Linear and Nonlinear Analysis
Volume 8, Number 1, 2022, 31–48
Yokohama Publishers
ISSN 2188-8167

STRONGLY CONVERGENT FIXED POINT ALGORITHM WITH APPLICATIONS TO STRUCTURED MONOTONE INCLUSION PROBLEMS

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Dedicated to the memory of Professor Wataru Takahashi with deep gratitude.

Abstract. As described in this paper, we study the strong convergence properties of variants of the primal–dual splitting algorithm for structured monotone inclusion problems. For this purpose, we first consider a strongly convergent fixed point algorithm in a real Hilbert space and then provide a convergence analysis under mild assumptions. Then, by making use of primal–dual techniques we can employ the proposed fixed point algorithms when solving monotone inclusion problems involving parallel sums and compositions of maximally monotone operators with linear continuous ones. We show strong convergence of the iteratively generated sequences to the solution.

1. Introduction

Many problems in the fields of engineering and applied mathematics such as signal and image processing, compressive sensing, machine learning, and statistics can be reduced to solving a structured monotone inclusion problem involving parallel sums and compositions with linear operators. To solve such problems, primal–dual splitting algorithms have been proposed and studied as described in the literature [15, 2, 16, 18, 31, 7, 9]. The perspective that we adopt for this paper is solving the structured monotone inclusion problem directly by characterizing the solutions of the original problem in terms of a system of fixed point equations via mappings of nonexpansive type in an appropriate product space with the use of primal–dual techniques investigated as described earlier in [16, 31, 9].

The primal–dual splitting algorithms in [10, 16, 31, 9] are existing algorithms that solve the structured monotone inclusion problems in real Hilbert spaces. The generated sequences show weak convergence to a solution. In general, the weak convergence of these algorithms cannot be improved to strong convergence without additional hypotheses on the operators such as those described in the literature as [10, Theorem 2.5 (iii)], [16, Theorem 3.1 (ii)(e) and (ii)(f)], [31, Theorem 3.1 (i) and (ii)], and [9, Theorem 2.1 (i)(c) and (ii)]. Indeed, in the special case in which the problem reduces to a problem of finding a zero of a maximal monotone operator, algorithms in [10, 16, 31] can be expressed in the form of the proximal point

2020 *Mathematics Subject Classification.* Primary 47H09, Secondary 47H05, 47J25, 90C25.
Key words and phrases. Primal-dual splitting algorithm, nonexpansive mapping, fixed point, strong convergence, Hilbert space.
This work was supported in part by the Ministry of Education, Culture, Sports, Science, and Technology [grant number 19K03639].

algorithm [27], which is known to converge weakly but not strongly [20]. Moreover, the algorithm in [9] has the structure of the Douglas–Rachford splitting algorithm [23, 19], which is known to converge weakly but not strongly [12]. Therefore, to ensure strong convergence, new techniques should be developed to analyze the convergence of algorithms for solving the problem.

Herein, we introduce and investigate strongly convergent primal–dual splitting algorithms without assuming restrictive properties for the involved operators. For this purpose, we first develop an error tolerant fixed point algorithm and study its convergence properties. The idea of the algorithm comes from the existing regularization method used for the proximal point algorithm [33, page 120], [28, 32] and the proximal gradient method [23, 25]. Because the proposed algorithm and the proximal gradient method have similar forms (see Remark 3.1), we further consider the inertial version of the proposed algorithm using the idea of the accelerated proximal gradient method [5]. Then, the proposed algorithms can be applied directly to solve the structured monotone inclusion problem. Indeed, using some techniques elaborated in earlier work [16, 31, 9], a solution of the structured monotone inclusion problem can be regarded as a fixed point of mappings of nonexpansive type. Relying on this, we derive primal–dual algorithms. We are able to guarantee strong convergence of the proposed algorithms. The main contributions of the present paper are described below.

- We present the error tolerant fixed point algorithm (Algorithm 3.1) designed for finding a common fixed point of a sequence of mappings of nonexpansive type. Under suitable conditions of the iterative parameters and error sequences (Assumption 3.3), it is guaranteed that the sequence generated by the algorithm converges strongly to the solution (Theorem 3.5).
- Making use of the proposed algorithms and primal–dual techniques [31, 16, 9], we then proceed to solve the structured monotone inclusion problem (Problem 4.1). We present the strongly convergent primal–dual algorithm (Algorithm 4.2). Under suitable conditions on the iterative parameters and error sequences (Assumption 4.4), it is guaranteed that the sequences generated by the algorithms converge strongly to the solution (Theorem 4.5). For convergence, our algorithms require no additional hypothesis on the operators, such as uniform monotonicity, which are necessary in [10, 16, 31, 9].

For related algorithms with strong convergence properties, we refer to [8]. Unlike [8], our algorithm incorporates numerical errors and can be applied to fixed point problem of a countable family of mappings of strongly (quasi)nonexpansive.

The remainder of this paper is organized as follows. After some preliminaries are presented in Section 2, we show convergence of the proposed algorithm in Section 3. In Section 4, we formulate the primal–dual splitting algorithm and study its convergence. Concrete problems and special cases of our algorithm are elucidated as described in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

The following notation is used for this study: $\mathbb{R}$ represents the set of real numbers; $\mathbb{R}_{++}$ denotes the set of strictly positive real numbers; $\overline{\mathbb{R}}$ stands for the extended real line, i.e., $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; $\mathbb{N} = \{1, 2, \dots\}$ represents the set of positive integers; $\mathcal{H}$ denotes a real Hilbert space; for any $x, y \in \mathcal{H}$, $\langle x, y\rangle$ denotes the inner product of x and y; for any $z \in \mathcal{H}$, $\|z\|$ denotes the norm of z, i.e., $\|z\| = \sqrt{\langle z, z\rangle}$; for any $\{x_k\} \subset \mathcal{H}$ and $x \in \mathcal{H}$, $x_k \to x$ and $x_k \rightharpoonup x$ stand for the strong and weak convergences of $\{x_k\}$ to x, respectively; I stands for the identity mapping on $\mathcal{H}$. For an arbitrary set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$, $\mathrm{dom}(A)$ represents the domain of A, i.e., $\mathrm{dom}(A) = \{x \in \mathcal{H} : A(x) \neq \emptyset\}$; $\mathrm{ran}(A)$ signifies the range of A, i.e., $\mathrm{ran}(A) = \bigcup\{A(x) : x \in \mathrm{dom}(A)\}$, $\mathrm{gr}(A)$ stands for the graph of A, i.e., $\mathrm{gr}(A) = \{(x, x^*) : x^* \in A(x)\}$; The set of zero points of A is expressed as $A^{-1}(0)$, i.e., $A^{-1}(0) = \{z \in \mathrm{dom}(A) : 0 \in A(z)\}$; For a function $f : \mathcal{H} \to \overline{\mathbb{R}}$, $\mathrm{dom}(f)$ represents the domain of f, i.e., $\mathrm{dom}(f) = \{x \in \mathcal{H} : f(x) \in \mathbb{R}\}$. $\Gamma(\mathcal{H})$ denotes the family of proper, convex and lower semicontinuous extended real-valued functions.

