eta\phi(x)$ and then it follows that $x = J_{\rho,P}^{\partial_\eta\phi}(x^*) = (P + \rho\partial_\eta\phi)^{-1}(x^*)$.

It should be pointed out that if $\eta(u, v) = u - v$ for all $u, v \in E$, then Definition 2.8 reduces to the following definition of a P-proximal mapping.

Definition 2.9 ([20])**.** Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper subdifferentiable (not necessarily convex) functional and $P : E \to E^*$ be a mapping. If for any given point $x^* \in E^*$ and $\rho > 0$, there exists a unique point $x \in E$ satisfying

$$\langle P(x) - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \qquad \forall y \in E,$$

then the mapping $x^* \to x$, denoted by $J_{\rho,P}^{\partial\phi}$, is said to be P-proximal mapping of ϕ. Clearly, invoking Definition 2.6, we have $x^* - P(x) \in \rho\partial\phi(x)$ and then it follows that $x = J_{\rho,P}^{\partial\phi}(x^*) = (P + \rho\partial\phi)^{-1}(x^*)$.

Definition 2.10. Let $P : E \to E^*$ and $\eta : E \times E \to E$ be two vector-valued mappings. P is said to be

(i) k-strongly η-monotone if there exists a constant $k > 0$ such that

$$\langle P(x) - P(y), \eta(x, y)\rangle \geq k\|x - y\|^2, \qquad \forall x, y \in E;$$

(ii) λ_P-Lipschitz continuous if there exists a constant $\lambda_P > 0$ such that

$$\|P(x) - P(y)\| \leq \lambda_P \|x - y\|, \qquad \forall x, y \in E.$$

Definition 2.11. A vector-valued mapping $\eta : E \times E \to E$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x,y)\| \leq \tau\|x - y\|$, for all $x, y \in E$.

A natural question then arises whether for given mappings $\eta : E \times E \to E$ and $P : E \to E^*$, an η-subdifferentiable (not necessarily convex) proper functional $\phi : E \to \mathbb{R} \cup \{+\infty\}$ and an arbitrary real constant $\rho > 0$, the P-η-proximal mapping associated with the mappings ϕ, P, η and the constant $\rho > 0$ is well defined necessarily? Under some appropriate conditions, an affirmative answer is given by Ahmad et al. [2] and Kazmi and Bhat [24] by the next theorem.

Theorem 2.12 ([2, 24])**.** *Let E be a reflexive Banach space, $\eta : E \times E \to E$ be a τ-Lipschitz continuous mapping such that $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in E$, and let $P : E \to E^*$ be a γ-strongly η-monotone continuous mapping. Suppose that for any given $x^* \in E^*$, the function*

$$h : (y, x) \in E \times E \to h(y, x) = \langle x^* - P(x), \eta(y, x)\rangle \in \mathbb{R} \cup \{+\infty\}$$

is 0-DQCV *in the first argument. Moreover, let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional on E, which may not be convex. Then for any given $\rho > 0$ and $x^* \in E^*$, there exists a unique point $x \in E$ such that*

$$\langle P(x) - x^*, \eta(y, x)\rangle \geq \rho\phi(x) - \rho\phi(y), \qquad \forall y \in E,$$

that is, $x = J^{\partial_\eta \phi}_{\rho,P}(x^*)$ and so the P-η-proximal mapping associated with ϕ, P, η and ρ is well defined.

It is very essential to note that by a careful reading the proof of Theorem 3.1 in [24] and by comparing it with the assumptions appeared in its context, we found that the mapping η must be τ-Lipschitz continuous. In fact, in the context of [24, Theorem 3.1], the continuity hypothesis of the mapping η must be replaced by the τ-Lipschitz continuity assumption, as we have done in the context of Theorem 2.12.

We now give a new example in which the existence of the two mappings $\eta : E \times E \to E$ and $P : E \to E^*$ satisfying all the conditions of Theorem 2.12 is shown.

Example 2.13. Consider $E = \mathbb{R}$ with the Euclidean norm $\|.\| = |.|$ and let the mappings $\eta : E \times E \to E$ and $P : E \to E^*$ be defined by

$$\eta(x,y) = \begin{cases} \alpha(e^{\sqrt[k]{|xy|}} + |xy|^m + \mu)(x - y), & \text{if } |xy| < p, \\ \beta(a^{\sqrt[n]{|xy|}} + \gamma log_b|xy|)(x - y), & \text{if } p \leq |xy| < q, \\ (\frac{\xi}{\sqrt[s]{|xy| + |xy|^l + b^{|xy|}}} + \varrho)(x - y), & \text{if } q \leq |xy|, \end{cases}$$

and $P(x) = \varsigma x$ for all $x, y \in E$, where $\alpha, \beta, \gamma, \xi, \varrho, \varsigma, m, l$ are arbitrary real constants that are strictly bigger than zero, $k, n, s \in \mathbb{N}\backslash\{1\}$ are arbitrary constants, and a, b, p, q are arbitrary real constants such that $a, b > 1$ and $q, p \geq 1$. It is easy to observe that $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in E$.

Taking into account that for all $x, y \in E$,

$$|\eta(x,y)| = \begin{cases} \alpha(e^{\sqrt[k]{|xy|}} + |xy|^m + \mu)|x-y|, & \text{if } |xy| < p, \\ \beta(a^{\sqrt[n]{|xy|}} + \gamma log_b|xy|)|x-y|, & \text{if } p \leq |xy| < q, \\ (\frac{\xi}{\sqrt[s]{|xy| + |xy|^l + b^{|xy|}}} + \varrho)|x-y|, & \text{if } q \leq |xy|, \end{cases}$$

with the help of the assumptions, we derive that

$$\mu < e^{\sqrt[k]{|xy|}} + |xy|^m + \mu < e^{\sqrt[k]{p}} + p^m + \mu, \quad \forall x, y \in E \text{ with } |xy| < p, \tag{2.5}$$

$$0 < a^{\sqrt[n]{p}} + \gamma log_b p \quad \leq a^{\sqrt[n]{|xy|}} + \gamma log_b|xy| < a^{\sqrt[n]{q}} + \gamma log_b q, \tag{2.6}$$

for all $x, y \in E$ with $p \leq |xy| < q$, and

$$\varrho < \frac{\xi}{\sqrt[s]{|xy|} + |xy|^l + b^{|xy|}} + \varrho < \frac{\xi}{\sqrt[s]{q} + q^l + b^q} + \varrho, \tag{2.7}$$

for all $x, y \in E$ with $|xy| \geq q$.

Making use of (2.5)–(2.7) and in view of the fact that $\alpha, \beta > 0$, we obtain

$$|\eta(x,y)| \leq \max\left\{\alpha(e^{\sqrt[k]{p}} + p^m + \mu), \beta(a^{\sqrt[n]{q}} + \gamma log_b q), \frac{\xi}{\sqrt[s]{q} + q^l + b^q} + \varrho\right\}|x-y|,$$

for all $x, y \in E$, which means that η is a $\max\left\{\alpha(e^{\sqrt[k]{p}} + p^m + \mu), \beta(a^{\sqrt[n]{a}} + \gamma log_b q), \frac{\xi}{\sqrt[s]{q} + q^l + b^q} + \varrho\right\}$-Lipschitz continuous mapping. Define, associated with each $z \in E$, a correspondence $h_z : E \times E \to \mathbb{R} \cup \{+\infty\}$ for each $(y, x) \in E \times E$ by

$$h_z(y,x) = \langle z - P(x), \eta(y,x)\rangle = (z - P(x))\eta(y,x).$$

Using proof by contradiction, we now prove that the function h_z is 0-DQCV in the first argument. For this end, suppose that there exist a finite set $\{y_1, y_2, \dots, y_n\}$ and $t = \sum_{i=1}^n \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that for each $i \in \{1, 2, \dots, n\}$,

$$0 < h_z(y_i, t) = \begin{cases} \alpha(z - \varsigma t)(e^{\sqrt[k]{|y_i t|}} + |y_i t|^m + \mu)(y_i - t), & \text{if } |y_i t| < p, \\ \beta(a^{\sqrt[n]{|y_i t|}} + \gamma log_b|y_i t|)(y_i - t), & \text{if } p \leq |y_i t| < q, \\ (\frac{\xi}{\sqrt[s]{|y_i t| + |y_i t|^l + b^{|y_i t|}}} + \varrho)(y_i - t), & \text{if } q \leq |y_i t|. \end{cases}$$

Therefore, $(z - \varsigma t)(y_i - t) > 0$ for each $i \in \{1, 2, \dots, n\}$ which ensures that

$$0 < \sum_{i=1}^n \lambda_i (z - \varsigma t)(y_i - t) = (z - \varsigma t)(\sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i t) = (z - \varsigma t)(t - t) = 0,$$

which leads to a contradiction. Hence, for any given $z \in E$ the function h_z is 0-DQCV in the first argument. Since for all $x, y \in E$,

$$\langle P(x) - P(y), \eta(x,y)\rangle = \begin{cases} \alpha\varsigma(e^{\sqrt[k]{|xy|}} + |xy|^m + \mu)|x-y|^2, & \text{if } |xy| < p, \\ \beta\varsigma(a^{\sqrt[n]{|xy|}} + \gamma log_b|xy|)|x-y|^2, & \text{if } p \leq |xy| < q, \\ \varsigma(\frac{\xi}{\sqrt[s]{|xy| + |xy|^l + b^{|xy|}}} + \varrho)|x-y|^2, & \text{if } q \leq |xy|, \end{cases}$$

utilizing (2.5)–(2.7), it follows that

$$\langle P(x)-P(y),\eta(x,y)\rangle \geq \alpha\varsigma\mu|x-y|^2, \quad \forall x,y\in E \text{ with } |xy|\in[0,p),$$

$$\langle P(x)-P(y),\eta(x,y)\rangle \geq \beta\varsigma(a^{\sqrt[n]{p}}+\gamma log_b p)|x-y|^2, \quad \forall x,y\in E \text{ with } |xy|\in[p,q)$$

and

$$\langle P(x)-P(y),\eta(x,y)\rangle \geq \varsigma\varrho|x-y|^2, \quad \forall x,y\in E \text{ with } |xy|\in[q,+\infty).$$

Consequently,

$$\langle P(x)-P(y),\eta(x,y)\rangle \geq \min\{\alpha\varsigma\mu,\beta\varsigma(a^{\sqrt[n]{p}}+\gamma log_b p),\varsigma\varrho\}|x-y|^2, \quad \forall x,y\in E,$$

that is, P is a $\min\{\alpha\varsigma\mu,\beta\varsigma(a^{\sqrt[n]{p}}+\gamma log_b p),\varsigma\varrho\}$-strongly η-monotone mapping.

Thereby, the two mappings P and η are satisfied all the conditions of Theorem 2.12.

