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A MODIFIED EXPLICIT ITERATIVE TECHNIQUES FOR SOLVING EQUILIBRIUM PROBLEMS

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ABSTRACT. A new extragradient method for solving a pseudomonotone equilibrium problem with a Lipschitz-type requirement is presented in this study. Variable stepsize is being used in the algorithm, which has been revised at each iteration and is dependent on earlier iterations. The algorithms have the feature of not requiring prior knowledge of Lipschitz-type constants or a linesearch technique. The algorithm's convergence is determined using only the most basic assumptions. Several basic experiments are included to demonstrate the numerical performance of the algorithms as well as compare them to others.

1. Introduction

Let Ξ represent a real Hilbert space and Δ represent a nonempty closed convex subset of Ξ. The research focuses on an iterative technique for addressing the equilibrium problem ((EP), to put it short). Let $\mathcal{F} : \Xi \times \Xi \to \mathbb{R}$ be a bifunction with $\mathcal{F}(y_1, y_1) = 0$, for each $y_1 \in \Delta$. A *equilibrium problem* for a given bifunction $\mathcal{F}$ on Δ is presented as follows: Find $u^* \in \Delta$ such that

$$\mathcal{F}(u^*, y_1) \geq 0, \ \forall y_1 \in \Delta. \tag{EP}$$

Let us represent a problem's solution set as $Sol(\mathcal{F}, \Delta)$, and we will assume in the following text that this solution set is not empty. This research looks at the numerical evaluation of the equilibrium problem given the following conditions. We suppose the following conditions have been met:

($\mathcal{F}1$) A bifunction $\mathcal{F}$ is said to be *pseudomonotone* [3, 5] i.e.,

$$\mathcal{F}(y_1, y_2) \geq 0 \Longrightarrow \mathcal{F}(y_2, y_1) \leq 0, \ \forall y_1, y_2 \in \Delta;$$

($\mathcal{F}2$) A bifunction $\mathcal{F}$ is said to be *Lipschitz-type continuous* [15] on Δ if there exist two constants $c_1, c_2 > 0$, such that

$$\mathcal{F}(y_1, y_3) \leq \mathcal{F}(y_1, y_2) + \mathcal{F}(y_2, y_3) + c_1\|y_1 - y_2\|^2 + c_2\|y_2 - y_3\|^2, \ \forall y_1, y_2, y_3 \in \Delta;$$

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($\mathcal{F}3$) For any sequence $\{y_k\} \subset \Delta$ satisfying $y_k \rightharpoonup y^*$, then the following inequality holds
$$\limsup_{k\to+\infty} \mathcal{F}(y_k, y_1) \le \mathcal{F}(y^*, y_1),\ \forall\, y_1 \in \Delta;$$

($\mathcal{F}4$) $\mathcal{F}(y_1, \cdot)$ is convex and subdifferentiable on Ξ for each fixed $y_1 \in \Xi$.

The equilibrium problem is of tremendous interest among researchers these days since it connects numerous mathematical problems, including fixed-point problems, the vector and scalar minimization problems, variational inequalities, the complementarity problems, the saddle point problems, the Nash equilibrium problems in non-cooperative games and the inverse optimization problems (for more details, see [4, 5, 9, 13, 16]). It also has various applications in economics [8], the dynamics of offer and demand [2] and continues to utilize the theoretical framework of non-cooperative games and Nash's equilibrium models [17, 18]. In the literature, the term "equilibrium problem" in its particular format was initially introduced in 1992 by Muu and Oettli [16] and further studied by Blum and Oettli [5].

The extragradient method developed by Flam et al. [10] and Quoc et al. [21] is one useful approach. The following is how this approach was constructed. Take an arbitrary initial points $u_0 \in \Xi$; using the current iteration u_k, take the next iteration as continues to follow:

$$\begin{cases} u_0 \in \Delta, \\ y_k = \arg\min_{y\in\Delta}\{\varkappa\mathcal{F}(u_k, y) + \frac{1}{2}\|u_k - y\|^2\}, \\ u_{k+1} = \arg\min_{y\in\Delta}\{\varkappa\mathcal{F}(y_k, y) + \frac{1}{2}\|u_k - y\|^2\}, \end{cases} \tag{1.1}$$

where $0 < \varkappa < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and c_1, c_2 are two Lipschitz-type constants. Due to Korpelevich's first contribution in [14] to handle saddle point problems, the iterative approaches in Flam et al. [10] and Quoc et al. [21] is also recognized as the two-step extragradient method. This method requires the solution of one optimization program on Δ as well as another optimization program on a feasible set, and its stepsize is fixed.