A *fixed point* of a mapping $T : \mathcal{H} \to \mathcal{H}$ is a point $x \in \mathcal{H}$ satisfying $T(x) = x$. The set of

$$\mathrm{Fix}(T) := \{x \in \mathcal{H} : T(x) = x\}$$

is called a *fixed point set* of T. $I - T$ is said to be *demiclosed at zero* if $p \in \mathrm{Fix}(T)$ whenever $\{x_k\}$ is a sequence in H such that $x_k \rightharpoonup p$ and $\lim_{k\to\infty} \|x_k - T(x_k)\| = 0$. Let $\alpha \in (0, 1)$. T is said to be

(i) *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\| \ (\forall x, y \in \mathcal{H});$$

(ii) *firmly nonexpansive* if

$$\|T(x) - T(y)\|^2 \leq \langle x - y, T(x) - T(y)\rangle \ (\forall x, y \in \mathcal{H});$$

(iii) *α-averaged* if there exists a nonexpansive mapping $R : \mathcal{H} \to \mathcal{H}$ such that $T = (1 - \alpha)I + \alpha R$;

(iv) *quasinonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|T(x) - u\| \leq \|x - u\| \ (\forall (x, u) \in \mathcal{H} \times \mathrm{Fix}(T));$$

(v) *strongly nonexpansive* if T is nonexpansive and $\|x_k - y_k - (T(x_k) - T(y_k))\| \to 0$ whenever $\{x_k\}, \{y_k\} \subset \mathcal{H}$ are bounded such that $\|x_k - y_k\| - \|T(x_k) - T(y_k)\| \to 0$:

(vi) *strongly quasinonexpansive* if T is quasinonexpansive and $\|x_k - T(x_k)\| \to 0$ whenever $\{x_k\} \subset \mathcal{H}$ is bounded such that $\|x_k - u\| - \|T(x_k) - u\| \to 0$ for some $u \in \mathrm{Fix}(T)$;

(vii) *cocoercive* if $\gamma > 0$ and

$$\langle x - y, T(x) - T(y)\rangle \geq \gamma\|T(x) - T(y)\|^2 \ (\forall x, y \in \mathcal{H}).$$

For properties and insights into these mappings, we refer to [13, 6, 30]. If T is nonexpansive, then it is known that $I - T$ is demiclosed at zero. The fixed point set of T is closed and convex [30, 6].

Remark 2.1.

- The class of nonexpansive mappings having a fixed point is an essential subclass of quasinonexpansive mappings. Moreover, the class of averaged mappings is a proper subclass of the class of strongly nonexpansive mappings [13, Section 2].
- As described in earlier reports [3, Section 4] and [22, Section 3], the problem of finding a common fixed point for a countable family of nonexpansive mappings can be transformed into the problem of finding a common fixed point for countable family of strongly nonexpansive mappings.
- As described in earlier reports [6, 20, 12], the weakly convergent algorithm for strongly nonexpansive mappings cannot be improved to strong convergence in general.

Example 2.2.

- The metric projection P_C[1] onto a nonempty, closed convex set $C(\subset \mathcal{H})$, is firmly nonexpansive with $\mathrm{Fix}(P_C) = C$ [6, Proposition 4.16 (4.19)].
- Let $f : \mathcal{H} \to \mathbb{R}$ be a Fréchet differentiable and convex function with the Lipschitz continuous gradient and C be a nonempty, closed, and convex set in $\mathcal{H}$. Denote by ∇f the gradient of f and L the Lipschitz constant of ∇f. By the Baillon–Haddad Theorem (see [6, Corollary 18.17]), ∇f is cocoercive. Moreover, for $\gamma \in (0, 2/L]$, $P_C \circ (I - \gamma\nabla f)$ is α-averaged for some $\alpha > 0$ and $\mathrm{Fix}(P_C \circ (I - \gamma\nabla f)) = \mathrm{argmin}_{x\in C} f(x)$ [6, Theorem 26.14].
- Let $g : \mathcal{H} \to \mathbb{R}$ be a continuous convex function. Let $\xi \in \mathbb{R}$ be such that $C = \mathrm{lev}_{\leq\xi} g$[2] $\neq \emptyset$, and let s be a selection of ∂g, where ∂g is defined in (2.1). The *subgradient projector* [6, Subchapter 29.6] onto C associated with (g, ξ, s) is

$$G : \mathcal{H} \to \mathcal{H} : x \mapsto \begin{cases} x + \frac{\xi - g(x)}{\|s(x)\|^2} s(x) & (g(x) > \xi); \\ x & (g(x) \leq \xi). \end{cases}$$

Then $\mathrm{Fix}(G) = C$ and G are firmly quasinonexpansive[3] [6, Proposition 29.41 (i), (iii)]. Particularly, G is strongly quasinonexpansive. Indeed, let $\{x_k\} \subset \mathcal{H}$ be bounded such that $\|x_k - u\| - \|G(x_k) - u\| \to 0$ for some $u \in C$. The firm quasinonexpansiveness of G yields

$$\|G(x_k) - x_k\|^2 \leq \|x_k - u\|^2 - \|G(x_k) - u\|^2 \to 0.$$

Moreover, $I - G$ is demiclosed at 0 under the additional assumption that g is bounded on every bounded subset of $\mathcal{H}$ [6, Proposition 29.41 (vii)].

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[1] Let $C(\subset \mathcal{H})$ be a nonempty, closed convex set. The *metric projection* $P_C : \mathcal{H} \to C$ onto C is defined for all $x \in \mathcal{H}$ by $P_C(x) \in C$ and $\|x - P_C(x)\| = \inf_{y\in C} \|x - y\|$ [6, Subchapter 3.2].

[2] The *lower level set* $\mathrm{lev}_{\leq\xi} g$ of g at height ξ is defined as $\mathrm{lev}_{\leq\xi} g = \{x \in \mathcal{H} : g(x) \leq \xi\}$ [6, Definition 1.4].

[3] $T : \mathcal{H} \to \mathcal{H}$ is said to be *firmly quasinonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and $\|T(x) - u\|^2 + \|T(x) - x\|^2 \leq \|x - u\|^2$ $(\forall (x, u) \in \mathcal{H} \times \mathrm{Fix}(T))$ [6, Definition 4.1 (iv)].

Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces and let $L : \mathcal{H} \to \mathcal{G}$ be a nonzero bounded linear operator with induced norm $\|L\| = \sup\{\|Lx\| : x \in \mathcal{H} \text{ with } \|x\| \le 1\}$. The *adjoint operator* $L^* : \mathcal{G} \to \mathcal{H}$ of L is defined as $\langle Lx, y\rangle = \langle x, L^*y\rangle$ for all $x \in \mathcal{H}$ and all $y \in \mathcal{G}$.

A set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be

(i) *monotone* if, for all $(x, x^*), (y, y^*) \in \mathrm{gr}(A)$,

$$\langle x - y, x^* - y^*\rangle \ge 0 :$$

(ii) *ν-strongly monotone*, for some $\nu > 0$, if, for all $(x, x^*), (y, y^*) \in \mathrm{gr}(A)$,

$$\langle x - y, x^* - y^*\rangle \ge \nu\|x - y\|^2 :$$

(iii) *maximal monotone* if A is monotone and $A = B$ whenever $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is a monotone mapping such that $\mathrm{gr}(A) \subset \mathrm{gr}(B)$.