Under some appropriate conditions, the Lipschitz continuity of the P-η-proximal mapping $J^{\partial_\eta\phi}_{\rho,P}$ associated with the mappings ϕ, P, η and the constant $\rho>0$ is proved in [2, 24] and an estimate of its Lipschitz constant is also computed as follows.

Theorem 2.14 ([2,24])**.** *Let E be a reflexive Banach space with the dual space E^*, $\eta: E\times E\to E$ be a τ-Lipschitz continuous mapping such that $\eta(x,y)+\eta(y,x)=0$ for all $x,y\in E$, and $P: E\to E^*$ be a γ-strongly η-monotone continuous mapping. Suppose that for given $x^*\in E^*$, the function $h:(y,x)\in E\times E\to h(y,x)=\langle x^*-P(x),\eta(y,x)\rangle\in\mathbb{R}\cup\{+\infty\}$ is* 0-DQCV *in the first argument, $\phi: E\to\mathbb{R}\cup\{+\infty\}$ is a lower semicontinuous η-subdifferentiable proper functional on E and $\rho>0$ is an arbitrary real constant. Then, the P-η-proximal mapping $J^{\partial_\eta\phi}_{\rho,P}: E^*\to E$ associated with ϕ, P, η and $\rho>0$ is $\frac{\tau}{\gamma}$-Lipschitz continuous, i.e.,*

$$\|J^{\partial_\eta\phi}_{\rho,P}(x^*)-J^{\partial_\eta\phi}_{\rho,P}(y^*)\|\leq\frac{\tau}{\gamma}\|x^*-y^*\|, \qquad \forall x^*,y^*\in E^*.$$

3. Formulation, algorithm and convergence theorem

Let $A,B,C,F,G: E\to CB(E^*)$ and $H: E\to CB(E)$ be set-valued mappings; and $f: E\to E^*$, $g: E\to E$, $\eta: E\times E\to E$, $M: E^*\times E^*\to E^*$ and $N: E^*\times E^*\times E^*\to E^*$ be single-valued mappings. Suppose that $\phi: E\times E\to\mathbb{R}\cup\{+\infty\}$ is an extended real-valued bifunction such that for each fixed $\nu\in E$, $\phi(.,\nu): E\to\mathbb{R}\cup\{+\infty\}$ is a proper lower semicontinuous and η-subdifferentiable functional on E with $g(E)\cap\operatorname{dom}\partial_\eta\phi(.,\nu)\neq\emptyset$. We consider the problem of finding $x\in E$, $u\in A(x)$, $v\in B(x)$, $w\in C(x)$, $s\in F(x)$, $t\in G(x)$ and $z\in H(x)$ such that $g(x)\in\operatorname{dom}\partial_\eta\phi(.,z)$ and

$$\langle N(u,v,w)+f(x)-M(s,t),\eta(y,g(x))\rangle\geq\phi(g(x),z)-\phi(y,z), \quad \forall y\in E, \tag{3.1}$$

which is called a *generalized set-valued nonlinear variational-like inequality problem* (in short, GSNVLIP).

If $f=M\equiv 0$ and $H=D$, then the GSNVLIP (3.1) reduces to the problem of finding $x\in E$, $u\in A(x)$, $v\in B(x)$, $w\in C(x)$ and $z\in D(x)$ such that

$$\langle N(u,v,w),\eta(y,g(x))\rangle\geq\phi(g(x),z)-\phi(y,z), \quad \forall y\in E,$$

which was introduced and studied by Kazmi and Bhat [24].

If $N \equiv 0$ and $H : E \to E$ is a single-valued mapping, then the GSNVLIP (3.1) becomes the problem of finding $x \in E$, $s \in F(x)$ and $t \in G(x)$ such that $g(x) \in \operatorname{dom} \partial_\eta \phi(.,x)$ and

$$\langle f(x) - M(s,t), \eta(y, g(x))\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in E,$$

which was introduced and studied by Ahmad et al. [2].

We remark that for a suitable choices of the mappings $A, B, C, F, G, H, M, N, \phi, f$, g, η and the underlying space E, a number of known problems of variational-like and variational inequalities can be obtained as special cases of the GSNVLIP (3.1), see, for example, [2, 17, 18, 20, 24, 31] and the references therein.

The following conclusion, which tells the GSNVLIP (3.1) is equivalent to a fixed point problem under some appropriate conditions, gives a characterization of the solution of the GSNVLIP (3.1).

Lemma 3.1. *Let E be a reflexive Banach space with its dual space E^*, and let $A, B, C, F, G, H, M, N, \phi, f, g$ be the same as in the* GSNVLIP (3.1)*. Assume that $\eta : E \times E \to E$ is a τ-Lipschitz continuous mapping such that $\eta(\hat{x}, \hat{y}) + \eta(\hat{y}, \hat{x}) = 0$ for all $\hat{x}, \hat{y} \in E$, and let $P : E \to E^*$ be an α-strongly η-monotone continuous mapping such that $g(E) \cap \operatorname{dom}(P) \neq \emptyset$. Suppose that for any $x^* \in E^*$, the function $h : (\hat{x}, \hat{y}) \in E \times E \to h(\hat{x}, \hat{y}) = \langle x^* - P(\hat{y}), \eta(\hat{x}, \hat{y})\rangle \in \mathbb{R} \cup \{+\infty\}$ is* 0-DQCV *in the first argument. Then $(x, u, v, w, s, t, z) \in E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ is a solution of the* GSNVLIP (3.1) *if and only if $g(x) \in \operatorname{dom} P$ and*

$$g(x) = J^{\partial_\eta \phi(.,z)}_{\rho,P}[(P \circ g)(x) - \rho(N(u,v,w) + f(x) - M(s,t))], \tag{3.2}$$

where $J^{\partial_\eta \phi(.,\hat{z})}_{\rho,P} = (P + \rho\partial_\eta \phi(., \hat{z}))^{-1}$ is P-η-proximal mapping of $\phi(., \hat{z})$ for each fixed $\hat{z} \in E$, $P \circ g$ denotes P composition g, and $\rho > 0$ is a constant.

Proof. Invoking Definitions 2.6 and 2.8, we deduce that $(x, u, v, w, s, t, z) \in E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ is a solution of the GSNVLIP (3.1) if and only if

$$\begin{aligned}
&\phi(y, z) - \phi(g(x), z) \geq \langle -(N(u, v, w) + f(x) - M(s, t)), \eta(y, g(x))\rangle, \quad \forall y \in E,\\
&\Leftrightarrow -(N(u, v, w) + f(x) - M(s, t)) \in \partial_\eta \phi(g(x), z)\\
&\Leftrightarrow (P \circ g)(x) - \rho(N(u, v, w) + f(x) - M(s, t)) \in (P \circ g)(x) + \rho\partial_\eta \phi(g(x), z)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = (P + \rho\partial_\eta \phi(., z))(g(x))\\
&\Leftrightarrow g(x) = J^{\partial_\eta \phi(.,z)}_{\rho,P}[(P \circ g)(x) - \rho(N(u, v, w) + f(x) - M(s, t))],
\end{aligned}$$

where $J^{\partial_\eta \phi(.,z)}_{\rho,P} = (P + \rho\partial_\eta \phi(., z))^{-1}$. □

Let $E, A, B, C, F, G, H, M, N, \phi, \eta, f, g, h$ be the same as in Lemma 3.1 such that $\operatorname{dom} P \cap J^{\partial_\eta \phi(.,\hat{z})}_{\rho,P}(E^*) \neq \emptyset$ for any $\hat{z} \in E$. Related to the GSNVLIP (3.1), we consider the problem of finding $p \in E^*$, $x \in E$, $u \in A(x)$, $v \in B(x)$, $w \in C(x)$, $s \in F(x)$, $t \in G(x)$ and $z \in H(x)$ such that

$$\rho(N(u, v, w) + f(x) - M(s, t) + \rho^{-1} R^{\partial_\eta \phi(.,z)}_{\rho,P}(p) = 0, \tag{3.3}$$

where $\rho > 0$ is a constant, $R^{\partial_\eta \phi(.,z)}_{\rho,P} = I - P \circ J^{\partial_\eta \phi(.,z)}_{\rho,P}$ and I is the identity mapping on E. The problem (3.3) is called a *generalized implicit Wiener-Hopf equation* (in short, GIWHE).

Accordance with the following conclusion, the GSNVLIP (3.1) and the GIWHE (3.3) are equivalent.

Lemma 3.2. *Let $E, A, B, C, F, G, H, M, N, P, \phi, \eta, f, g, h$ be the same as in Lemma 3.1 and let all the conditions of Lemma 3.1 hold. Moreover, let* $\operatorname{dom} P \cap J^{\partial_\eta \phi(.,\hat{z})}_{\rho,P}(E^*) \neq \emptyset$ *for any $\hat{z} \in E$. Then, $(x, u, v, w, s, t, z) \in E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ is a solution of the* GSNVLIP (3.1) *if and only if the* GIWHE (3.3) *has a solution $(p, x, u, v, w, s, t, z) \in E^* \times E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ satisfying*

$$\left\{ \begin{array}{l} g(x) = J^{\partial_\eta \phi(.,z)}_{\rho,P}(p), \\ p = (P \circ g)(x) - \rho(N(u, v, w) + f(x) - M(s, t)), \end{array} \right. \tag{3.4}$$

where $J^{\partial_\eta \phi(.,z)}_{\rho,P}$, $P \circ g$ and ρ are the same as in Lemma 3.1.

Proof. Suppose that $(x, u, v, w, s, t, z) \in E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ is a solution of the GSNVLIP (3.1). From Lemma 3.1 it follows that

$$g(x) = J^{\partial_\eta \phi(.,z)}_{\rho,P}[(P \circ g)(x) - \rho(N(u, v, w) + f(x) - M(s, t))].$$

Taking $p = (P \circ g)(x) - \rho(N(u, v, w) + f(x) - M(s, t))$ in the latter equation, we conclude that

$$\begin{aligned} g(x) = J^{\partial_\eta \phi(.,z)}_{\rho,P}(p) &\Leftrightarrow (P \circ g)(x) = (P \circ J^{\partial_\eta \phi(.,z)}_{\rho,P})(p) \\ &\Leftrightarrow \rho(N(u, v, w) + f(x) - M(s, t)) = (P \circ J^{\partial_\eta \phi(.,z)}_{\rho,P})(p) \\ &\quad - (P \circ g)(x) + \rho(N(u, v, w) + f(x) - M(s, t)) \\ &\Leftrightarrow \rho(N(u, v, w) + f(x) - M(s, t)) = -(p - (P \circ J^{\partial_\eta \phi(.,z)}_{\rho,P})(p)) \\ &\Leftrightarrow \rho(N(u, v, w) + f(x) - M(s, t)) = -(I - P \circ J^{\partial_\eta \phi(.,z)}_{\rho,P})(p) \\ &\Leftrightarrow \rho(N(u, v, w) + f(x) - M(s, t)) = -R^{\partial_\eta \phi(.,z)}_{\rho,P}(p) \\ &\Leftrightarrow N(u, v, w) + f(x) - M(s, t) + \rho^{-1} R^{\partial_\eta \phi(.,z)}_{\rho,P}(p) = 0, \end{aligned}$$

where $R^{\partial_\eta \phi(.,z)}_{\rho,P} = I - P \circ J^{\partial_\eta \phi(.,z)}_{\rho,P}$ and I is the identity mapping on E^*. Thereby, every solution of the GSNVLIP (3.1) is a solution of the GIWHE (3.3) and vice versa. Hence, the two problems (3.1) and (3.3) are equivalent. The proof is finished. $\square$

Lemma 3.3. [27] *Let E be a complete metric space and $T : E \to CB(E)$ be a set-valued mapping. Then for any $\varepsilon > 0$ and for any given $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq (1 + \varepsilon) D(T(x), T(y)), \tag{3.5}$$

where $D(.,.)$ is the Hausdorff metric on $CB(E)$ defined by

$$D(A,B)=\max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\sup_{y\in B}\inf_{x\in A}\|x-y\|\right\},\quad \forall A,B\in CB(E).$$

The fixed Point formulation (3.4) and Nadler's technique [27] allow us to construct the following iterative algorithm for approximating a solution of the GSNVLIP (3.1).