The primary goal is to present a new explicit-type approach that demonstrates that the gradient approach still yields a weak convergence sequence when solving equilibrium problems involving pseudomonotone bifunctions using a nonmonotone variable stepsize rule. We will present new extragradient-type methods for the problem (EP) in an infinite-dimensional real Hilbert space, inspired by the work of Censor et al. [7] and Hieu et al. [12].

Our main contributions to this work, in particular, are as follows:

- ⋄ We develop a subgradient extragradient approach with a non-monotone variable stepsize rule to solve equilibrium problems in a real Hilbert space and demonstrate that the resulting sequence is weakly convergent.
- ⋄ To solve equilibrium problems, we offer another extragradient approach that leverages a variable nonmonotone stepsize rule that is independent of the Lipschitz constants.

- ◇ Some conclusions are drawn in order to solve two types of equilibrium problems in real Hilbert space.
- ◇ We present numerical demonstrations of the proposed approaches for verifying theoretical findings and compare them to published results. Our numerical results indicate that the new methods are useful and outperform the existing ones.

The following is how the paper is organized: Preliminary results were reported in Section 2. All new methods and their convergence analysis are presented in Section 3. Finally, Section 4 provides numerical findings to explain the practical efficacy of the proposed methods.

2. Preliminaries

In this section, we will go over some elementary identities as well as key lemmas and definitions. A *metric projection* $P_{\Delta}(y_1)$ of $y_1 \in \Xi$ is defined by: $P_{\Delta}(y_1) = \arg\min\{\|y_1 - y_2\| : y_2 \in \Delta\}$. The key characteristics of projection mapping are described below.

Lemma 2.1 ([11])**.** *Let $P_{\Delta} : \Xi \to \Delta$ be a metric projection. Then we have the following characteristics:*

(i)

$$\|y_1 - P_{\Delta}(y_2)\|^2 + \|P_{\Delta}(y_2) - y_2\|^2 \leq \|y_1 - y_2\|^2,\ y_1 \in \Delta, y_2 \in \Xi;$$

(ii) $y_3 = P_{\Delta}(y_1)$ *if and only if*

$$\langle y_1 - y_3, y_2 - y_3\rangle \leq 0,\ \forall\, y_2 \in \Delta;$$

(iii)

$$\|y_1 - P_{\Delta}(y_1)\| \leq \|y_1 - y_2\|,\ y_2 \in \Delta, y_1 \in \Xi.$$

Lemma 2.2 ([11])**.** *For any $y_1, y_2 \in \Xi$ and $\ell \in \mathbb{R}$. Then the following conditions are met:*

(i)

$$\|\ell y_1 + (1-\ell)y_2\|^2 = \ell\|y_1\|^2 + (1-\ell)\|y_2\|^2 - \ell(1-\ell)\|y_1 - y_2\|^2.$$

(ii)

$$\|y_1 + y_2\|^2 \leq \|y_1\|^2 + 2\langle y_2, y_1 + y_2\rangle.$$

The *normal cone of* Δ at $y_1 \in \Delta$ is defined by

$$N_{\Delta}(y_1) = \{y_3 \in \Xi : \langle y_3, y_2 - y_1\rangle \leq 0, \forall\, y_2 \in \Delta\}.$$

Assume that $\mho : \Delta \to \mathbb{R}$ is a convex function and *subdifferential of* $\mho$ at $y_1 \in \Delta$ is defined by

$$\partial\mho(y_1) = \{y_3 \in \Xi : \mho(y_2) - \mho(y_1) \geq \langle y_3, y_2 - y_1\rangle, \forall\, y_2 \in \Delta\}.$$

Lemma 2.3 ([20]). *Let $\mho : \Delta \to \mathbb{R}$ be a subdifferentiable, convex and lower semi-continuous function on Δ. An element $u \in \Delta$ is a minimizer of a function $\mho$ if and only if*

$$0 \in \partial\mho(u) + N_\Delta(u),$$

where $\partial\mho(u)$ stands for the sub-differential of $\mho$ at $u \in \Delta$ and $N_\Delta(u)$ the normal cone of Δ at u.