For set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$, and for $\gamma \in \mathbb{R}_{++}$, the *resolvent* $J_{\gamma A} : \mathcal{H} \rightrightarrows \mathcal{H}$ of A is defined as $J_{\gamma A} = (I + \gamma A)^{-1}$. Moreover, if A is maximal monotone, then $J_{\gamma A}$ is single-valued and $\mathrm{dom}(J_A) = \mathcal{H}$. The resolvent of the inverse operator of maximal monotone operator A can be computed as shown below [6, Proposition 23.20]:

$$I = J_{\gamma A} + \gamma J_{\gamma^{-1}A^{-1}} \circ \gamma^{-1} I.$$

For set-valued operators $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$, the *parallel sum* is defined as

$$A \Box B := (A^{-1} + B^{-1})^{-1}.$$

For a function $f \in \Gamma(\mathcal{H})$, the *subdifferential* $\partial f : \mathcal{H} \rightrightarrows \mathcal{H}$ of f at $x \in \mathcal{H}$ is defined as

$$\partial f(x) = \{x^* \in \mathcal{H} : f(y) \ge f(x) + \langle y - x, x^*\rangle \ (\forall y \in \mathcal{H})\}. \tag{2.1}$$

We know that the subdifferential ∂f is maximal monotone (see [30, Theorem 4.6.6], [6, Theorem 20.40]). Its resolvent is given as $J_{\gamma\partial f} = \mathrm{prox}_{\gamma f}$ (see [6]), where $\mathrm{prox}_{\gamma f}(x) = \mathrm{argmin}_{y \in H}\{f(y) + \frac{1}{2\gamma}\|y - x\|^2\}$ denotes the *proximal mapping* of f. We say that f is *ν-strongly convex* for some $\nu > 0$ if $f - \nu\|\cdot\|^2/2$ is convex. The *conjugate* of f is $f^* : \mathcal{H} \to f^*(p) = \sup\{\langle p, x\rangle - f(x) : x \in \mathcal{H}\}$ for all $p \in \mathcal{H}$. Moreover, if $f \in \Gamma(\mathcal{H})$, then $f^* \in \Gamma(\mathcal{H})$, as well, and $(\partial f)^{-1} = \partial f^*$. For $g \in \Gamma(\mathcal{H})$, the *infimal convolution* $f \Box g : \mathcal{H} \to \overline{\mathbb{R}}$ of f and g is defined as $f \Box g(x) = \inf_{y \in \mathcal{H}}\{f(y) + g(x - y)\}$ for all $x \in \mathcal{H}$.

3. Fixed point algorithm and convergence results

We provide a strongly convergent fixed point algorithm together with convergence results. We consider the following iterative algorithm.

Algorithm 3.1.

$$x_{k+1} = T_k(\alpha_k x + (1 - \alpha_k)x_k + e_k)$$

where $x, x_1 \in \mathcal{H}$, $\{e_k\} \subset \mathcal{H}$, $\{\alpha_k\} \subset (0,1)$, and where $\{T_k\}$ is a sequence of mappings of $\mathcal{H}$ into itself that satisfies the following condition:

$$\begin{cases} \text{if } \{x_{k_j}\} \subset \mathcal{H} \text{ and } \{T_{k_j}\} \subset \{T_k\} \text{ such that} \\ x_{k_j} \rightharpoonup x \in \mathcal{H} \text{ and } x_{k_j} - T_{k_j}(x_{k_j}) \to 0, \text{ then } x \in \cap_{k=1}^{\infty} \mathrm{Fix}(T_k) \end{cases} \tag{3.1}$$

(see [22, page 1564]). The condition (3.1) can be regarded as a generalization of the demiclosedness property for one mapping.

Remark 3.2.

- Assume that T_k is the resolvent of a maximal monotone operator. Then Algorithm 3.1 has the same structure of the error-tolerant regularization proximal point algorithm [33]. It is noteworthy that the algorithm in [33] and the strongly convergent proximal point algorithm in [21, Algorithm (3.3)] are the same (see [28, Remark 3]).
- In the error free case (i.e. $e_k \equiv 0$), Algorithm 3.1 can be written equivalently as

$$x_{k+1} = T_k(x_k - \alpha_k \nabla h(x_k)), \tag{3.2}$$

  where $h : \mathcal{H} \to \mathbb{R}, h(y) = (1/2)\|y - x\|^2$. Assume that T_k is the proximal mapping of a proper, convex, and lower semicontinuous function. Then (3.2) has the structure of the proximal gradient algorithm [25, 23].
- The following variant of the Krasnoselskiĭ–Mann fixed point algorithm has been considered in [8]:

$$x_{k+1} = \beta_k x_k + \lambda_k (T(\beta_k x_k) - \beta_k x_k), \tag{3.3}$$

  where $x_0 \in \mathcal{H}$, $\{\lambda_k\} \subset (0, \infty)$ and $\{\beta_k\} \subset (0, \infty)$. In case $T_k \equiv (1 - \lambda_k)I + \lambda_k T$, (3.3) becomes

$$x_{k+1} = T_k(\beta_k x_k), \tag{3.4}$$

  which can be considered as the particular case of Algorithm 3.1 when $x = 0$ and $e_k \equiv 0$.

3.1. **Convergence analysis.** To establish strong convergence of the sequence generated by Algorithm 3.1, we require the following assumptions:

Assumption 3.3. We assume that $\{\alpha_k\}$ and $\{e_k\}$ satisfy the following conditions:

(A1) $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(A2) (i) $\sum_{k=1}^{\infty} \|e_k\| < \infty$, or (ii) $\lim_{k\to\infty} \|e_k\|/\alpha_k = 0$.

Remark 3.4. Assumption 3.3 is investigated in [32, Theorem 3] and [28, Theorem 3.1] to demonstrate strong convergence of the regularization method for the proximal point algorithm.

We present the following result, which is the main result of this section.

Theorem 3.5. *Let $\{T_k\}$ be a sequence of strongly nonexpansive mappings $T_k : \mathcal{H} \to \mathcal{H}$ such that $\cap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$ and (3.1) is satisfied. Assume that Assumption 3.3 holds for $\{\alpha_k\}$ and $\{e_k\}$. Then the sequence $\{x_k\}$ generated by Algorithm 3.1 converges strongly to $P_{\cap_{k=1}^{\infty} Fix(T_k)}(x)$.*

Proof. We consider the following algorithm:

$$y_{k+1} = \alpha_k x + (1 - \alpha_k) T_k(y_k) + e_k, \tag{3.5}$$

where $y_1 \in H$. Algorithm (3.5) is known as the error tolerant Halpern fixed point algorithm [22, page 1567]. Under the assumptions that $\lim_{k\to\infty} \alpha_k = 0$ and $\lim_{k\to\infty} \|e_k\| = 0$. Algorithm 3.1 and (3.5) are equivalent, i.e., $\{x_k\}$ generated by Algorithm 3.1 converges if and only if $\{y_k\}$ generated by (3.5) does. Indeed, assume that the convergence of the sequence generated by Algorithm 3.1 is guaranteed. Let $\{y_k\}$ be generated by (3.5) and letting $\{z_k\}$ be defined as

$$z_k := T_k(y_k) \text{ and } \hat{T}_k := T_{k+1},$$

for every $k \in \mathbb{N}$, then we observe that $\{z_k\}$ can be written as

$$z_{k+1} = T_{k+1}(y_{k+1}) = \hat{T}_k(\alpha_k x + (1 - \alpha_k) z_k + e_k).$$