Algorithm 3.1. Let $E, A, B, C, F, G, H, M, N, P, \phi, \eta, f, g, h$ be the same as in Lemma 3.1 such that g is an onto mapping. For any given $p_0 \in E^*$, $x_0 \in E$, $u_0 \in A(x_0)$, $v_0 \in B(x_0)$, $w_0 \in C(x_0)$, $s_0 \in F(x_0)$, $t_0 \in G(x_0)$ and $z_0 \in H(x_0)$, define the iterative sequences $\{p_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{s_n\}$ $\{t_n\}$ and $\{z_n\}$ in the following way:

$$(3.6)\quad \begin{cases} g(x_n)=J_{\rho,P}^{\partial_\eta\phi(.,z_n)}(p_n),\\ p_{n+1}=(1-\lambda)p_n+\lambda[(P\circ g)(x_n)-\rho(N(u_n,v_n,w_n)+f(x_n)\\ \qquad -M(s_n,t_n))]+\lambda e_n+r_n,\\ u_n\in A(x_n);\|u_{n+1}-u_n\|\le(1+(1+n)^{-1})D(A(x_{n+1}),A(x_n)),\\ v_n\in B(x_n);\|v_{n+1}-v_n\|\le(1+(1+n)^{-1})D(B(x_{n+1}),B(x_n)),\\ w_n\in C(x_n);\|w_{n+1}-w_n\|\le(1+(1+n)^{-1})D(C(x_{n+1}),C(x_n)),\\ s_n\in F(x_n);\|s_{n+1}-s_n\|\le(1+(1+n)^{-1})D(F(x_{n+1}),F(x_n)),\\ t_n\in G(x_n);\|t_{n+1}-t_n\|\le(1+(1+n)^{-1})D(G(x_{n+1}),G(x_n)),\\ z_n\in H(x_n);\|z_{n+1}-z_n\|\le(1+(1+n)^{-1})D(H(x_{n+1}),H(x_n)),\end{cases}$$

where $n=0,1,2,\dots$; $\rho>0$ is a constant, $\lambda\in(0,1]$ is a relaxation parameter, $D(.,.)$ is the Hausdorff metric on $CB(E)$ and $\{e_n\}$, $\{r_n\}$ are two sequences in E^* to take into account a possible inexact computation of the P-η-proximal mapping points satisfying the following conditions:

$$(3.7)\quad \begin{cases} \lim_{n\to\infty}\|e_n\|=\lim_{n\to\infty}\|r_n\|=0,\\ \sum_{n=0}^{\infty}\|e_n-e_{n-1}\|<\infty,\sum_{n=0}^{\infty}\|r_n-r_{n-1}\|<\infty.\end{cases}$$

Before proceeding to the main result of this paper, let us to recall the following definitions which will be used in the sequel.

Recall that a mapping $J: E\to 2^{E^*}$ satisfying the condition

$$J(x)=\{x^*\in E^*:\langle x,x^*\rangle=\|x\|^2=\|x^*\|_*^2\},\quad \forall x\in E,$$

is called the normalized duality mapping on E. The Hahn-Banach theorem guarantees that $J(x)\neq\emptyset$ for every $x\in E$. In the sequel, we shall denote a selection of the normalized duality mapping J by j.

Definition 3.4. A set-valued mapping $T: E\to CB(E)$ is said to be D-Lipschitz continuous with constant λ_T (or λ_T-D-Lipschitz continuous) if there exists a constant $\lambda_T>0$ such that

$$D(T(x),T(y))\le\lambda_T\|x-y\|,\quad \forall x,y\in E,$$

where $D(.,.)$ is the Hausdorff metric on $CB(E)$.

Definition 3.5. A mapping $g : E \to E$ is said to be k-strongly accretive, if there exists a constant $k > 0$ such that for any $x, y \in E$,

$$\langle g(x) - g(y), j(x - y)\rangle \geq k\|x - y\|^2, \quad \forall j(x - y) \in J(x - y),$$

where J is the normalized duality mapping from real Banach space E into its dual space E^*.

Definition 3.6. The mapping $N : E \times E \times E \to E$ is said to be

(i) λ_{N_1}-Lipschitz continuous in the first argument if there exists a constant $\lambda_{N_1} > 0$ such that

$$\|N(x, y, z) - N(\hat{x}, y, z)\| \leq \lambda_{N_1}\|x - \hat{x}\|, \quad \forall x, \hat{x}, y, z \in E;$$

(ii) λ_{N_2}-Lipschitz continuous in the second argument if there exists a constant $\lambda_{N_2} > 0$ such that

$$\|N(x, y, z) - N(x, \hat{y}, z)\| \leq \lambda_{N_2}\|y - \hat{y}\|, \quad \forall x, y, \hat{y}, z \in E;$$

(iii) λ_{N_3}-Lipschitz continuous in the third argument if there exists a constant $\lambda_{N_3} > 0$ such that

$$\|N(x, y, z) - N(x, y, \hat{z})\| \leq \lambda_{N_3}\|z - \hat{z}\|, \quad \forall x, y, z, \hat{z} \in E.$$

Similarly, one can define the Lipschitz continuity of a bifunction $M : E \times E \to E$ in the first and second arguments.

This section is closed by the following theorem in which under sufficient conditions, the strong convergence of the sequences generated by Algorithm 3.1 to a solution of the GSNVLIP (3.1) is proved.

Theorem 3.7. *Let E be a reflexive Banach space with the dual space E^* and $\eta : E \times E \to E$ be a τ-Lipschitz continuous mapping such that $\eta(\hat{x}, \hat{y}) + \eta(\hat{y}, \hat{x}) = 0$ for all $\hat{x}, \hat{y} \in E$. Suppose that $g : E \to E$ is a k-strongly accretive and λ_g-Lipschitz continuous onto mapping and $P : E \to E^*$ is a γ-strongly η-monotone and λ_P-Lipschitz continuous mapping such that $g(E) \cap \operatorname{dom} P \neq \emptyset$. Let for any given $x^* \in E^*$, the function $h : (y, \hat{x}) \in E \times E \to h(y, \hat{x}) = \langle x^* - P(\hat{x}), \eta(y, \hat{x})\rangle \in \mathbb{R} \cup \{+\infty\}$ be 0-DQCV in the first argument and $\phi : E \times E \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued bifunction such that for each fixed $\nu \in E$, $\phi(., \nu) : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous and η-subdifferentiable functional on E with $g(E) \cap \operatorname{dom} \partial_\eta \phi(., \nu) \neq \emptyset$. Let $f : E \to E^*$ be a λ_f-Lipschitz continuous mapping, $M : E^* \times E^* \to E^*$ be a λ_{M_1}-Lipschitz continuous and λ_{M_2}-Lipschitz continuous mapping in the first and second arguments, respectively, and $N : E^* \times E^* \times E^* \to E^*$ be a λ_{N_1}-Lipschitz continuous, λ_{N_2}-Lipschitz continuous and λ_{N_3}-Lipschitz continuous in the first, second and third arguments, respectively. Suppose that the set-valued mappings $A, B, C, F, G : E \to CB(E^*)$ and $H : E \to CB(E)$ are D-Lipschitz continuous with constants λ_A, λ_B, λ_C, λ_F, λ_G and λ_H, respectively. If there exist constants $\varsigma \in (0, \frac{k}{2\lambda_H})$ and $\rho > 0$ such that*

$$\|J_{\rho,P}^{\partial_\eta \phi(.,\hat{x})}(\hat{p}) - J_{\rho,P}^{\partial_\eta \phi(.,\hat{y})}(\hat{p})\| \leq \varsigma\|\hat{x} - \hat{y}\|, \quad \forall \hat{x}, \hat{y} \in E, \hat{p} \in E^* \tag{3.8}$$

and

$$\left\{\begin{array}{l}\rho(\lambda_{N_1}\lambda_A+\lambda_{N_2}\lambda_B+\lambda_{N_3}\lambda_C+\lambda_{M_1}\lambda_F+\lambda_{M_2}\lambda_G)\tau\\ <\gamma(k-\varsigma\lambda_H)-(\lambda_P\lambda_g+\lambda_f)\tau,\\ \gamma(k-\varsigma\lambda_H)>(\lambda_P\lambda_g+\lambda_f)\tau,\\ k>\varsigma\lambda_H,\end{array}\right. \tag{3.9}$$

then the iterative sequences $\{p_n\},\{x_n\},\{u_n\},\{v_n\},\{w_n\},\{s_n\},\{t_n\}$ *and* $\{z_n\}$ *generated by Algorithm 3.1 converge strongly to* p,x,u,v,w,s,t *and* z*, respectively, and* (p,x,u,v,w,s,t,z) *is a solution of the* GIWHE *(3.3).*

Proof. Using (3.6), (3.8), Theorem 2.14 and λ_H-D-Lipschtiz continuity of the mapping H, we derive that for all $n\geq 0$,

$$\begin{aligned}\|g(x_{n+1})-g(x_n)\|&=\|J_{\rho,P}^{\partial_\eta\phi(.,z_{n+1})}(p_{n+1})-J_{\rho,P}^{\partial_\eta\phi(.,z_n)}(p_n)\|\\ &\leq\|J_{\rho,P}^{\partial_\eta\phi(.,z_{n+1})}(p_{n+1})-J_{\rho,P}^{\partial_\eta\phi(.,z_n)}(p_{n+1})\|\\ &\quad+\|J_{\rho,P}^{\partial_\eta\phi(.,z_n)}(p_{n+1})-J_{\rho,P}^{\partial_\eta\phi(.,z_n)}(p_n)\|\end{aligned}$$

$$\begin{aligned}&\leq\varsigma\|z_{n+1}-z_n\|+\frac{\tau}{\gamma}\|p_{n+1}-p_n\|\\ &\leq\varsigma(1+(1+n)^{-1})D(H(x_{n+1}),H(x_n))+\frac{\tau}{\gamma}\|p_{n+1}-p_n\|\\ &\leq\varsigma(1+(1+n)^{-1})\lambda_H\|x_{n+1}-x_n\|+\frac{\tau}{\gamma}\|p_{n+1}-p_n\|.\end{aligned}\tag{3.10}$$