Lemma 2.4 ([19]). *Let Δ be a nonempty subset of Ξ and $\{u_k\}$ be a sequence in Ξ satisfying the two conditions:*

(i) *for each $u \in \Delta$, $\lim_{k\to+\infty} \|u_k - u\|$ exists;*
(ii) *each sequentially weak cluster point of $\{u_k\}$ belongs to Δ.*

Then, sequence $\{u_k\}$ weakly converges to an element in Δ.

3. Main results

In this section, we introduce a numerical iterative approach for speeding the rate of convergence of an iterative sequence that contains two strong convex optimization problems with a non-monotone stepsize rule. We offer a method for solving equilibrium problems:

Algorithm 1

STEP 0: Choose $\varkappa_0 > 0$, $u_0 \in \Xi$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$. Choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$. First, we have to compute

STEP 1: Compute

$$y_k = \arg\min_{y\in\Delta}\{\varkappa_k \mathcal{F}(u_k, y) + \frac{1}{2}\|u_k - y\|^2\}.$$

STEP 2: Given the current iterates u_k, y_k. Firstly choose $\omega_k \in \partial_2\mathcal{F}(u_k, y_k)$ satisfying $u_k - \varkappa_k\omega_k - y_k \in N_\Delta(y_k)$ and generate a half-space

$$\Xi_k = \{z \in \Xi : \langle u_k - \varkappa_k\omega_k - y_k, z - y_k\rangle \le 0\}.$$

STEP 3: Compute

$$u_{k+1} = \arg\min_{y\in\Xi_k}\{\varkappa_k \mathcal{F}(y_k, y) + \frac{1}{2}\|u_k - y\|^2\}.$$

STEP 4: Compute

$$\varkappa_{k+1} = \begin{cases} \min\left\{\varkappa_k + p_k,\ \frac{(2-\sqrt{2}-\theta)\mu\|u_k-y_k\|^2 + (2-\sqrt{2}-\theta)\mu\|u_{k+1}-y_k\|^2}{2[\mathcal{F}(u_k,u_{k+1}) - \mathcal{F}(u_k,y_k) - \mathcal{F}(y_k,u_{k+1})]}\right\} \\ \quad \text{if} \quad \mathcal{F}(u_k, u_{k+1}) - \mathcal{F}(u_k, y_k) - \mathcal{F}(y_k, u_{k+1}) > 0, \\ \varkappa_k + p_k, \qquad \text{otherwise.} \end{cases} \tag{3.1}$$

STEP 5: If $u_k = y_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Lemma 3.1. *A sequence* $\{\varkappa_k\}$ *is converge to* $\varkappa$ *and* $\min\left\{\frac{\mu(2-\sqrt{2}-\theta)}{\max\{2c_1,2c_2\}},\varkappa_0\right\} \leq \varkappa \leq \varkappa_0 + P$*, where* $P = \sum_{k=1}^{+\infty}\varphi_k$.

Proof. Assume that $\mathcal{F}(u_k,u_{k+1}) - \mathcal{F}(u_k,y_k) - \mathcal{F}(y_k,u_{k+1}) > 0$ such that

$$\begin{aligned}
&\frac{\mu(2-\sqrt{2}-\theta)(\|u_k-y_k\|^2+\|u_{k+1}-y_k\|^2)}{2[\mathcal{F}(u_k,u_{k+1})-\mathcal{F}(u_k,y_k)-\mathcal{F}(y_k,u_{k+1})]}\\
&\geq \frac{\mu(2-\sqrt{2}-\theta)(\|u_k-y_k\|^2+\|u_{k+1}-y_k\|^2)}{2[c_1\|u_k-y_k\|^2+c_2\|u_{k+1}-y_k\|^2]}\\
&\geq \frac{\mu(2-\sqrt{2}-\theta)}{2\max\{c_1,c_2\}}. \qquad (3.2)
\end{aligned}$$

Applying mathematical induction on the concept of $\varkappa_{k+1}$, we have

$$\min\left\{\frac{\mu(2-\sqrt{2}-\theta)}{\max\{2c_1,2c_2\}},\varkappa_0\right\} \leq \varkappa_k \leq \varkappa_0 + P.$$

Suppose that $[\varkappa_{k+1}-\varkappa_k]^+ = \max\{0,\varkappa_{k+1}-\varkappa_k\}$ and $[\varkappa_{k+1}-\varkappa_k]^- = \max\{0,-(\varkappa_{k+1}-\varkappa_k)\}$. Due to the definition of $\{\varkappa_k\}$, we get

$$\sum_{k=1}^{+\infty}(\varkappa_{k+1}-\varkappa_k)^+ = \sum_{k=1}^{+\infty}\max\{0,\varkappa_{k+1}-\varkappa_k\} \leq P < +\infty. \qquad (3.3)$$