Also, $\{z_k\}$ converges to some $z^* \in H$ because $\{z_k\}$ has the structure of Algorithm 3.1. In this case,

$$\begin{aligned}\|y_{k+1} - z^*\| &= \|\alpha_k x + (1 - \alpha_k) z_k + e_k - z^*\| \\ &\leq \alpha_k \|x - z^*\| + (1 - \alpha_k)\|z_k - z^*\| + \|e_k\| \to 0 \ (k \to \infty).\end{aligned}$$

Conversely, assume that the convergence of the sequence generated by (3.5) is guaranteed. Let $\{x_k\}$ be generated by Algorithm 3.1 and $\{w_k\}$ be defined as

$$w_k := \alpha_k x + (1 - \alpha_k) x_k + e_k, \hat{\alpha}_k := \alpha_{k+1} \text{ and } \hat{e}_k := e_{k+1}$$

for every $k \in \mathbb{N}$. We observe that $\{w_k\}$ can be written as

$$\begin{aligned} w_{k+1} &= \alpha_{k+1} x + (1 - \alpha_{k+1}) x_{k+1} + e_{k+1} \\ &= \hat{\alpha}_k x + (1 - \hat{\alpha}_k) T_k(w_k) + \hat{e}_k. \end{aligned} \tag{3.6}$$

Also, $\{w_k\}$ converges to some $w^* \in H$ because $\{w_k\}$ has the structure of (3.5). In this case,

$$w_k - w^* = \alpha_k (x - w^*) + (1 - \alpha_k)(x_k - w^*) + e_k.$$

Therefore,

$$\|x_k - w^*\| \leq \frac{1}{1 - \alpha_k} (\|w_k - w^*\| + \alpha_k \|x - w^*\| + \|e_k\|) \to 0 \ (k \to \infty).$$

Assumption 3.3 guarantees that $\lim_{k\to\infty} \alpha_k = 0$ and $\lim_{k\to\infty} \|e_k\| = 0$. Moreover, according to [22, Theorem 3.3], the sequence $\{y_k\}$ generated by (3.5) converges strongly to $P_{\cap_{k=1}^{\infty} \mathrm{Fix}(T_k)}(x)$. Consequently, $\{x_k\}$ converges strongly to $P_{\cap_{k=1}^{\infty} \mathrm{Fix}(T_k)}(x)$ □

Remark 3.6. In the Banach space setting, (3.5) has been considered in [3] when $e_k \equiv 0$ and T_k is nonexpansive.

Assumption of $\{T_k\}$ can be relaxed to strongly quasinonexpansive. The result is straightforward (e.g., using [22, Theorem 3.1] and [26, Lemma 5.3]). Therefore, we omit the proof of the following result.

Theorem 3.7. *Let $\{T_k\}$ be a sequence of strongly quasinonexpansive mappings $T_k : \mathcal{H} \to \mathcal{H}$ such that $\cap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$ and (3.1) is satisfied. Assume that (A1) and (A2) (i) of Assumption 3.3 hold for $\{\alpha_k\}$ and $\{e_k\}$. Then the sequence $\{x_k\}$ generated by Algorithm 3.1 converges strongly to $P_{\cap_{k=1}^{\infty} Fix(T_k)}(x)$.*

Remark 3.8. Algorithm 3.1 and (3.3) are designed to solve the same type of problem and have the strong convergence property. Algorithm 3.1 incorporates numerical errors and the assumptions on the iterative parameters are mild.

3.2. **Fixed point algorithm involving inertial terms.** This section considers an inertial variant of Algorithm 3.1. Using the idea of an inertial variant of the proximal gradient method [5], we consider the following iterative algorithm.

Algorithm 3.9.

$$\begin{cases} \overline{x}_k = x_k + t_k(x_k - x_{k-1}) \\ x_{k+1} = T_k\left(\alpha_k x + (1-\alpha_k)\overline{x}_k\right) \end{cases}$$

where $x_0, x_1, x \in \mathcal{H}$, $\{t_k\} \subset [0,\infty)$, and $\{\alpha_k\} \subset (0,1)$.

Remark 3.10.

- Algorithm 3.9 can be written equivalently as

$$\begin{cases} \overline{x}_k = x_k + t_k(x_k - x_{k-1}) \\ x_{k+1} = T_k\left(\overline{x}_k - \alpha_k \nabla h(\overline{x}_k)\right), \end{cases} \tag{3.7}$$

  where $h : \mathcal{H} \to \mathbb{R}, h(y) = (1/2)\|y - x\|^2$. On the assumption that T is the proximal mapping of a proper, convex, and lower semicontinuous function, then (3.7) has the same structure of the accelerated proximal gradient method [5].
- For related fixed point algorithm with inertial terms, we refer to [29]. Numerical examples in [29, Section 7] illustrate the attractive properties of inertia compared to existing algorithms. Unlike [29], our algorithm can be applied to a problem of finding a common fixed point for a countable family of strongly (quasi)nonexpansive mappings.

To establish convergence of the sequence generated by Algorithm 3.9, the following assumptions must be made.

Assumption 3.11. Let $\{x_k\}$ be a sequence generated by Algorithm 3.9, one can also assume that $\{t_k\}$ satisfies the following condition:

(i) $\sum_{k=1}^{\infty} t_k\|x_k - x_{k-1}\| < \infty$, or (ii) $\lim_{k\to\infty} t_k\|x_k - x_{k-1}\|/\alpha_k = 0$.

Remark 3.12. Let us provide the sufficient conditions of Assumption 3.11.

- Let $\{t_k\}$ be defined as

$$t_k := \begin{cases} \theta_k/\|x_k - x_{k-1}\| & (x_k \neq x_{k-1}) \\ 0 & (\text{otherwise}) \end{cases} \tag{3.8}$$

  where $\{\theta_k\} \subset [0, \infty)$. When $\theta_k := 1/k^2$ $(k \in \mathbb{N})$, $\{t_k\}$ satisfies (i) of Assumption 3.11. When $\theta_k := 1/k^a$ $(k \in \mathbb{N}, a > 1)$, and $\alpha_k := 1/k$ $(k \in \mathbb{N})$, $\{t_k\}$ satisfies (ii) of Assumption 3.11. Consequently, Assumption 3.11 is implemented easily.
- (3.8) can be modified as

$$t_k := \begin{cases} \min\{\gamma_k, \theta_k/\|x_k - x_{k-1}\|\} & (x_k \neq x_{k-1}) \\ 0 & (\text{otherwise}) \end{cases} \tag{3.9}$$

  where $\{\gamma_k\}, \{\theta_k\} \subset [0, \infty)$. When $\theta_k := 1/k^2$ $(k \in \mathbb{N})$, $\{t_k\}$ satisfies (i) of Assumption 3.11. When $\theta_k := 1/k^a$ $(k \in \mathbb{N}, a > 1)$, and $\alpha_k := 1/k$ $(k \in \mathbb{N})$, $\{t_k\}$ satisfies (ii) of Assumption 3.11. It is noteworthy that we can choose $\{\gamma_k\}$ as the inertial sequences adopted in earlier studies of [1, 5, 14, 2] without extra assumptions about the iterative parameters or on the inertial sequences.

Using Theorems 3.5 and 3.7, the following result can be derived.