Since g is k-strongly accretive, it follows that for each $n\geq 0$,

$$\begin{aligned}\|g(x_{n+1})-g(x_n)\|\|x_{n+1}-x_n\|&=\|g(x_{n+1})-g(x_n)\|\|j(x_{n+1}-x_n)\|\\ &\geq\langle g(x_{n+1})-g(x_n),j(x_{n+1}-x_n)\rangle\\ &\geq k\|x_{n+1}-x_n\|^2,\quad\forall j(x_{n+1}-x_n)\in J(x_{n+1}-x_n),\end{aligned}$$

from which yields

$$\|g(x_{n+1})-g(x_n)\|\geq k\|x_{n+1}-x_n\|.\tag{3.11}$$

Making use of (3.10) and (3.11), we conclude that

$$k\|x_{n+1}-x_n\|\leq\varsigma(1+(1+n)^{-1})\lambda_H\|x_{n+1}-x_n\|+\frac{\tau}{\gamma}\|p_{n+1}-p_n\|.\tag{3.12}$$

In virtue of the fact that $k>2\varsigma\lambda_H$, using (3.12) we deduce that

$$\|x_{n+1}-x_n\|\leq\frac{\tau}{\gamma(k-\varsigma(1+(1+n)^{-1})\lambda_H)}\|p_{n+1}-p_n\|.\tag{3.13}$$

Applying (3.6) and taking into account that the mapping P is λ_P-Lipschitz continuous, N is λ_{N_1}-Lipschitz continuous, λ_{N_2}-Lipschitz continuous and λ_{N_3}-Lipschitz continuous in the first, second and third arguments, respectively, M is λ_{M_1}-Lipschitz continuous and λ_{M_2}-Lipschitz continuous in the first and second arguments, respectively, and the mappings A, B, C, F and G are D-Lipschitz continuous with constants λ_A, λ_B, λ_C, λ_F and λ_G, respectively, and the mappings f and g are λ_f-Lipschitz continuous and λ_g-Lipschitz continuous, respectively, for each $n \geq 0$, we yield

$$
\begin{aligned}
\|p_{n+2} - p_{n+1}\| &= \|(1-\lambda)p_{n+1} + \lambda[(P \circ g)(x_{n+1}) - \rho(N(u_{n+1}, v_{n+1}, w_{n+1}) \\
&\quad + f(x_{n+1}) - M(s_{n+1}, t_{n+1}))] + \lambda e_{n+1} + r_{n+1} \\
&\quad - ((1-\lambda)p_n + \lambda[(P \circ g)(x_n) - \rho(N(u_n, v_n, w_n) \\
&\quad + f(x_n) - M(s_n, t_n))] + \lambda e_n + r_n)\| \\
&\leq (1-\lambda)\|p_{n+1} - p_n\| + \lambda\Big(\|(P \circ g)(x_{n+1}) - (P \circ g)(x_n)\| \\
&\quad + \rho(\|N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_n)\| \\
&\quad + \|f(x_{n+1}) - f(x_n)\| + \|M(s_{n+1}, t_{n+1}) - M(s_n, t_n)\|)\Big) \\
&\quad + \lambda\|e_{n+1} - e_n\| + \|r_{n+1} - r_n\| \\
&\leq (1-\lambda)\|p_{n+1} - p_n\| + \lambda\Big(\lambda_P\|g(x_{n+1}) - g(x_n)\| \\
&\quad + \rho(\|N(u_{n+1}, v_{n+1}, w_{n+1}) - N(u_n, v_{n+1}, w_{n+1})\| \\
&\quad + \|N(u_n, v_{n+1}, w_{n+1}) - N(u_n, v_n, w_{n+1})\| \\
&\quad + \|N(u_n, v_n, w_{n+1}) - N(u_n, v_n, w_n)\| + \lambda_f\|x_{n+1} - x_n\| \\
&\quad + \|M(s_{n+1}, t_{n+1}) - M(s_n, t_{n+1})\| + \|M(s_n, t_{n+1}) - M(s_n, t_n)\|)\Big) \\
&\quad + \lambda\|e_{n+1} - e_n\| + \|r_{n+1} - r_n\| \\
&\leq (1-\lambda)\|p_{n+1} - p_n\| + \lambda\Big(\lambda_P\lambda_g\|x_{n+1} - x_n\| \\
&\quad + \rho(\lambda_{N_1}\|u_{n+1} - u_n\| + \lambda_{N_2}\|v_{n+1} - v_n\| + \lambda_{N_3}\|w_{n+1} - w_n\| \\
&\quad + \lambda_f\|x_{n+1} - x_n\| + \lambda_{M_1}\|s_{n+1} - s_n\| + \lambda_{M_2}\|t_{n+1} - t_n\|)\Big) \\
&\quad + \lambda\|e_{n+1} - e_n\| + \|r_{n+1} - r_n\| \\
&\leq (1-\lambda)\|p_{n+1} - p_n\| + \lambda\Big(\lambda_P\lambda_g\|x_{n+1} - x_n\| \qquad (3.14) \\
&\quad + \rho(\lambda_{N_1}(1 + (1+n)^{-1})D(A(x_{n+1}), A(x_n)) \\
&\quad + \lambda_{N_2}(1 + (1+n)^{-1})D(B(x_{n+1}), B(x_n)) \\
&\quad + \lambda_{N_3}(1 + (1+n)^{-1})D(C(x_{n+1}), C(x_n)) \\
&\quad + \lambda_f\|x_{n+1} - x_n\| + \lambda_{M_1}(1 + (1+n)^{-1})D(F(x_{n+1}), F(x_n)) \\
&\quad + \lambda_{M_2}(1 + (1+n)^{-1})D(G(x_{n+1}), G(x_n))\Big) \\
&\quad + \lambda\|e_{n+1} - e_n\| + \|r_{n+1} - r_n\| \\
&\leq (1-\lambda)\|p_{n+1} - p_n\| + \lambda\Big(\lambda_P\lambda_g\|x_{n+1} - x_n\| \\
&\quad + \rho(\lambda_{N_1}(1 + (1+n)^{-1})\lambda_A\|x_{n+1} - x_n\| \\
&\quad + \lambda_{N_2}(1 + (1+n)^{-1})\lambda_B\|x_{n+1} - x_n\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \lambda_{N_3}(1+(1+n)^{-1})\lambda_C\|x_{n+1}-x_n\| + \lambda_f\|x_{n+1}-x_n\| \\
&+ \lambda_{M_1}(1+(1+n)^{-1})\lambda_F\|x_{n+1}-x_n\| \\
&+ \lambda_{M_2}(1+(1+n)^{-1})\lambda_G\|x_{n+1}-x_n\|)\Big) \\
&+ \lambda\|e_{n+1}-e_n\| + \|r_{n+1}-r_n\| \\
= &(1-\lambda)\|p_{n+1}-p_n\| + \lambda\Big(\lambda_P\lambda_g + \rho(\lambda_{N_1}\lambda_A \\
&+ \lambda_{N_2}\lambda_B + \lambda_{N_3}\lambda_C + \lambda_{M_1}\lambda_F \\
&+ \lambda_{M_2}\lambda_G)(1+(1+n)^{-1}) + \lambda_f\Big)\|x_{n+1}-x_n\| \\
&+ \lambda\|e_{n+1}-e_n\| + \|r_{n+1}-r_n\| \\
= &(1-\lambda)\|p_{n+1}-p_n\| + \lambda\varrho(n)\|x_{n+1}-x_n\| \\
&+ \lambda\|e_{n+1}-e_n\| + \|r_{n+1}-r_n\|,
\end{aligned}
$$

where for each $n \geq 0$,

$$\varrho(n) = \lambda_P\lambda_g + \rho(\lambda_{N_1}\lambda_A + \lambda_{N_2}\lambda_B + \lambda_{N_3}\lambda_C + \lambda_{M_1}\lambda_F + \lambda_{M_2}\lambda_G)(1+(1+n)^{-1}) + \lambda_f.$$

It follows from (3.13) and (3.14) that for each $n \geq 0$,

$$
\begin{aligned}
\|p_{n+2}-p_{n+1}\| \leq &(1-\lambda)\|p_{n+1}-p_n\| \\
&+ \frac{\lambda\varrho(n)\tau}{\gamma(k-\varsigma(1+(1+n)^{-1})\lambda_H)}\|p_{n+1}-p_n\| \\
&+ \lambda\|e_{n+1}-e_n\| + \|r_{n+1}-r_n\| \\
= &(1-\lambda(1-\vartheta(n))\|p_{n+1}-p_n\| + \lambda\|e_{n+1}-e_n\| + \|r_{n+1}-r_n\|,
\end{aligned}
\tag{3.15}
$$

where for each $n \geq 0$,

$$\vartheta(n) = \frac{\varrho(n)\tau}{\gamma(k-\varsigma(1+(1+n)^{-1})\lambda_H)}.$$

Clearly $\vartheta(n) \to \vartheta$, as $n \to \infty$, where $\vartheta = \frac{\varrho\tau}{\gamma(k-\varsigma\lambda_H)}$ and

$$\varrho = \lambda_P\lambda_g + \rho(\lambda_{N_1}\lambda_A + \lambda_{N_2}\lambda_B + \lambda_{N_2}\lambda_B + \lambda_{N_3}\lambda_C + \lambda_{M_1}\lambda_F + \lambda_{M_2}\lambda_G) + \lambda_f.$$

Letting $\varphi(n) = 1-\lambda(1-\vartheta(n))$ for each $n \geq 0$, we know that $\varphi(n) \to \varphi$, as $n \to \infty$, where $\varphi = 1-\lambda(1-\vartheta)$. Clearly, (3.9) implies that $\vartheta \in (0,1)$, and so $\varphi \in (0,1)$. Thus, there exist $\hat{\varphi} \in (0,1)$ (take $\hat{\varphi} = \frac{\varphi+1}{2} \in (\varphi,1)$) and $n_0 \in \mathbb{N}$ such that $\varphi(n) \leq \hat{\varphi}$, for all $n \geq n_0$. Then, by (3.15), for all $n > n_0$, we obtain

$$
\begin{aligned}
\|p_{n+1} - p_n\| &\leq \hat{\varphi}\|p_n - p_{n-1}\| + \lambda\|e_n - e_{n-1}\| + \|r_n - r_{n-1}\| \\
&\leq \hat{\varphi}(\hat{\varphi}\|p_{n-1} - p_{n-2}\| + \lambda\|e_{n-1} - e_{n-2}\| + \|r_{n-1} - r_{n-2}\|) \\
&\quad + \lambda\|e_n - e_{n-1}\| + \|r_n - r_{n-1}\| \\
&= \hat{\varphi}^2\|p_{n-1} - p_{n-2}\| + \lambda(\hat{\varphi}\|e_{n-1} - e_{n-2}\| + \|e_n - e_{n-1}\|) \\
&\quad + \hat{\varphi}\|r_{n-1} - r_{n-2}\| + \|r_n - r_{n-1}\| \\
&\leq \cdots \\
&\leq \hat{\varphi}^{n-n_0}\|p_{n_0+1} - p_{n_0}\| + \lambda\sum_{i=1}^{n-n_0}\hat{\varphi}^{i-1}\|e_{n-(i-1)} - e_{n-i}\| \\
&\quad + \sum_{i=1}^{n-n_0}\hat{\varphi}^{i-1}\|r_{n-(i-1)} - r_{n-i}\|.
\end{aligned}
\tag{3.16}
$$