That is, the series $\sum_{k=1}^{+\infty}(\varkappa_{k+1}-\varkappa_k)^+$ is convergent. The convergence must now be proven of $\sum_{k=1}^{+\infty}(\varkappa_{k+1}-\varkappa_k)^-$. Let $\sum_{k=1}^{+\infty}(\varkappa_{k+1}-\varkappa_k)^- = +\infty$. Due to the fact that $\varkappa_{k+1}-\varkappa_k = (\varkappa_{k+1}-\varkappa_k)^+ - (\varkappa_{k+1}-\varkappa_k)^-$. We could get

$$\varkappa_{k+1}-\varkappa_0 = \sum_{k=0}^{k}(\varkappa_{k+1}-\varkappa_k) = \sum_{k=0}^{k}(\varkappa_{k+1}-\varkappa_k)^+ - \sum_{k=0}^{k}(\varkappa_{k+1}-\varkappa_k)^-. \qquad (3.4)$$

Letting $k\to+\infty$ in (3.4), we have $\varkappa_k\to-\infty$ as $k\to+\infty$. This is an absurdity. As a result of the series convergence $\sum_{k=0}^{k}(\varkappa_{k+1}-\varkappa_k)^+$ and $\sum_{k=0}^{k}(\varkappa_{k+1}-\varkappa_k)^-$ taking $k\to+\infty$ in (3.4), we obtain $\lim_{k\to+\infty}\varkappa_k = \varkappa$. This brings the proof to a conclusion. □

Lemma 3.2. *We can derive the following important inequality using Algorithm 1*

$$\varkappa_k\mathcal{F}(y_k,y) - \varkappa_k\mathcal{F}(y_k,u_{k+1}) \geq \langle u_k-u_{k+1}, y-u_{k+1}\rangle,\ \forall\, y\in\Xi_k.$$

Proof. Due to the use of Lemma 2.3, we have

$$0 \in \partial_2\Big\{\varkappa_k\mathcal{F}(y_k,\cdot) + \frac{1}{2}\|u_k-\cdot\|^2\Big\}(u_{k+1}) + N_{\Xi_k}(u_{k+1}).$$

Thus, $\upsilon\in\partial\mathcal{F}(y_k,u_{k+1})$ there exists a vector $\overline{\upsilon}\in N_{\Xi_k(}u_{k+1})$ such that

$$\varkappa_k\upsilon + u_{k+1} - u_k + \overline{\upsilon} = 0.$$

As a result, we have

$$\langle u_k-u_{k+1}, y-u_{k+1}\rangle = \varkappa_k\langle\upsilon, y-u_{k+1}\rangle + \langle\overline{\upsilon}, y-u_{k+1}\rangle,\ \forall\, y\in\Xi_k.$$

Since $\overline{\upsilon} \in N_{\Xi_k}(u_{k+1})$ we have $\langle \overline{\upsilon}, y - u_{k+1}\rangle \leq 0$ for all $y \in \Xi_k$. It implies that

$$\langle u_k - u_{k+1}, y - u_{k+1}\rangle \leq \varkappa_k \langle \upsilon, y - u_{k+1}\rangle, \ \forall\, y \in \Xi_k. \tag{3.5}$$

Since $\upsilon \in \partial \mathcal{F}(y_k, u_{k+1})$, we have

$$\mathcal{F}(y_k, y) - \mathcal{F}(y_k, u_{k+1}) \geq \langle \upsilon, y - u_{k+1}\rangle, \ \forall\, y \in \Xi. \tag{3.6}$$

We obtain by combining the formulas (3.5) and (3.6)

$$\varkappa_k \mathcal{F}(y_k, y) - \varkappa_k \mathcal{F}(y_k, u_{k+1}) \geq \langle u_k - u_{k+1}, y - u_{k+1}\rangle, \ \forall\, y \in \Xi_k.$$

□

Lemma 3.3. *From Algorithm 1 we also have the following useful inequality*

$$\varkappa_k \mathcal{F}(u_k, y) - \varkappa_k \mathcal{F}(u_k, y_k) \geq \langle u_k - y_k, y - y_k\rangle, \ \forall\, y \in \Delta.$$