Corollary 3.13. *Let $\{T_k\}$ be a sequence of mappings such that $\cap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$ and (3.1) is satisfied and let $\{x_k\}$ be a sequence generated by Algorithm 3.9. Then the following hold:*

(a) *Assume that $\{T_k\}$ is a sequence of strongly nonexpasive mappings, (A1) of Assumption 3.3 holds for $\{\alpha_k\}$ and Assumption 3.11 holds for $\{t_k\}$. Then $\{x_k\}$ converges strongly to $P_{\cap_{k=1}^{\infty} Fix(T_k)}(x)$.*

(b) *Assume that $\{T_k\}$ is a sequence of strongly quasinonexpasive mappings, (A1) of Assumption 3.3 holds for $\{\alpha_k\}$ and (i) of Assumption 3.11 holds for $\{t_k\}$. Then $\{x_k\}$ converges strongly to $P_{\cap_{k=1}^{\infty} Fix(T_k)}(x)$.*

Proof. By the definition of $\{x_k\}$, we have

$$\begin{aligned} \alpha_k x + (1 - \alpha_k)\overline{x}_k &= \alpha_k x + (1 - \alpha_k)(x_k + t_k(x_k - x_{k-1})) \\ &= \alpha_k x + (1 - \alpha_k)x_k + t_k(1 - \alpha_k)(x_k - x_{k-1}). \end{aligned}$$

Therefore,

$$x_{k+1} = T(\alpha_k x + (1 - \alpha_k)x_k + t_k(1 - \alpha_k)(x_k - x_{k-1})),$$

which can be considered as the particular case of Algorithm 3.1 when $e_k := t_k(1 - \alpha_k)(x_k - x_{k-1})$.

The claim follows from Theorems 3.5 and 3.7 because Algorithm 3.9 is the special instance of Algorithm 3.1. □

4. Primal-dual splitting algorithms and convergence results

In this section, we consider the following structured monotone inclusion problem.

Problem 4.1. Let m be a strictly positive integer and let $I := \{1, 2, \dots, m\}$, we consider the following primal inclusion problem [15, 31]

$$\text{find } x \in \mathcal{H} \text{ such that } z \in Ax + \sum_{i=1}^{m} L_i^* \left((B_i \Box D_i) (L_i x - r_i) \right) + Cx, \tag{4.1}$$

and its dual inclusion problem

$$\text{find } v_1 \in \mathcal{G}_1, \dots, v_m \in \mathcal{G}_m \text{ such that}$$

$$(\exists x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^{m} L_i^* v_i \in Ax + Cx \\ v_i \in (B_i \Box D_i)(L_i x - r_i), \; i = 1, 2, \dots, m, \end{cases} \tag{4.2}$$

where

- $\mathcal{H}, \mathcal{G}_1, \dots, \mathcal{G}_m$ are real Hilbert spaces.
- $z \in \mathcal{H}$ and $(r_1, \dots, r_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$.
- $A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i$ $(i \in I)$ are maximal monotone operators.
- $C : \mathcal{H} \to \mathcal{H}$ is cocoercive for some $\mu > 0$.
- $D_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i$ $(i \in I)$ is ν_i-strongly monotone for some $\nu_i \in (0, \infty)$
- $L_i : \mathcal{H} \to \mathcal{G}$ $(i \in I)$ is a nonzero bounded linear operator with adjoint L_i^*.

It can be said that $(\overline{x}, \overline{v}_1, \dots, \overline{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a primal–dual solution to Problem 4.1 if

$$z - \sum_{j=1}^{m} L_j^* \overline{v}_j \in A\overline{x} + C\overline{x} \text{ and } \overline{v}_i \in (B_i \Box D_i)(L_i \overline{x} - r_i) \; i = 1, 2, \dots, m. \tag{4.3}$$

If $\overline{x}$ is a solution to (4.1), then there exists $(\overline{v}_1, \dots, \overline{v}_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ such that $(\overline{x}, \overline{v}_1, \dots, \overline{v}_m)$ is a primal–dual solution to Problem 4.1, and if $(\overline{v}_1, \dots, \overline{v}_m)$ is a solution to (4.2), then there exists $\overline{x} \in \mathcal{H}$ such that $(\overline{x}, \overline{v}_1, \dots, \overline{v}_m)$ is a primal–dual solution to Problem 4.1. If $(\overline{x}, \overline{v}_1, \dots, \overline{v}_m)$ is a primal–dual solution to Problem 4.1, then $\overline{x}$ is a solution to (4.1) and $(\overline{v}_1, \dots, \overline{v}_m)$ is a solution to (4.2).

Example 4.1. Problem 4.1 is very useful in applications to many practical problems. Indeed, one can consider the convex optimization problems of the form

$$\min_{x \in \mathcal{H}} \left\{ f(x) + \sum_{i=1}^{m} g_i(L_i x) + h(x) \right\}, \tag{4.4}$$

where $f \in \Gamma(\mathcal{H})$ and $h : \mathcal{H} \to \overline{\mathbb{R}}$ are differentiable with the Lipschitz continuous gradient, for every $i \in I$, $g_i \in \Gamma(\mathcal{G}_i)$ and $L_i : \mathcal{H} \to \mathcal{G}_i$ is a bounded linear operator. Under mild assumptions (see [16, Proposition 4.3]) the equivalent monotone inclusion problem takes the form of

$$\text{find } u \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \sum_{i=1}^{m} L_i^* \partial g_i(L_i x) + \nabla h(x), \tag{4.5}$$

which is a special instance of (4.1).

In order to show the results, we need the following notation. We consider the Hilbert space $\boldsymbol{\mathcal{G}} := \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ endowed with the inner product and associated norm defined for $\boldsymbol{u} = (u_1, \ldots, u_m), \boldsymbol{v} = (v_1, \ldots, v_m) \in \boldsymbol{\mathcal{G}}$ as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{G}}} := \sum_{i=1}^{m} \langle u_i, v_i \rangle_{\mathcal{G}_i} \text{ and } \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{G}}} := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{\boldsymbol{\mathcal{G}}}},$$

respectively. Furthermore, we let $\boldsymbol{\mathcal{K}} = \mathcal{H} \times \boldsymbol{\mathcal{G}}$ be the Hilbert space endowed with inner product and associated norm defined for every $(x, \boldsymbol{u}), (y, \boldsymbol{v}) \in \boldsymbol{\mathcal{K}}$ as

$$\langle (x, \boldsymbol{u}), (y, \boldsymbol{v}) \rangle_{\boldsymbol{\mathcal{K}}} := \langle x, y \rangle_{\mathcal{H}} + \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{G}}} \text{ and } \|(x, \boldsymbol{u})\|_{\boldsymbol{\mathcal{K}}} := \sqrt{\langle (x, \boldsymbol{u}), (x, \boldsymbol{u}) \rangle_{\boldsymbol{\mathcal{K}}}}, \tag{4.6}$$

respectively. Furthermore, we consider the set-valued operator

$$\boldsymbol{M} : \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}} : (x, v_1, \ldots, v_m) \mapsto (-z + Ax, r_1 + B_1^{-1}(v_1), \ldots, r_m + B_m^{-1}(v_m)), \tag{4.7}$$

We next consider the linear continuous operator of

$$\boldsymbol{S} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} : (x, v_1, \ldots, v_m) \mapsto \left(\sum_{i=1}^{m} L_i^* v_i, -L_1 x, \ldots, -L_m x \right). \tag{4.8}$$

We consider the single-valued operator

$$\boldsymbol{Q} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} : (x, v_1, \ldots, v_m) \mapsto \left(Cx, D_1^{-1} v_1, \ldots, D_m^{-1} v_m \right). \tag{4.9}$$

According to [31, page 672], $\boldsymbol{Q}$ is β-cocoercive with

$$\beta = \min\{\mu, \nu_1, \ldots, \nu_m\}.$$

We next introduce the bounded linear operator as

$$\boldsymbol{V} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} : (x, v_1, \ldots, v_m) \mapsto \left(\frac{1}{\tau} x - \sum_{i=1}^{m} L_i^* v_i, -L_1 x + \frac{1}{\sigma_1} v_1, \ldots, -L_m x + \frac{1}{\sigma_m} v_m \right). \tag{4.10}$$

Then $\boldsymbol{V}$ is self-adjoint, with ρ-strongly positive for

$$\rho := \min\left\{ \tau^{-1}, \sigma_1^{-1}, \ldots, \sigma_m^{-1} \right\} \left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2} \right) > 0$$

when (4.11) holds (see [31]).