The preceding inequality (3.16) implies that for any $m \geq n > n_0$,

$$
\begin{aligned}
\|p_m - p_n\| &\leq \sum_{j=n}^{m-1}\|p_{j+1} - p_j\| \\
&\leq \sum_{j=n}^{m-1}\hat{\varphi}^{j-n_0}\|p_{n_0+1} - p_{n_0}\| \\
&\quad + \lambda\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}\hat{\varphi}^{i-1}\|e_{j-(i-1)} - e_{j-i}\| \\
&\quad + \sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}\hat{\varphi}^{i-1}\|r_{j-(i-1)} - r_{j-i}\|.
\end{aligned}
\tag{3.17}
$$

Since $\hat{\varphi} < 1$, from (3.7) and (3.17), we infer that for any $m \geq n > n_0$, $\|p_m - p_n\| \to 0$ as $n \to \infty$. Consequently, $\{p_n\}$ is a Cauchy sequence in E^* and so the completeness of E^* ensures the existence of a point $p \in E^*$ such that $p_n \to p$, as $n \to \infty$. Making use of (3.13), we deduce that the sequence $\{x_n\}$ is also a Cauchy sequence in E and so relying on the completeness of E, $x_n \to x$ for some $x \in E$, as $n \to \infty$. Taking into consideration the fact that the mapping A is λ_A-D-Lipschitz continuous, by (3.6), it follows that for each $n \geq 0$,

$$
\begin{aligned}
\|u_{n+1} - u_n\| &\leq (1 + (1+n)^{-1})D(A(x_{n+1}), A(x_n)) \\
&\leq (1 + (1+n)^{-1})\lambda_A\|x_{n+1} - x_n\|,
\end{aligned}
$$

which implies that $\{u_n\}$ is a Cauchy sequence in E^*. Accordingly, there is a point $u \in E^*$ such that $u_n \to u$ as $n \to \infty$. Since $u_n \in A(x_n)$ for each $n \geq 0$, we have

$$\begin{aligned} d(u, A(x)) &= \inf\{\|u - q\| : q \in A(x)\} \\ &\leq \|u - u_n\| + d(u_n, A(x)) \\ &\leq \|u - u_n\| + d(A(x_n), A(x)) \\ &\leq \|u - u_n\| + \lambda_A \|x_n - x\|. \end{aligned}$$

The right-hand side of the above inequality tends to zero as $n \to \infty$. Now, the fact that $A(x)$ is closed implies that $u \in A(x)$. Following the same argument, one can prove that $\{v_n\}$, $\{w_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{z_n\}$ are Cauchy sequences in E^* and E, respectively, and $v_n \to v$, $w_n \to w$, $s_n \to s$, $t_n \to t$ and $z_n \to z$, as $n \to \infty$, for some $v \in B(x)$, $w \in C(x)$, $s \in F(x)$, $t \in G(x)$ and $z \in H(x)$.

On the other hand, for each $n \geq 0$, we obtain

$$\begin{aligned} \|J_{\rho,P}^{\partial_\eta \phi(.,z_n)}(p_n) - J_{\rho,P}^{\partial_\eta \phi(.,z)}(p)\| &\leq \|J_{\rho,P}^{\partial_\eta \phi(.,z_n)}(p_n) - J_{\rho,P}^{\partial_\eta \phi(.,z)}(p_n)\| \\ &\quad + \|J_{\rho,P}^{\partial_\eta \phi(.,z)}(p_n) - J_{\rho,P}^{\partial_\eta \phi(.,z)}(p)\| \\ &\leq \varsigma \|z_n - z\| + \frac{\tau}{\gamma}\|p_n - p\|. \end{aligned} \tag{3.18}$$

Owing to the fact that $z_n \to z$ and $p_n \to p$ as $n \to \infty$, it follows that the right-hand side of (3.18) approaches zero, as $n \to \infty$. Thereby,

$$J_{\rho,P}^{\partial_\eta \phi(.,z_n)}(p_n) \to J_{\rho,P}^{\partial_\eta \phi(.,z)}(p), \text{ as } n \to \infty.$$

Making use of (3.6), it follows that $g(x) = J_{\rho,P}^{\partial_\eta \phi(.,z)}(p)$. Now, in the light of the above-mentioned facts, we deduce that $(p, x, u, v, w, s, t, z) \in E^* \times E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ is a solution of the GIWHE (3.3) and so according to Lemma 3.2, $(x, u, v, w, s, t, z) \in E \times A(x) \times B(x) \times C(x) \times F(x) \times G(x) \times H(x)$ is a solution of the GSNVLIP (3.1). This completes the proof. □

4. Comments on co-proximal operators

This section is devoted to the investigation and analysis of the concept of co-proximal operator introduced in [1]. Some facts relating to co-proximal operator and the results appeared in [1] are also pointed out. Before dealing with analysis of the results presented in the above-mentioned paper, we need to recall the following concepts.

Recall that a normed space E is called *strictly convex* if the unit sphere in E is strictly convex. that is, the inequality $\|x+y\| < 2$ holds, for all distinct unit vectors x and y in E. It is said to be *smooth* if for every unit vector x in E there exists a unique $x^* \in E^*$ such that $\|x^*\| = \langle x, x^* \rangle = 1$.

It is known that E is smooth if E^* is strictly convex, and that E is strictly convex if E^* is smooth. A normed space E is said to be *uniformly convex* if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if x and y are unit vectors in E with $\|x - y\| \geq 2\varepsilon$, then the average $(x+y)/2$ has norm at most $1-\delta$. In other words, E is uniformly convex if for any two distinct points x and y on the unit sphere centred at the origin the midpoint of the line segment joining x and y is never on the sphere but is close to the sphere only if x and y are sufficiently close to each other.

The function $\delta_E : [0,2] \to [0,1]$ given by

$$\delta_E(\varepsilon) := \inf\{1 - \frac{1}{2}\|x+y\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\}$$

is called the *modulus of convexity* of E. The function δ_E is continuous and increasing on the interval $[0,2]$ and $\delta_E(0) = 0$. Obviously, in the light of the definition of the function δ_E, a normed space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$.

A normed space E is said to be *uniformly smooth* if, for all $\varepsilon > 0$, there is a $\tau > 0$ such that if x and y are unit vectors in E with $\|x - y\| \leq 2\tau$, then the average $(x+y)/2$ has norm at least $1 - \varepsilon\tau$.

The function $\rho_E : [0, +\infty) \to [0, +\infty)$ given by

$$\rho_E(\tau) := \sup\{\frac{1}{2}(\|x + \tau y\| + \|x - \tau y\|) - 1 : x, y \in E, \|x\| = \|y\| = 1\}$$

is called the *modulus of smoothness* of E. Note that the function ρ_E is convex, continuous and increasing on the interval $[0, +\infty)$ and $\rho_E(0) = 0$. In addition, $\rho_E(\tau) \leq \tau$ for all $\tau \geq 0$. Invoking the definition of the function ρ_E, a normed space E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$. It is important to emphasize that in the definitions of $\delta_E(\varepsilon)$ and $\rho_E(\tau)$, we can as well take the infimum and supremum over all vectors $x, y \in E$ with $\|x\|, \|y\| \leq 1$. Any uniformly convex and any uniformly smooth Banach space is reflexive. A Banach space E is uniformly convex (resp., uniformly smooth) if and only if E^* is uniformly smooth (resp., uniformly convex).

The spaces l^p, L^p and W_m^p, $1 < p < \infty$, $m \in \mathbb{N}$, are uniformly convex as well as uniformly smooth, see [16, 22, 26]. At the same time, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W_m^p, $1 < p < \infty$, $m \in \mathbb{N}$, can be found in [16, 22, 26].

For an arbitrary but fixed real number $q > 1$, the set-valued mapping $J_q : E \to 2^{E^*}$ given by

$$J_q(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

is called the *generalized duality mapping* of E. In particular, $J_2 = J$ is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$, for all $x \neq 0$. Note that J_q is single-valued if E is uniformly smooth or equivalently E^* is strictly convex. If E is a Hilbert space, then J_2 becomes the identity mapping on E.

For a real constant $q > 1$, a Banach space E is called q-*uniformly smooth* if there exists a constant $C > 0$ such that $\rho_E(\tau) \leq C\tau^q$, for all $\tau \in \mathbb{R}^+$.

It is well known that (see e.g. [35]) L_q (or l_q) is q-uniformly smooth for $1 \leq q \leq 2$ and is 2-uniformly smooth if $q > 2$.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [35] proved the following result.

Lemma 4.1. *Let E be a real uniformly smooth Banach space. For a real constant $q > 1$, E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Throughout the rest of this paper, as it is assumed in [1], E is a real q-uniformly smooth Banach space with the dual space E^*.

Definition 4.2 ([1, Definition 2.8])**.** Let $J_q : E \to 2^{E^*}$ (the generalized duality mapping) and $H, g : E \to E$ be the mappings. Then

(i) H is said to be Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|H(x) - H(y)\| \leq \tau\|x - y\|, \quad \forall x, y \in E;$$

(ii) J_q is said to be cocoercive with respect to H if, there exists a constant $\gamma_1 > 0$ such that

$$\langle J_q(H(x)) - J_q(H(y)), x - y\rangle \geq \gamma_1\|H(x) - H(y)\|^q, \quad \forall x, y \in E;$$

(iii) J_q is said to be relaxed cocoercive with respect to H if, there exists a constant $\gamma_2 > 0$ such that

$$\langle J_q(H(x)) - J_q(H(y)), x - y\rangle \geq -\gamma_2\|H(x) - H(y)\|^q, \quad \forall x, y \in E;$$

(iv) g is said to be strongly accretive if, there exists a constant $\delta_g > 0$ such that

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq \delta_g\|x - y\|^q, \quad \forall x, y \in E.$$

In support of Definition 4.2 (that is, [1, Definition 2.8]), Ahmad et al. [1] gave an example as follows.