Proof. The proof is analogous to the proof of Lemma 3.2. Substituting $y = u_{k+1}$, we have

$$\varkappa_k \big\{\mathcal{F}(u_k, u_{k+1}) - \mathcal{F}(u_k, y_k)\big\} \geq \langle u_k - y_k, u_{k+1} - y_k\rangle. \tag{3.7}$$

□

Theorem 3.4. *Let $\{u_k\}$ be a sequence constructed by Algorithm 1 and the items $(\mathcal{F}1)$–$(\mathcal{F}4)$ are held. Then, sequence $\{u_k\}$ weakly converges to u^*. Also, $\lim_{k\to+\infty} P_{Sol(\mathcal{F},\Delta)}(u_k) = u^*$.*

Proof. By substituting $y = u^*$ into Lemma 3.2, we have

$$\varkappa_k \mathcal{F}(y_k, u^*) - \varkappa_k \mathcal{F}(y_k, u_{k+1}) \geq \langle u_k - u_{k+1}, u^* - u_{k+1}\rangle. \tag{3.8}$$

By the use of condition $(\mathcal{F}2)$ we obtain

$$\langle u_k - u_{k+1}, u_{k+1} - u^*\rangle \geq \varkappa_k \mathcal{F}(y_k, u_{k+1}). \tag{3.9}$$

From expression (3.28) we obtain

$$\mathcal{F}(u_k, u_{k+1}) - \mathcal{F}(u_k, y_k) - \mathcal{F}(y_k, u_{k+1}) \leq \frac{(2-\sqrt{2}-\theta)\mu\big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big)}{2\varkappa_{k+1}},$$

which after multiplying both sides by $\varkappa_k > 0$ implies that

$$\begin{aligned}\varkappa_k \mathcal{F}(y_k, u_{k+1}) \geq{}& \varkappa_k \mathcal{F}(u_k, u_{k+1}) - \varkappa_k \mathcal{F}(u_k, y_k)\\ &- \frac{(2-\sqrt{2}-\theta)\varkappa_k\mu\big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big)}{2\varkappa_{k+1}}.\end{aligned} \tag{3.10}$$

Combining expressions (3.9) and (3.10) we obtain

$$\begin{aligned}\langle u_k - u_{k+1}, u_{k+1} - u^*\rangle \geq{}& \varkappa_k\{\mathcal{F}(u_k, u_{k+1}) - \mathcal{F}(u_k, y_k)\}\\ &- \frac{(2-\sqrt{2}-\theta)\varkappa_k\mu\big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big)}{2\varkappa_{k+1}}.\end{aligned} \tag{3.11}$$

By using expression (3.7), we have

$$\varkappa_k \big\{\mathcal{F}(u_k, u_{k+1}) - \mathcal{F}(u_k, y_k)\big\} \geq \langle u_k - y_k, u_{k+1} - y_k\rangle. \tag{3.12}$$

Combining expressions (3.11) and (3.12) we have

$$\langle u_k - u_{k+1}, u_{k+1} - u^* \rangle \geq \langle u_k - y_k, u_{k+1} - y_k \rangle - \frac{(2-\sqrt{2}-\theta)\varkappa_k \mu \big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big)}{2\varkappa_{k+1}}. \tag{3.13}$$

The following facts are available to us:

$$2\langle u_k - u_{k+1}, u_{k+1} - u^* \rangle = \|u_k - u^*\|^2 - \|u_{k+1} - u_k\|^2 - \|u_{k+1} - u^*\|^2,$$

$$2\langle y_k - u_k, y_k - u_{k+1} \rangle = \|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2 - \|u_k - u_{k+1}\|^2.$$

As a result, we have

$$\|u_{k+1} - u^*\|^2 \leq \|u_k - u^*\|^2 - \|u_k - y_k\|^2 - \|u_{k+1} - y_k\|^2 + \frac{(2-\sqrt{2}-\theta)\varkappa_k \mu \big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big)}{\varkappa_{k+1}}. \tag{3.14}$$

Due to $\varkappa_k \to \varkappa$, there exists a fixed natural number $k_1 \in \mathbb{N}$ such that

$$\lim_{k\to+\infty} \frac{\mu \varkappa_k}{\varkappa_{k+1}} \leq 1.$$

Thus, we have

$$\|u_{k+1} - u^*\|^2 \leq \|u_k - u^*\|^2 - \|u_k - y_k\|^2 - \|u_{k+1} - y_k\|^2 + (2-\sqrt{2}-\theta)\big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big). \tag{3.15}$$