4.1. **Strongly convergent primal–dual splitting algorithm and convergence analysis.** This subsection presents the following algorithm for solving Problem 4.1.

Algorithm 4.2. (primal–dual splitting algorithm of forward-backward-type)

$$\begin{cases} \overline{x}_k = \alpha_k x + (1 - \alpha_k) x_k + e_k \\ \overline{v}_{i,k} = \alpha_k v_i + (1 - \alpha_k) v_{i,k} + e_{i,k} \ (\forall i \in I) \\ x_{k+1} = J_{\tau A} \left(\overline{x}_k - \tau \left(\sum_{i=1}^{m} L_i^* \overline{v}_{i,k} + C \overline{x}_k - z \right) \right) \\ v_{i,k+1} = J_{\sigma_i B_i^{-1}} \left(\overline{v}_{i,k} + \sigma_i \left(L_i (2x_{k+1} - \overline{x}_k) - D_i^{-1} \overline{v}_{i,k} - r_i \right) \right) \ (\forall i \in I) \end{cases}$$

where $(x, v_1, \ldots, v_m), (x_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{(e_k, e_{1,k}, \ldots, e_{m,k})\} \subset \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{\alpha_k\} \subset (0,1)$, and $\tau, \sigma_1, \ldots, \sigma_m > 0$ such that

$$2 \cdot \min\{\tau^{-1}, \sigma_1^{-1}, \ldots, \sigma_m^{-1}\} \cdot \min\{\mu, \nu_1, \ldots, \nu_m\} \cdot \left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2}\right) > 1. \tag{4.11}$$

Remark 4.3. Assume that $m = 1$, $z = 0$, $\alpha_k \equiv 0$, $e_k \equiv 0$, $e_{1,k} \equiv 0$, $r_1 = 0$, $C = 0$, and $D_1 : \mathcal{G}_1 \rightrightarrows \mathcal{G}$, $D_1(0) = \mathcal{G}$, and $D_1(v) = \emptyset$ for all $v \in \mathcal{G} \setminus \{0\}$. Then Algorithm 4.2 can be written as

$$\begin{cases} x_{k+1} = J_{\tau A}(x_k - \tau L_1^* v_{1,k}), \\ v_{1,k+1} = J_{\sigma_1 B_1^{-1}}(v_{1,k} + \sigma_1 L_1(2x_{k+1} - x_k)). \end{cases} \tag{4.12}$$

Algorithm (4.12) is known as the primal–dual splitting algorithm [7, Algorithm 1]. Also, (4.12) has been introduced into [15] in the context of the convex optimization problem. Its fundamental convergence properties have been investigated in [7, 18, 31]. Weak convergence of this type of algorithm is obtained in [31, Theorem 3.1 (i)] and [18, Theorem 3.2]. Moreover, the uniform monotonicity of A and B_i^{-1} ($i \in I$) ensures strong convergence of (4.12) [31, Theorem 3.1 (ii) and (iii)]. It is noteworthy that Algorithm 4.2 we propose has the strong convergence property without assuming restrictive properties for the involved operators.

By making use of primal–dual techniques [16, 31], we derive strong convergence of Algorithm 4.2. To establish convergence of the algorithm, we need the following assumption.

Assumption 4.4. We assume that $\{\alpha_k\}$ and $\{(e_k, e_{1,k}, \ldots, e_{m,k})\}$ satisfy the following conditions:

(A1) $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(A2) (i) $\sum_{k=1}^{\infty} \|e_k\| < \infty$ and $\sum_{k=1}^{\infty} \|e_{i,k}\| < \infty$ ($\forall i \in I$), or (ii) $\lim_{k\to\infty} \|e_k\|/\alpha_k = 0$ and $\lim_{k\to\infty} \|e_{i,k}\|/\alpha_k = 0$ ($\forall i \in I$).

We show the following result.

Theorem 4.5. *In Problem 4.1, presume that*

$$z \in ran \left(A + \sum_{i=1}^{m} L_i^* \left((B_i \Box D_i)(L_i \cdot - r_i)\right) + C\right). \tag{4.13}$$

Let $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ be a sequence generated by Algorithm 4.2. Assume that Assumption 4.4 holds for $\{\alpha_k\}$ and $\{(e_k, e_{1,k}, \ldots, e_{m,k})\}$. Then there exists a primal–dual solution $\overline{\boldsymbol{v}} = (\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ to Problem 4.1 such that $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ converges strongly to $\overline{\boldsymbol{v}}$.

Proof. By using an argument similar to that in [31, Theorem 3.1], Algorithm 4.2 becomes

$$\boldsymbol{v}_{k+1} = J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})(\alpha_k \boldsymbol{v} + (1 - \alpha_k)\boldsymbol{v}_k + \boldsymbol{e}_k), \tag{4.14}$$

where $\boldsymbol{v}_k := (x_k, v_{1,k}, \ldots, v_{m,k}), \boldsymbol{v} := (x, v_1, \ldots, v_m)$, $\boldsymbol{e}_k := (e_k, e_{1,k}, \ldots, e_{m,k})$, $\boldsymbol{A} := \boldsymbol{V}^{-1}(\boldsymbol{M} + \boldsymbol{S})$ and $\boldsymbol{B} := \boldsymbol{V}^{-1}\boldsymbol{Q}$. Consequently, it has the structure of Algorithm 3.1 when $T := J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})$. Therefore, it is sufficient to check the convergence conditions of Algorithm 3.1 to demonstrate our claim.

One can consider a Hilbert space $\boldsymbol{\mathcal{K}_V}$ endowed with the inner product and norm defined for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{K}}$ as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{\mathcal{K}_V}} := \langle \boldsymbol{x}, \boldsymbol{V}\boldsymbol{y} \rangle_{\boldsymbol{K}} \text{ and } \|\boldsymbol{x}\|_{\boldsymbol{\mathcal{K}_V}} := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\boldsymbol{\mathcal{K}_V}}} \tag{4.15}$$

respectively. Because operators $\boldsymbol{M} + \boldsymbol{S}$ and $\boldsymbol{Q}$ are maximal monotone on $\boldsymbol{\mathcal{K}}$, the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ are maximal monotone on $\boldsymbol{\mathcal{K}_V}$ [31]. Furthermore, $\boldsymbol{B}$ is $\beta\rho$-cocoercive on $\boldsymbol{\mathcal{K}_V}$. Additionally, we have $(\boldsymbol{A} + \boldsymbol{B})^{-1}(\boldsymbol{0}) = (\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q})^{-1}(\boldsymbol{0})$. Set $\alpha := 2\beta\rho/(4\beta\rho - 1) \in (0, 1)$. Then it follows from (4.11) that $T = J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})$ is α-averaged and $\mathrm{Fix}(T) = (\boldsymbol{A}+\boldsymbol{B})^{-1}(\boldsymbol{0}) = (\boldsymbol{M}+\boldsymbol{S}+\boldsymbol{Q})^{-1}(\boldsymbol{0})$ (see [6, Proposition 26.1 (iv)]). According to [11, Proposition 1.3], T is strongly nonexpansive. Moreover, because $\boldsymbol{V}$ is self-adjoint and ρ-strongly positive, weak and strong convergence in $\boldsymbol{\mathcal{K}_V}$ are respectively equivalent with weak and strong convergence in $\boldsymbol{\mathcal{K}}$.