Example 4.3 ([1, Example 2.9])**.** Let $E = \mathbb{R} = E^*$ with usual inner product and let the mappings $g, H : E \to E$ and $J_2 : E \to E^*$ be defined by $H(x) = -x$, $g(x) = x + a$ and $J_2(x) = \frac{x}{2}$ for all $x \in E$, where $a > 0$ is an arbitrary constant. They showed that H is an n-Lipschitz continuous mapping for $n = 1, 2, \ldots$; g is $\frac{1}{n}$-strongly accretive and n-relaxed cocoercive with respect to H for $n = 2, 3, 4, \ldots$, and J_2 is n-relaxed cocoercive with respect to H for $n = 1, 2, 3, \ldots$.

But, it should be pointed out that since $E = \mathbb{R}$ is a Hilbert space, we have $J_2 = I$, that is, the identity mapping on E. In other words, the normalized duality mapping J_2 from $E = \mathbb{R}$ into $E^* = \mathbb{R}$ cannot be defined as $J(x) = \frac{x}{2}$ for all $x \in E$. Accordingly, it seems that there is a fatal error in Example 4.3 (that is, [1, Example 2.9]) and contrary to the claim of the authors in [1], this example cannot be considered in support of Definition 4.2.

Ahmad et al. [1] defined the notion of co-proximal operator as a new proximal operator as follows.

Definition 4.4 ([1, Definition 2.10])**.** Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper and subdifferentiable functional, $J_q : E \to E^*$ (the generalized duality mapping) and

$H : E \to E$ be the mappings. If for any $x^* \in E^*$ and $\rho > 0$, there exists a unique $x \in E$ satisfying

$$\langle J_q(I-H)x - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in E,$$

then, the mapping $x^* \to x$, denoted by $J_q^{\partial\phi}(x^*)$, is said to be co-proximal operator of ϕ, where I is the identity mapping on E. Then we have $x^* - J_q(I-H)x \in \rho\partial\phi(x)$, and it follows that

$$J_q^{\partial\phi}(x^*) = [J_q(I-H) + \rho\partial\phi]^{-1}(x^*).$$

Defining the mapping $P : E \to E^*$ by $P(x) = J_q(I-H)x$ for all $x \in E$, we have

$$\langle J_q(I-H)x - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) = \langle P(x) - x^*, y - x\rangle + \rho\phi(y) - \rho(x) \geq 0,$$

for all $y \in E$. Then invoking Definition 2.9, the mapping $x^* \to x$, denoted by $J^{\partial\phi}_{\rho,P} = J^{\partial\phi}_{\rho,J_q(I-H)} = J_q^{\partial\phi}$ is the $P = J_q(I-H)$-proximal mapping of ϕ. Clearly, in the light of Definition 2.5, we have $x^* - P(x) = x^* - J_q(I-H)x \in \rho\partial\phi(x)$ and then it follows that

$$\begin{aligned} x = J_q^{\partial\phi}(x^*) = J^{\partial\phi}_{\rho,J_q(I-H)}(x^*) = J^{\partial\phi}_{\rho,P}(x^*) &= (P + \rho\partial\phi)^{-1}(x^*) \\ &= (J_q(I-H) + \rho\partial\phi)^{-1}(x^*). \end{aligned}$$

Hence, in virtue of the above-mentioned fact, contrary to the claim in [1], the notion of co-proximal operator introduced in [1] is the same concept of P-proximal mapping introduced by Ding and Xia [20], and is not a new proximal operator.

By presenting [1, Theorem 2.11], the authors claimed that under some appropriate conditions, the co-proximal operator of a given proper, lower semicontinuous and subdifferentiable functional $\phi : E \to \mathbb{R} \cup \{+\infty\}$ is well defined necessarily.

Theorem 4.5 ([1, Theorem 2.11]). *Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and subdifferentiable functional. Let $H : E \to E$ be a δ_H-strongly accretive and λ_H-Lipschitz continuous mapping such that $q\delta_H - c_q\lambda_H^q > 1$, where $c_q > 0$ is a constant guaranteed by Lemma 4.1. Let $J_q : E \to E^*$ be the generalized duality mapping such that J_q is γ-relaxed cocoercive with respect to $I - H$, where I is the identity mapping on E. Let for any $x \in E$, the functional $h(y, x) = \langle x^* - J_q(I-H)x, y - x\rangle$ be 0-DQCV in y. Then, for any $\rho > 0$ and any $x^* \in E^*$, there exists a unique point $x \in E$ such that*

$$\langle J_q(I-H)x - x^*, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \qquad \text{for all } y \in E; \tag{4.1}$$

i.e., $x = J_q^{\partial\phi}(x^*)$, and so the co-proximal operator ϕ is well defined.

It should be remarked that by a careful reading the proof of Theorem 2.1 of [1], we found that there is a small mistake in the inequality (2.1) of [1]. In fact, the inequality

$$\langle J_q(I-H)x, y - x\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \qquad \text{for all } y \in E$$

in [1] must be replaced by (4.1), as is done in Theorem 4.5.

At the same time, it is also remarkable that there is a fatal error in its proof which makes it incorrect. Now, in order to detect and point out the fatal error in the proof of Theorem 2.11 of [1], let us analyze its proof.

For any $\rho > 0$ and $x^* \in E^*$, the authors [1] defined a functional $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ by

$$f(y, x) = \langle x^* - J_q(I - H)x, y - x\rangle + \rho\phi(x) - \rho\phi(y), \quad \text{for all } x, y \in E,$$

and took a point $\bar{y} \in \operatorname{dom}\phi$ arbitrarily but fixed. Taking into account that ϕ is subdifferentiable at $\bar{y}$, they deduced the existence of a point $f^*_{\bar{y}} \in E^*$ such that

$$\phi(x) - \phi(y) \geq \langle f^*_{\bar{y}}, x - \bar{y}\rangle, \quad \text{for all } x \in E. \tag{4.2}$$

Then, on page 1098 of [1], making use of (4.2) and Lemma 4.1, and in the light of the facts that J_q is γ-relaxed cocoercive with respect to $I - H$, and H is δ_H-strongly accretive and λ_H-Lipschitz continuous, they obtained the following relations:

$$\begin{aligned} f(\bar{y}, x) &= \langle x^* - J_q(I - H)x, \bar{y} - x\rangle + \rho\phi(x) - \rho\phi(\bar{y}) \\ &\geq -\gamma\|(I - H)\bar{y} - (I - H)x\|^q - [\|x^*\| \\ &\quad + \|J_q(I - H)(\bar{y})\| + \rho\|f^*_{\bar{y}}\|]\|\bar{y} - x\| \\ &\geq [\gamma(q\delta_H - c_q\lambda^q_H - 1)\|\bar{y} - x\|^{q-1} - [\|x^*\| \\ &\quad + \|J_q(I - H)(\bar{y})\| + \rho\|f^*_{\bar{y}}\|]\|\bar{y} - x\|. \end{aligned} \tag{4.3}$$

It follows from (4.2) that

$$-\gamma\|(I - H)\bar{y} - (I - H)x\|^q \geq \gamma(q\delta_H - c_q\lambda^q_H - 1)\|\bar{y} - x\|^q,$$

which, because $\gamma(q\delta_H - c_q\lambda^q_H - 1) > 0 > -\gamma$, is a contradiction.

Hence, contrary to the claim in [1], the second inequality in (4.2) does not hold. Thereby, under the assumptions mentioned in the context of Theorem 4.5 (that is, [1, Theorem 2.11]), the co-proximal operator of ϕ is not well defined necessarily.

Let E be a real q-uniformly smooth Banach space, $A, B : E \to CB(E)$ be the set-valued mappings, and $J_q : E \to E^*$ (the generalized duality mapping) and $P, f, g : E \to E$ be the single-valued mappings. Let $\phi : E \times E \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued bifunction such that for each $x \in E$, $\phi(., x) : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous and subdifferentiable functional satisfying $g(E) \in \operatorname{dom}\partial\phi(., x) \neq \emptyset$. At the first of section 3 of [1], the authors considered the problem of finding $x \in E$, $u \in A(x)$ and $v \in B(x)$ such that $g(x) \in \operatorname{dom}\partial\phi(., x)$ and

$$\langle J_q(P(u) - f(v)), y - g(x)\rangle \geq \phi(g(x), x) - \phi(y, x), \quad \text{for all } y \in E. \tag{4.4}$$

With the aim of obtaining a characterization of a solution of the problem (4.4) (that is, [1, problem (3.1)]), they deduced the following conclusion in which the equivalence between the problem (4.4) and a fixed point problem is asserted.

Theorem 4.6 ([1, Theorem 3.1]). *The triplet (x, u, v), where $x \in E$, $u \in A(x)$ and $v \in B(x)$ is a solution of the co-variational inclusion problem (4.4) if and only if it satisfies the following relation:*

$$g(x) = J_q^{\partial\phi(.,x)}\{J_q(I - H)g(x) - \rho(J_q(P(u) - f(v)))\}, \tag{4.5}$$

where $\rho > 0$ is a constant and $J_q^{\partial\phi(.,x)} = [J_q(I-H) + \rho\partial\phi(.,x)]^{-1}$ is co-proximal operator of $\phi(.,x)$.

Using the fixed point formulation (4.5) (that is, [1, the fixed point formulation (3.5)]), Ahmad et al. [1] suggested an iterative algorithm for finding a solution of the problem (4.4) as follows.

Algorithm 4.1 ([1, Algorithm 3.2])**.** Let $A, B : E \to CB(E)$ be the set-valued mappings, and $P, f, g : E \to E$ and $J_q : E \to E^*$ be the single-valued mappings with $g(E) = E$. Let $\phi : E \times E \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued bifunction such that for each fixed $x \in E$, $\phi(.,x) : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, lower semicontinuous and subdifferentiable functional satisfying $g(E) \cap \operatorname{dom} \partial\phi(.,x) \neq \emptyset$.

Step 1. Choose an arbitrary initial point $x_0 \in E$ such that $u_0 \in A(x_0)$ and $v_0 \in B(x_0)$.

Step 2. Since $g(E) = E$, there exists a point $x_1 \in E$ such that

$$x_1 = (1-t)x_0 + t[x_0 - g(x_0) + J_q^{\partial\phi(.,x_0)}\{J_q(I-H)g(x_0) - \rho(J_q(P(u_0) - f(v_0)))\}],$$

where $t \in (0,1]$ and $\rho > 0$ are constants.

By Nadler's theorem [27], there exist $u_1 \in A(x)$ and $v_1 \in B(x)$ such that

$$\|u_0 - u_1\| \leq D(A(x_0), A(x_1)),$$
$$\|v_0 - v_1\| \leq D(B(x_0), B(x_1)).$$

Step 3. Let

$$x_2 = (1-t)x_1 + t[x_1 - g(x_1) + J_q^{\partial\phi(.,x_1)}\{J_q(I-H)g(x_1) - \rho(J_q(P(u_1) - f(v_1)))\}],$$

and continue the above process inductively.