Furthermore, it implies that

$$\|u_{k+1} - u^*\|^2 \leq \|u_k - u^*\|^2 - (\sqrt{2}-1)\|u_k - y_k\|^2 - (\sqrt{2}-1)\|u_{k+1} - y_k\|^2 - \theta\big(\|u_k - y_k\|^2 + \|u_{k+1} - y_k\|^2\big). \tag{3.16}$$

From expression (3.16), we obtain

$$\|u_{k+1} - u^*\|^2 \leq \|u_k - u^*\|^2, \ \forall\, k \geq k_1. \tag{3.17}$$

Therefore, we deduce that the sequence $\{u_k\}$ is bounded. Let $m \geq k_1$ and take eq. (3.16) for $k_1, k_1+1, \cdots, m$. Summing up them, we obtain

$$\begin{aligned} \|u_{m+1} - u^*\|^2 &\leq \|u_{k_1} - u^*\|^2 - \sum_{k=k_1}^{m} (\sqrt{2}-1)\|u_k - y_k\|^2 \\ &\quad - \sum_{k=k_1}^{m} (\sqrt{2}-1)\|u_{k+1} - y_k\|^2 \\ &\leq \|u_{k_1} - u^*\|^2. \end{aligned} \tag{3.18}$$

Letting $k \to +\infty$ in (3.18), we obtain

$$\sum_{k=1}^{+\infty} \|u_k - y_k\|^2 < +\infty \Longrightarrow \lim_{k\to+\infty} \|u_k - y_k\| = 0, \tag{3.19}$$

and

$$\sum_{k=1}^{+\infty} \|u_{k+1} - y_k\|^2 < +\infty \Longrightarrow \lim_{k\to+\infty} \|u_{k+1} - y_k\| = 0. \tag{3.20}$$

Due to the equations (3.19), (3.20) and using Cauchy inequality, we obtain

$$\lim_{k\to+\infty} \|u_{k+1} - u_k\| \Longrightarrow 0. \tag{3.21}$$

Next, consider that $\hat{u}$ is a weak limit point of $\{u_k\}$, i.e., a subsequence, represented by $\{u_{k_k}\}$ of $\{u_k\}$ converges weakly to $\hat{u}$. Then, $\{y_{k_k}\}$ is also weakly converges to $\hat{u}$ and $\hat{u} \in \Delta$. From (3.10), the definition of $\varkappa_{k+1}$ and inequality (3.13), we have

$$\begin{aligned}
\varkappa_{k_k}\mathcal{F}(y_{k_k}, y) &\geq \varkappa_{k_k}\mathcal{F}(y_{k_k}, u_{k_k+1}) + \langle u_{k_k} - u_{k_k+1}, y - u_{k_k+1}\rangle \\
&\geq \varkappa_{k_k}\mathcal{F}(u_{k_k}, u_{k_{k+1}}) - \varkappa_{k_k}\mathcal{F}(u_{k_k}, y_{k_k}) - \frac{\mu\varkappa_{k_k}}{2\varkappa_{k_k+1}}\|u_{k_k} - y_{k_k}\|^2 \\
&\quad - \frac{\mu\varkappa_{k_k}}{2\varkappa_{k_k+1}}\|y_{k_k} - u_{k_k+1}\|^2 + \langle u_{k_k} - u_{k_k+1}, y - u_{k_k+1}\rangle \\
&\geq \langle u_{k_k} - y_{k_k}, u_{k_k+1} - y_{k_k}\rangle - \frac{\mu\varkappa_{k_k}}{2\varkappa_{k_k+1}}\|u_{k_k} - y_{k_k}\|^2 \\
&\quad - \frac{\mu\varkappa_{k_k}}{2\varkappa_{k_k+1}}\|y_{k_k} - u_{k_k+1}\|^2 + \langle u_{k_k} - u_{k_k+1}, y - u_{k_k+1}\rangle,
\end{aligned} \tag{3.22}$$

where $y \in \Xi_k$. It pursue from (3.19), (3.20), (3.21) and due to boundedness of $\{u_k\}$ provides that the right-hand side of above relation converge to zero. Using $\varkappa_{k_k} > 0$, condition $(\mathcal{F}3)$ and $y_{k_k} \rightharpoonup \hat{u}$, we obtain

$$0 \leq \limsup_{k\to+\infty} \mathcal{F}(y_{k_k}, y) \leq \mathcal{F}(\hat{u}, y),\ \forall\, y \in \Delta.$$

Due to $\Delta \subset \Xi_k$ implies that $\hat{u} \in \Delta$ and $\mathcal{F}(\hat{u}, y) \geq 0$, for all $y \in \Delta$. It proves that $\hat{u} \in Sol(\mathcal{F}, \Delta)$. Therefore, Lemma 2.4, guarantee that $\{u_k\}$ and $\{y_k\}$ converges weakly to u^* as $k \to +\infty$.