(A2) (i) and (A2) (ii) of Assumption 4.4 made on the error sequences $\{e_k, e_{1,k}, \ldots, e_{m,k}\}$ yield

$$\sum_{k=0}^{\infty} \|\boldsymbol{e}_k\|_{\boldsymbol{\mathcal{K}}} < \infty, \text{ and } \lim_{k\to\infty} \|\boldsymbol{e}_k\|_{\boldsymbol{\mathcal{K}}}/\alpha_k = 0.$$

By the linearity and boundedness of $\boldsymbol{V}$ it follows that

$$\sum_{k=0}^{\infty} \|\boldsymbol{e}_k\|_{\boldsymbol{K_V}} < \infty, \text{ and } \lim_{k\to\infty} \|\boldsymbol{e}_k\|_{\boldsymbol{\mathcal{K}_V}}/\alpha_k = 0.$$

According to Theorem 3.5, the sequence $\{\boldsymbol{v}_k\}$ converges strongly in $\boldsymbol{\mathcal{K}_V}$ and, consequently, in $\boldsymbol{\mathcal{K}}$ to $\overline{\boldsymbol{v}} := P_{\mathrm{Fix}(T)}(\boldsymbol{v}) \in (\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q})^{-1}(\boldsymbol{0})$. This result implies that $\overline{\boldsymbol{v}}$ is a primal–dual solution to Problem 4.1. The conclusion follows. □

Remark 4.6.

(a) When $\alpha_k \equiv 0$ and $\boldsymbol{e}_k \equiv \boldsymbol{0}$, (4.14) reduces to

$$\boldsymbol{v}_{k+1} = J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})(\boldsymbol{v}_k),$$

which is the forward–backward splitting algorithm [23, 25].

(b) When $C \equiv 0$ and for every $i = 1, \cdots, m$, one takes $D_i(0) = \mathcal{G}_i$ and $D_i(v) = \emptyset$ for all $v \in \mathcal{G}_i \setminus \{0\}$. The results of Theorem 4.5 are valid with condition (4.11) replaced by

$$\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2 < 1$$

(see [31, Remark 3.3]).

(c) Algorithm 4.2 and a primal–dual algorithm of forward–backward-type with Tikhonov regularization terms in [8] are designed to solve the same type of problem and to have the strong convergence property. Unlike [8, Theorem 14], Algorithm 4.2 incorporates numerical errors. The assumptions on the iterative parameters are mild.

When $e_k := t_k(1-\alpha_k)(x_k - x_{k-1})$ and $e_{i,k} := t_k(1-\alpha_k)(v_{i,k} - v_{i,k-1})$, Algorithm 4.2 reduces to the following algorithm:

Algorithm 4.7. (inertial primal–dual splitting algorithm of forward–backward-type).

$$\begin{cases} \overline{x}_k = \alpha_k x + (1-\alpha_k)x_k + t_k(1-\alpha_k)(x_k - x_{k-1}) \\ \overline{v}_{i,k} = \alpha_k v_i + (1-\alpha_k)v_{i,k} + t_k(1-\alpha_k)(v_{i,k} - v_{i,k-1}) \ (\forall i \in I) \\ x_{k+1} = J_{\tau A}\left(\overline{x}_k - \tau\left(\sum_{i=1}^m L_i^* \overline{v}_{i,k} + C\overline{x}_k - z\right)\right) \\ v_{i,k+1} = J_{\sigma_i B_i^{-1}}\left(\overline{v}_{i,k} + \sigma_i\left(L_i(2x_{k+1} - \overline{x}_k) - D_i^{-1}\overline{v}_{i,k} - r_i\right)\right) \ (\forall i \in I) \end{cases}$$

where $(x, v_1, \ldots, v_m), (x_0, v_{1,0}, \ldots, v_{m,0}), (x_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{(e_k, e_{1,k}, \ldots, e_{m,k})\} \subset \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{\alpha_k\} \subset (0,1)$, and $\tau, \sigma_1, \ldots, \sigma_m > 0$ such that (4.11) holds.

Establishing convergence of Algorithm 4.7 requires the following assumption:

Assumption 4.8. Let $\{x_k\}$ and $\{(v_{1,k}, \ldots, v_{m,k})\}$ be sequences generated by Algorithm 4.7. We assume that $\{t_k\}$ satisfies the following condition:

(i) $\sum_{k=1}^\infty t_k\|x_k - x_{k-1}\| < \infty$ and $\sum_{k=1}^\infty t_k\|v_{i,k} - v_{i,k-1}\| < \infty$ $(\forall i \in I)$ or (ii) $\lim_{k\to\infty} t_k\|x_k - x_{k-1}\|/\alpha_k = 0$ and $\lim_{k\to\infty} t_k\|v_{i,k} - v_{i,k-1}\|/\alpha_k = 0$ $(\forall i \in I)$.

Using Corollary 3.13 and Theorems 3.5 and 4.5, we present the following result.

Corollary 4.9. *For Problem 4.1, we presume that (4.11) and (4.13) hold. Let $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ be a sequence generated by Algorithm 4.7. Assume that (A1) of Assumption 4.4 holds for $\{\alpha_k\}$ and Assumption 4.8 holds for $\{t_k\}$. Then there exists a primal–dual solution $\overline{\boldsymbol{v}} = (\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ to Problem 4.1 such that $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ converges strongly to $\overline{\boldsymbol{v}}$.*

Proof. For the proof, we use Theorem 4.5 in the same setting as in the proof of Theorem 4.5. In this setting, Algorithm 4.7 can be written equivalently in the form of

$$\boldsymbol{v}_{k+1} = J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})(\alpha_k \boldsymbol{v} + (1-\alpha_k)\boldsymbol{v}_k + (1-\alpha_k)t_k(\boldsymbol{v}_k - \boldsymbol{v}_{k-1})), \tag{4.16}$$

where

$$\boldsymbol{v}_k := (x_k, v_{1,k}, \ldots, v_{m,k}) \text{ and } \boldsymbol{v} := (x, v_1, \ldots, v_m),$$

for every $k \in \mathbb{N}$. Set $e_k := (1-\alpha_k)t_k(x_k - x_{k-1})$ and $e_{i,k} := (1-\alpha_k)t_k(v_{i,k} - v_{i,k-1})$ for every $k \in \mathbb{N}$.