Step 4. Compute the sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ by the following scheme:

$$x_{n+1} = (1-t)x_n + t[x_n - g(x_n) + J_q^{\partial\phi(.,x_n)}\{J_q(I-H)g(x_n) - \rho(J_q(P(u_n) - f(v_n)))\}], \tag{4.6}$$

$$u_n \in A(x_n); \|u_n - u_{n-1}\| \leq D(A(x_n), A(x_{n-1})), \tag{4.7}$$

$$v_n \in B(x_n); \|v_n - v_{n-1}\| \leq D(B(x_n), B(x_{n-1})). \tag{4.8}$$

Step 5. If $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ satisfy Step 4 to an amount accuracy, then stop. Otherwise, set $n := n+1$ and repeat the above process.

As it was pointed out before, for each fixed $x \in E$, the co-proximal operator $J_q^{\partial\phi(.,x)}$ is not well defined necessarily. Hence, the fixed point formulation (4.5) (that is, [1, (3.5)]) does not hold in general. Accordingly, Algorithm 3.2 in [1] which has been constructed based on the fixed point formulation (3.5) of [1] is not well defined necessarily. Even, without considering this fact, by a careful reading Algorithm 3.2 in [1], we found that the sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ generated by Algorithm 3.2 in [1] are not well defined necessarily. In fact, for any given $x_0 \in E$, $u_0 \in A(x_0)$ and $v_0 \in B(x_0)$, the authors computed $x_n \in E$ by induction on n using the iterative scheme (4.6), and then they claimed that one can choose $u_n \in A(x_n)$ and $v_n \in B(x_n)$ satisfying (4.7) and (4.8), respectively.

Invoking Lemma 3.3, if E is a complete metric space and $T : E \to CB(E)$ is a set-valued mapping, then for any $\varepsilon > 0$ and for any given $x, y \in E$, $u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u, v) \leq (1 + \varepsilon)D(T(x), T(y)).$$

But, for given $x, y \in E$ and $u \in T(x)$, there may not be a point $v \in T(y)$ such that

$$d(u, v) \leq D(T(x), T(y)).$$

This fact is illustrated in the following example.

Example 4.7. Consider $E = l^\infty(\mathbb{Z}) = \{z = \{z_n\}_{n=-\infty}^{\infty} | \sup_{n\in\mathbb{Z}} |z_n| < \infty, z_n \in \mathbb{C}\}$, the Banach space consisting of all bounded complex sequences $z = \{z_n\}_{n=-\infty}^{\infty}$ with the supremum norm $\|z\|_\infty = \sup_{n\in\mathbb{Z}} |z_n|$. An arbitrary element $z = \{z_n\}_{n=-\infty}^{\infty} \in l^\infty(\mathbb{Z})$ can be written as follows:

$$\begin{aligned}
z &= \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \\
&= \sum_{k\in\{\pm1,\pm3,\dots\}} \Big[(\dots,0,\dots,0,x_{2k-1}+iy_{2k-1},0,x_{2k+1}+iy_{2k+1},0,\dots) \\
&\quad + (\dots,0,\dots,0,x_{2k}+iy_{2k},0,x_{2k+2}+iy_{2k+2},0,\dots)\Big] \\
&= \sum_{k\in\{\pm1,\pm3,\dots\}} \Big[\frac{y_{2k-1}+y_{2k+1}-i(x_{2k-1}+x_{2k+1})}{2}(\dots,0,\dots,0,i_{2k-1},0,i_{2k+1},0,\dots) \\
&\quad + \frac{y_{2k-1}-y_{2k+1}-i(x_{2k-1}-x_{2k+1})}{2}(\dots,0,\dots,0,i_{2k-1},0,-i_{2k+1},0,\dots) \\
&\quad + \frac{y_{2k}+y_{2k+2}-i(x_{2k}+x_{2k+2})}{2}(\dots,0,\dots,0,i_{2k},0,i_{2k+2},0,\dots) \\
&\quad + \frac{y_{2k}-y_{2k+2}-i(x_{2k}-x_{2k+2})}{2}(\dots,0,\dots,0,i_{2k},0,-i_{2k+2},0,\dots)\Big] \\
&= \sum_{k\in\{\pm1,\pm3,\dots\}} \Big[\frac{y_{2k-1}+y_{2k+1}-i(x_{2k-1}+x_{2k+1})}{2}\delta_{2k-1,2k+1} \\
&\quad + \frac{y_{2k-1}-y_{2k+1}-i(x_{2k-1}-x_{2k+1})}{2}\delta'_{2k-1,2k+1} \\
&\quad + \frac{y_{2k}+y_{2k+2}-i(x_{2k}+x_{2k+2})}{2}\delta_{2k,2k+2} \\
&\quad + \frac{y_{2k}-y_{2k+2}-i(x_{2k}-x_{2k+2})}{2}\delta'_{2k,2k+2}\Big],
\end{aligned}$$

where for each $k \in \{\pm1, \pm3, \dots\}$, $\delta_{2k-1,2k+1} = (\dots,0,\dots,0,i_{2k-1},0,i_{2k+1},0,\dots)$, i at the $(2k-1)$th and $(2k+1)$th coordinates, and all other coordinates are zero, $\delta'_{2k-1,2k+1} = (\dots,0,\dots,0,i_{2k-1},0,-i_{2k+1},0,\dots)$, i and $-i$ at the $(2k-1)$th and $(2k+1)$th coordinates, respectively, and all other coordinates are zero, $\delta_{2k,2k+2} = (\dots,0,\dots,0,i_{2k},0,i_{2k+2},0,\dots)$, i at the $(2k)$th and $(2k+2)$th coordinates, and all other coordinates are zero, and $\delta'_{2k,2k+2} = (\dots,0,\dots,0,i_{2k},0,-i_{2k+2},0,\dots)$, i and

$-i$ at the $(2k)$th and $(2k+2)$th coordinates, respectively, and all other coordinates are zero. Thus, the set

$$\mathfrak{B} = \Big\{\delta_{2k-1,2k+1}, \delta'_{2k-1,2k+1}, \delta_{2k,2k+2}, \delta'_{2k,2k+2} : k = \pm 1, \pm 3, \dots\Big\}$$

spans the Banach space $l^\infty(\mathbb{Z})$. It is easy to see that the set $\mathfrak{B}$ is linearly independent and so it is a Schauder basis for the Banach space $l^\infty(\mathbb{Z})$. Let $T : E \to CB(E)$ be a set-valued mapping defined by

$$T(x) = \begin{cases} \{\{\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}} i\}_{n=-\infty}^{\infty}, \delta'_{2k-1,2k+1}, \delta'_{2k,2k+2} : k = \pm 1, \pm 3, \dots\}, & x \neq \delta'_{2m-1,2m+1}, \\ \{\delta_{2k-1,2k+1}, \delta_{2k,2k+2} : k = \pm 1, \pm 3, \dots\}, & x = \delta'_{2m-1,2m+1}, \end{cases}$$

where $\varrho \in [-1, 0)$ and $\beta > 1$ are arbitrary but fixed real numbers, $p \in \mathbb{N} \backslash \{1\}$ is an arbitrary but fixed natural number, α and γ are arbitrary but fixed even natural numbers, and $m \in \{\pm 1, \pm 3, \dots\}$ is chosen arbitrarily but fixed.

Take $\delta'_{2m-1,2m+1} \neq x \in E$ arbitrarily but fixed, $y = \delta'_{2m-1,2m+1}$ and $u = \{\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}} i\}_{n=-\infty}^{\infty}$.

If $a = \{\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}} i\}_{n=-\infty}^{\infty}$, then because $\varrho < 0$, for any $k \in \{\pm 1, \pm 3, \dots\}$, we yield

$$\begin{aligned} d(a, \delta_{2k-1,2k+1}) &= \|\{\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}} i\}_{n=-\infty}^{\infty} - \delta_{2k-1,2k+1}\|_\infty \\ &= \sup\{|\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}}|, |\frac{\varrho}{\sqrt[p]{(2k-1)^\alpha!\beta^{(2k-1)^\gamma!}}} - 1|, \\ &\quad |\frac{\varrho}{\sqrt[p]{(2k+1)^\alpha!\beta^{(2k+1)^\gamma!}}} - 1| : n \in \mathbb{Z}, n \neq 2k-1, 2k+1\} \\ &= \begin{cases} |\frac{\varrho}{\sqrt[p]{(2k-1)^\alpha!\beta^{(2k-1)^\gamma!}}} - 1|, & \text{if } k \in \{2\sigma+1 | \sigma \in \mathbb{N} \cup \{0\}\}, \\ |\frac{\varrho}{\sqrt[p]{(2k+1)^\alpha!\beta^{(2k+1)^\gamma!}}} - 1|, & \text{if } k \in \{-(2\sigma+1) | \sigma \in \mathbb{N} \cup \{0\}\}, \end{cases} \\ &= \begin{cases} 1 - \frac{\varrho}{\sqrt[p]{(2k-1)^\alpha!\beta^{(2k-1)^\gamma!}}}, & \text{if } k \in \{2\sigma+1 | \sigma \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\varrho}{\sqrt[p]{(2k+1)^\alpha!\beta^{(2k+1)^\gamma!}}}, & \text{if } k \in \{-(2\sigma+1) | \sigma \in \mathbb{N} \cup \{0\}\}, \end{cases} \end{aligned}$$

and

$$\begin{aligned} d(a, \delta_{2k,2k+2}) &= \|\{\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}} i\}_{n=-\infty}^{\infty} - \delta_{2k,2k+2}\|_\infty \\ &= \sup\{|\frac{\varrho}{\sqrt[p]{n^\alpha!\beta^{n^\gamma!}}}|, |\frac{\varrho}{\sqrt[p]{(2k)^\alpha!\beta^{(2k)^\gamma!}}} - 1|, \\ &\quad |\frac{\varrho}{\sqrt[p]{(2k+2)^\alpha!\beta^{(2k+2)^\gamma!}}} - 1| : n \in \mathbb{Z}, n \neq 2k, 2k+2\} \\ &= \begin{cases} |\frac{\varrho}{\sqrt[p]{(2k)^\alpha!\beta^{(2k)^\gamma!}}} - 1|, & \text{if } k \in \{2\sigma+1 | \sigma \in \mathbb{N} \cup \{0\}\}, \\ |\frac{\varrho}{\sqrt[p]{(2k+2)^\alpha!\beta^{(2k+2)^\gamma!}}} - 1|, & \text{if } k \in \{-(2\sigma+1) | \sigma \in \mathbb{N} \cup \{0\}\}, \end{cases} \\ &= \begin{cases} 1 - \frac{\varrho}{\sqrt[p]{(2k)^\alpha!\beta^{(2k)^\gamma!}}}, & \text{if } k \in \{2\sigma+1 | \sigma \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\varrho}{\sqrt[p]{(2k+2)^\alpha!\beta^{(2k+2)^\gamma!}}}, & \text{if } k \in \{-(2\sigma+1) | \sigma \in \mathbb{N} \cup \{0\}\}. \end{cases} \end{aligned}$$