The final step is to demonstrate that $\lim_{k\to+\infty} P_{Sol(\mathcal{F},\Delta)}(u_k) = u^*$. Let $\Im_k := P_{Sol(\mathcal{F},\Delta)}(u_k)$ be defined for each $k \in \urcorner$. Consider the following example to demonstrate the boundedness of the $\Im_k$ such that

$$\|\Im_k\| \leq \|\Im_k - u_k\| + \|u_k\| \leq \|u^* - u_k\| + \|u_k\|. \tag{3.23}$$

As a result, $\{\Im_k\}$ is bounded sequence. We write (3.16) for $k \geq 1$. We have

$$\|u_{k+1} - \Im_{k+1}\|^2 \leq \|u_{k+1} - \Im_k\|^2 \leq \|u_k - \Im_k\|^2,\ \forall\, k \geq k_1. \tag{3.24}$$

The formula above suggests that the sequence $\|u_k - \Im_k\|$ is convergent. We can write $m > k \geq k_1$ using (3.16) for

$$\|\Im_k - u_m\|^2 \leq \|\Im_k - u_{m-1}\|^2 \leq \cdots \leq \|\Im_k - u_k\|^2. \tag{3.25}$$

Allow $\Im_m, \Im_k \in Sol(\mathcal{F}, \Delta)$, and Lemma 2.1 (i) with (3.25) that $m > k \geq k_1$, we have

$$\|\Im_k - \Im_m\|^2 \leq \|\Im_k - u_m\|^2 - \|\Im_m - u_m\|^2 \leq \|\Im_k - u_k\|^2 - \|\Im_m - u_m\|^2. \tag{3.26}$$

The occurrence of $\lim_{k\to+\infty}\|\Im_k - u_k\|$ implies that $\lim_{m,k\to+\infty}\|\Im_k - \Im_m\| = 0$, for all $m > k$. The solution set $Sol(\mathcal{F},\Delta)$ is closed set and $\{\Im_k\}$ is a Cauchy sequence. Thus, $\{\Im_k\} \to \hat{\Pi} \in Sol(\mathcal{F},\Delta)$. From Lemma 2.1 (ii) and $u^*, \hat{\Pi} \in Sol(\mathcal{F},\Delta)$,

$$\langle u_k - \Im_k, u^* - \Im_k\rangle \le 0. \tag{3.27}$$

Since $\Im_k \to \hat{\Pi}$ and $u_k \rightharpoonup u^*$, imply $\langle u^* - \hat{\Pi}, u^* - \hat{\Pi}\rangle \le 0$, that gives $u^* = \hat{\Pi} = \lim_{k\to+\infty} P_{Sol(\mathcal{F},\Delta)}(u_k)$. Next, $\|u_k - y_k\| \to 0$, implies $\lim_{k\to+\infty} P_{Sol(\mathcal{F},\Delta)}(y_k) = u^*$. The proof of the theorem is now complete. □

We now provide an iterative method that consists of a non-monotone variable stepsize rule and two strongly convex minimization problems. The description of the second main result is as follows.

Algorithm 2

STEP 0: Choose $\varkappa_0 > 0$, $u_0 \in \Xi$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$. Choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$. First, we have to compute

STEP 1: Compute

$$y_k = \arg\min_{y\in\Delta}\{\varkappa_k \mathcal{F}(u_k, y) + \frac{1}{2}\|u_k - y\|^2\}.$$

STEP 2: Compute

$$u_{k+1} = \arg\min_{y\in\Delta}\{\varkappa_k \mathcal{F}(y_k, y) + \frac{1}{2}\|u_k - y\|^2\}.$$

STEP 3: Compute

$$\varkappa_{k+1} = \begin{cases} \min\left\{\varkappa_k + p_k, \ \frac{(2-\sqrt{2}-\theta)\mu\|u_k-y_k\|^2 + (2-\sqrt{2}-\theta)\mu\|u_{k+1}-y_k\|^2}{2[\mathcal{F}(u_k,u_{k+1}) - \mathcal{F}(u_k,y_k) - \mathcal{F}(y_k,u_{k+1})]}\right\} \\ \quad \text{if} \quad \mathcal{F}(u_k,u_{k+1}) - \mathcal{F}(u_k,y_k) - \mathcal{F}(y_k,u_{k+1}) > 0, \\ \\ \varkappa_k + p_k, \qquad\qquad \text{otherwise.} \end{cases} \tag{3.28}$$