Both (i) and (ii) of Assumption 4.8 made on the sequence $\{t_k\}$ yield

$$\sum_{k=0}^{\infty}\|e_k\| < \infty \text{ and } \sum_{k=0}^{\infty}\|e_{i,k}\| < \infty \ (\forall i \in I)$$

or

$$\lim_{k\to\infty}\|e_k\|/\alpha_k = 0 \text{ and } \lim_{k\to\infty}\|e_{i,k}\|/\alpha_k = 0 \ (\forall i \in I).$$

According to (a) of Theorem 4.5, the sequence $\{\boldsymbol{v}_k\}$ converges strongly to a primal–dual solution to Problem 4.1. The conclusion follows.

□

Remark 4.10. Using the same idea as that expressed in [9, 8], a strongly convergent Douglas–Rachford primal–dual splitting algorithm and its inertial variant are straightforward. For that reason, we omit them here.

5. Applications

In this section, we present a concrete problem that reduces to Problem 4.1. We apply the proposed algorithm to the following optimization [16, Problem 4.1].

Problem 5.1. Let $f \in \Gamma(\mathcal{H})$ and $h : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function with a μ^{-1}-Lipschitz continuous gradient, for some $\mu > 0$. For every $i \in I$, let $\mathcal{G}_i$ be a real Hilbert space. Also, let $r_i \in \mathcal{G}_i$ and let $g_i, l_i \in \Gamma(\mathcal{G}_i)$ such that l_i is ν_i-strongly convex, for some $\nu_i \geq 0$. Let $L_i : \mathcal{H} \to \mathcal{G}_i$ be a nonzero bounded linear operator. Consider primal problem

$$\min_{x\in\mathcal{H}}\left\{f(x) + \sum_{i=1}^{m}(g_i\Box l_i)(L_ix - r_i) + h(x) - \langle x, z\rangle_{\mathcal{H}}\right\} \tag{5.1}$$

and dual problem

$$\min_{v_1\in\mathcal{G}_1,\ldots,v_m\in\mathcal{G}_m}\left\{(f^*\Box h^*)\left(z - \sum_{i=1}^{m}L_i^*v_i\right) + \sum_{i=1}^{m}(g_i^*(v_i) + l_i^*(v_i) + \langle v_i, r_i\rangle_{\mathcal{G}_i})\right\}. \tag{5.2}$$

Remark 5.1. In Problem 5.1, if $z = 0$, and if l_i and r_i are respectively indicator functions of $\{0\}$ and $r_i = 0$, then (5.1) reduces to (4.4).

Corollary 5.2. *In Problem 5.1, presume that*

$$z \in ran\left(\partial f + \sum_{i=1}^{m}L_i^*\left((\partial g_i\Box\partial l_i)\left(L_i\cdot - r_i\right)\right) + \nabla h\right). \tag{5.3}$$

Let $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ *be a sequence generated by*

$$\begin{cases} \overline{x}_k = \alpha_k x + (1-\alpha_k)x_k + e_k \\ \overline{v}_{i,k} = \alpha_k v_i + (1-\alpha_k)v_{i,k} + e_{i,k} \ (\forall i \in I) \\ x_{k+1} = \operatorname{prox}_{\tau f}\left(\overline{x}_k - \tau\left(\sum_{i=1}^{m}L_i^*\overline{v}_{i,k} + \nabla h(\overline{x}_k) - z\right)\right) \\ v_{i,k+1} = \operatorname{prox}_{\sigma_i g_i^*}\left(\overline{v}_{i,k} + \sigma_i\left(L_i(2x_{k+1} - \overline{x}_k) - \nabla l_i^*\overline{v}_{i,k} - r_i\right)\right) \ (\forall i \in I), \end{cases}$$

where $(x, v_1, \ldots, v_m), (x_1, v_{1,1}, \ldots, v_{m,1}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{(e_k, e_{1,k}, \ldots, e_{m,k})\} \subset \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{\alpha_k\} \subset (0, 1)$, *and* $\tau, \sigma_1, \ldots, \sigma_m > 0$ *such that*

$$2 \cdot \min\{\tau^{-1}, \sigma_1^{-1}, \ldots, \sigma_m^{-1}\} \cdot \min\{\mu, \nu_1, \ldots, \nu_m\} \cdot \left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2}\right) > 1.$$

Assume that Assumption 4.4 holds for $\{\alpha_k\}$ *and* $\{(e_k, e_{1,k}, \ldots, e_{m,k})\}$. *Then there exists* $\overline{\boldsymbol{v}} = (\overline{x}, \overline{v}_1, \ldots, \overline{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ *such that* $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ *converges strongly to* $\overline{\boldsymbol{v}}$, $\overline{x}$ *is a solution of problem (5.1), and* $(\overline{v}_1, \ldots, \overline{v}_m)$ *is a solution of problem (5.2).*

Proof. We can define

$$A := \partial f, C = \nabla h, B_i = \partial g_i \text{ and } D := \partial l_i \ (i \in I).$$

It follows from [6, Theorem 20.40] that operators A and B_i $(i \in I)$ are maximal monotone. Moreover, the Baillon–Haddad Theorem (see [6, Corollary 18.16]) ensures that C is μ-cocoercive. Because l_i is ν_i-strongly convex, D_i is ν_i-strongly monotone for every $i \in I$. However, for every $i \in I$, it follows from the ν_i-strong convexity of l_i and [6, Corollary 13.38 and Theorem 18.15] that l_i^* is Fréchet differentiable on $\mathcal{G}_i$ with a $1/\nu_i$-Lipschitz continuous gradient, and from [6, Corollary 16.30] that $D_i^{-1} = \nabla l_i^*$. The strong convexity of the functions l_i guarantees that $g_i \square l_i \in \Gamma(\mathcal{G}_i)$ (see [6, Corollary 11.17,Proposition 12.14]) and $\partial(g_i \square l_i) = \partial g_i \square \partial l_i$ $(i \in I)$ (see [6, Proposition 15.7]). It follows from Theorem 4.9 that the sequence $\{x_k\}$ converges strongly to some $\overline{x} \in \mathcal{H}$ such that

$$z \in \partial f(\overline{x}) + \sum_{i=1}^{m} L_i^*((\partial g_i \square \partial l_i))(L_i \overline{x} - r_i) + \nabla h(\overline{x}),$$

and that the sequence $\{(v_{1,k}, \ldots, v_{m,k})\}$ converges strongly to some $(\overline{v}_1, \ldots, \overline{v}_m)$ such that

$$(\exists x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^{m} L_i^* \overline{v}_i \in Ax + Cx \\ \overline{v}_i \in (B_i \square D_i)(L_i x - r_i), \ i = 1, 2, \ldots, m. \end{cases}$$

Then $\overline{x}$ is a solution of the problem (5.1). Moreover, $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution of the problem (5.2) (see [16, Theorem 4.2]). □

6. Conclusions

As described in this paper, we have proposed an error-tolerant fixed point algorithm and studied its convergence properties. The proposed algorithm is applicable for finding a common fixed point of a sequence of mappings of strongly (quasi)nonexpansive in a real Hilbert space. Moreover, by making use of the proposed algorithm and primal–dual techniques, we present the stronglyconvergent primal–dual splitting algorithm. As described in [1, 2, 4, 5, 14, 15, 29, 8], inertial terms are known to contribute to the acceleration of the convergence behavior of the algorithms. To the best of our knowledge, this is the first paper to investigate the error-tolerant fixed point algorithm with strong convergence properties, including inertial algorithms as a special case. Directions for future work include convergence speed, the practical error case, and the case when inertial sequence goes to 1.

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Manuscript received 23 November 2021
revised 7 March 2022

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