Since $\varrho \in [-1, 0)$, it follows that

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = \inf \Big\{ 1 - \frac{\varrho}{\sqrt[p]{(2k+r)^{\alpha}! \beta^{(2k+r)^{\gamma}!}}} : \\ r = 0, \pm 1, 2; k = \pm 1, \pm 3, \dots \Big\} = 1.$$

For the case when $a = \delta'_{2t-1,2t+1}$ for some $t \in \{\pm 1, \pm 3, \dots\}$, then for each $k \in \{\pm 1, \pm 3, \dots\}$, we get

$$d(a, \delta_{2k-1,2k+1}) = \begin{cases} \|\delta'_{2t-1,2t+1} - \delta_{2t-1,2t+1}\|_{\infty}, & k = t, \\ \|\delta'_{2t-1,2t+1} - \delta_{2k-1,2k+1}\|_{\infty}, & k \neq t, \end{cases}$$

$$= \begin{cases} 2, & k = t, \\ 1, & k \neq t, \end{cases}$$

and

$$d(a, \delta_{2k,2k+2}) = \|\delta'_{2t-1,2t+1} - \delta_{2k,2k+2}\|_{\infty} = 1.$$

Therefore,

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = 1.$$

If $a = \delta'_{2j,2j+2}$ for some $j \in \{\pm 1, \pm 3, \dots\}$, relying on the facts that for each $k \in \{\pm 1, \pm 3, \dots\}$,

$$d(a, \delta_{2k-1,2k+1}) = \|\delta'_{2j,2j+2} - \delta_{2k-1,2k+1}\|_{\infty} = 1$$

and

$$d(a, \delta_{2k,2k+2}) = \begin{cases} \|\delta'_{2j,2j+2} - \delta_{2j,2j+2}\|_{\infty}, & k = j, \\ \|\delta'_{2j,2j+2} - \delta_{2k,2k+2}\|_{\infty}, & k \neq j, \end{cases}$$

$$= \begin{cases} 2, & k = j, \\ 1, & k \neq j, \end{cases}$$

we deduce that

$$d(a, T(y)) = \inf_{b \in T(y)} d(a, b) = 1$$

and so

$$\sup_{a \in T(x)} d(a, T(y)) = 1.$$

For the case when $b = \delta_{2s-1,2s+1}$ for some $s \in \{\pm 1, \pm 3, \dots\}$, thanks to the fact that $\varrho < 0$, we yield

$$\begin{aligned} & d(\{\frac{\varrho}{\sqrt[p]{n^{\alpha}! \beta^{n^{\gamma}!}}} i\}_{n=-\infty}^{\infty}, \delta_{2s-1,2s+1}) \\ & = \|\{\frac{\varrho}{\sqrt[p]{n^{\alpha}! \beta^{n^{\gamma}!}}} i\}_{n=-\infty}^{\infty} - \delta_{2s-1,2s+1}\|_{\infty} \\ & = \sup\{|\frac{\varrho}{\sqrt[p]{n^{\alpha}! \beta^{n^{\gamma}!}}}|, |\frac{\varrho}{\sqrt[p]{(2s-1)^{\alpha}! \beta^{(2s-1)^{\gamma}!}}} - 1|, \\ & \qquad |\frac{\varrho}{\sqrt[p]{(2s+1)^{\alpha}! \beta^{(2s+1)^{\gamma}!}}} - 1| : n \in \mathbb{Z}, n \neq 2s-1, 2s+1\} \end{aligned}$$

$$= \begin{cases} |\frac{\varrho}{\sqrt[p]{(2s-1)^{\alpha}!\beta^{(2s-1)^{\gamma}!}}} - 1|, & \text{if } s \in \{2\sigma+1|\sigma \in \mathbb{N}\cup\{0\}\}, \\ |\frac{\varrho}{\sqrt[p]{(2s+1)^{\alpha}!\beta^{(2s+1)^{\gamma}!}}} - 1|, & \text{if } s \in \{-(2\sigma+1)|\sigma \in \mathbb{N}\cup\{0\}\}, \end{cases}$$

$$= \begin{cases} 1 - \frac{\varrho}{\sqrt[p]{(2s-1)^{\alpha}!\beta^{(2s-1)^{\gamma}!}}}, & \text{if } s \in \{2\sigma+1|\sigma \in \mathbb{N}\cup\{0\}\}, \\ 1 - \frac{\varrho}{\sqrt[p]{(2s+1)^{\alpha}!\beta^{(2s+1)^{\gamma}!}}}, & \text{if } s \in \{-(2\sigma+1)|\sigma \in \mathbb{N}\cup\{0\}\}, \end{cases}$$

and for each $k \in \{\pm1, \pm3, \dots\}$,

$$d(\delta'_{2k-1,2k+1}, \delta_{2s-1,2s+1}) = \begin{cases} \|\delta'_{2s-1,2s+1} - \delta_{2s-1,2s+1}\|_\infty, & k = s, \\ \|\delta'_{2k-1,2k+1} - \delta_{2s-1,2s+1}\|_\infty, & k \neq s, \end{cases}$$

$$= \begin{cases} 2, & k = s, \\ 1, & k \neq s, \end{cases}$$

and

$$d(\delta'_{2k,2k+2}, \delta_{2s-1,2s+1}) = \|\delta'_{2k,2k+2} - \delta_{2s-1,2s+1}\|_\infty = 1.$$

Taking into account that $\varrho < 0$, we conclude that

$$d(T(x), b) = \inf_{a\in T(x)} d(a, b) = 1.$$

If $b = \delta_{2q,2q+2}$ for some $q \in \{\pm1, \pm3, \dots\}$, then in virtue of the fact that $\varrho < 0$, we get

$$d(\{\frac{\varrho}{\sqrt[p]{n^{\alpha}!\beta^{n^{\gamma}!}}} i\}_{n=-\infty}^{\infty}, \delta_{2q,2q+2})$$
$$= \|\{\frac{\varrho}{\sqrt[p]{n^{\alpha}!\beta^{n^{\gamma}!}}} i\}_{n=-\infty}^{\infty} - \delta_{2q,2q+2}\|_\infty$$
$$= \sup\{|\frac{\varrho}{\sqrt[p]{n^{\alpha}!\beta^{n^{\gamma}!}}}|, |\frac{\varrho}{\sqrt[p]{(2q)^{\alpha}!\beta^{(2q)^{\gamma}!}}} - 1|,$$

$$|\frac{\varrho}{\sqrt[p]{(2q+2)^{\alpha}!\beta^{(2q+2)^{\gamma}!}}} - 1| : n \in \mathbb{Z}, n \neq 2q, 2q+2\}$$

$$= \begin{cases} |\frac{\varrho}{\sqrt[p]{(2q)^{\alpha}!\beta^{(2q)^{\gamma}!}}} - 1|, & \text{if } q \in \{2\sigma+1|\sigma \in \mathbb{N}\cup\{0\}\}, \\ |\frac{\varrho}{\sqrt[p]{(2q+2)^{\alpha}!\beta^{(2q+2)^{\gamma}!}}} - 1|, & \text{if } q \in \{-(2\sigma+1)|\sigma \in \mathbb{N}\cup\{0\}\}, \end{cases}$$

$$= \begin{cases} 1 - \frac{\varrho}{\sqrt[p]{(2q)^{\alpha}!\beta^{(2q)^{\gamma}!}}}, & \text{if } q \in \{2\sigma+1|\sigma \in \mathbb{N}\cup\{0\}\}, \\ 1 - \frac{\varrho}{\sqrt[p]{(2q+2)^{\alpha}!\beta^{(2q+2)^{\gamma}!}}}, & \text{if } q \in \{-(2\sigma+1)|\sigma \in \mathbb{N}\cup\{0\}\}, \end{cases}$$

and for each $k \in \{\pm1, \pm3, \dots\}$,

$$d(\delta'_{2k-1,2k+1}, \delta_{2q,2q+2}) = \|\delta'_{2k-1,2k+1} - \delta_{2q,2q+2}\|_\infty = 1$$

and

$$d(\delta'_{2k,2k+2}, \delta_{2q,2q+2}) = \begin{cases} \|\delta'_{2q,2q+2} - \delta_{2q,2q+2}\|_\infty, & k = q, \\ \|\delta'_{2k,2k+2} - \delta_{2q,2q+2}\|_\infty, & k \neq q, \end{cases}$$

$$= \begin{cases} 2, & k = q, \\ 1, & k \neq q. \end{cases}$$

Owing to the fact that $\varrho < 0$, it follows that

$$d(T(x), b) = \inf_{a \in T(x)} d(a, b) = 1.$$

Accordingly,

$$\sup_{b \in T(y)} d(T(x), b) = 1.$$

Then, we have

$$D(T(x), T(y)) = \max\Big\{ \sup_{a \in T(x)} d(a, T(y)), \sup_{b \in T(y)} d(T(x), b) \Big\} = 1.$$

Since $\varrho \in [-1, 0)$, we conclude that for each $k \in \{\pm 1, \pm 3, \dots\}$,

$$\|\{\frac{\varrho}{\sqrt[p]{n^{\alpha}!\beta^{n^{\gamma}!}}} i\}_{n=-\infty}^{\infty} - \delta_{2k-1,2k+1}\|_{\infty}$$

$$= \begin{cases} 1 - \frac{\varrho}{\sqrt[p]{(2k-1)^{\alpha}!\beta^{(2k-1)^{\gamma}!}}} > 1, & \text{if } k \in \{2\sigma + 1 | \sigma \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\varrho}{\sqrt[p]{(2k=1)^{\alpha}!\beta^{(2k+1)^{\gamma}!}}} > 1, & \text{if } k \in \{-(2\sigma + 1) | \sigma \in \mathbb{N} \cup \{0\}\}, \end{cases}$$

and

$$\|\{\frac{\varrho}{\sqrt[p]{n^{\alpha}!\beta^{n^{\gamma}!}}} i\}_{n=-\infty}^{\infty} - \delta_{2k,2k+2}\|_{\infty}$$

$$= \begin{cases} 1 - \frac{\varrho}{\sqrt[p]{(2k)^{\alpha}!\beta^{(2k)^{\gamma}!}}} > 1, & \text{if } k \in \{2\sigma + 1 | \sigma \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\varrho}{\sqrt[p]{(2k+2)^{\alpha}!\beta^{(2k+2)^{\gamma}!}}} > 1, & \text{if } k \in \{-(2\sigma + 1) | \sigma \in \mathbb{N} \cup \{0\}\}. \end{cases}$$

These facts imply that for any $v \in T(y)$,

$$d(u, v) = \|u - v\|_{\infty} > D(T(x), T(y)).$$

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J. Balooee
School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran
E-mail address: `javad.balooee@gmail.com`