STEP 4: If $u_k = y_k$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

In this part, we use the results from our primary results to solve variational inequalities and fixed point problems. The expressions (3.29) and (3.31) are used to get the following results. More precisely, we consider two applications for the problem (EP). (i) A *variational inequality problem* for an operator $\mathcal{M} : \Delta \to \Xi$ is stated as follows: Find $u^* \in \Delta$ such that

$$\big\langle \mathcal{M}(u^*), y_1 - u^* \big\rangle \ge 0, \ \forall\, y_1 \in \Delta. \tag{VIP}$$

Let us define a bifunction $\mathcal{F}$ as

$$\mathcal{F}(y_1, y_2) := \big\langle \mathcal{M}(y_1), y_2 - y_1 \big\rangle, \ \forall\, y_1, y_2 \in \Delta. \tag{3.29}$$

Then, the equilibrium problem convert into the problem of variational inequalities defined in (VIP) and Lipschitz constants of the mapping $\mathcal{M}$ is $L = 2c_1 = 2c_2$. (ii) Let a mapping $\mathcal{N} : \Delta \to \Delta$ is said to κ-strict pseudocontraction [6] if there exists a constant $\kappa \in (0, 1)$ such that

$$\|\mathcal{N}y_1 - \mathcal{N}y_2\|^2 \leq \|y_1 - y_2\|^2 + \kappa\|(y_1 - \mathcal{N}y_1) - (y_2 - \mathcal{N}y_2)\|^2, \ \forall\, y_1, y_2 \in \Delta. \tag{3.30}$$

A fixed point problem (FPP) for $\mathcal{N} : \Delta \to \Delta$ is to find $u^* \in \Delta$ such that $\mathcal{N}(u^*) = u^*$. Let us define a bifunction $\mathcal{F}$ as

$$\mathcal{F}(y_1, y_2) = \langle y_1 - \mathcal{N}y_1, y_2 - y_1\rangle, \ \forall\, y_1, y_2 \in \Delta. \tag{3.31}$$

It can be easily seen in [22] that the expression (3.31) satisfies the conditions $\mathcal{F}1$-$\mathcal{F}5$ as well as the value of Lipschitz constants are $c_1 = c_2 = \frac{3-2\kappa}{2-2\kappa}$.

Corollary 3.5. *Assume that $\mathcal{M} : \Delta \to \Xi$ is a pseudomonotone, weakly continuous and L-Lipschitz continuous operator and the solution set $Sol(\mathcal{M}, \Delta) \neq \emptyset$. Choose $\varkappa_0 > 0$, $u_0 \in \Xi$, $\mu \in (0, 1)$, $\theta \in (0, 2 - \sqrt{2})$. Choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$. First, we have to compute*

$$y_k = P_\Delta(u_k - \varkappa_k \mathcal{M}(u_k)).$$

Given u_k, y_k, and construct a half-space

$$\Xi_k = \{z \in \Xi : \langle u_k - \varkappa_k\mathcal{M}(u_k) - y_k, z - y_k\rangle \leq 0\} \text{ for each } k \geq 0.$$

Compute

$$u_{k+1} = P_{\Xi_k}(u_k - \varkappa_k\mathcal{M}(y_k)).$$

The stepsize must be modified as follows:

$$\varkappa_{k+1} = \begin{cases} \min\left\{\varkappa_k, \ \frac{(2-\sqrt{2}-\theta)\mu\|u_k-y_k\|^2 + (2-\sqrt{2}-\theta)\mu\|u_{k+1}-y_k\|^2}{2\left\langle \mathcal{M}(u_k)-\mathcal{M}(y_k), \ u_{k+1}-y_k\right\rangle}\right\} \\ \qquad\qquad if \qquad \left\langle \mathcal{M}(u_k) - \mathcal{M}(y_k), \ u_{k+1} - y_k\right\rangle > 0, \\ \varkappa_k, \qquad\qquad\qquad\qquad otherwise. \end{cases}$$

Then, the sequences $\{u_k\}$ converge weakly to $u^* \in Sol(\mathcal{M}, \Delta)$.

Corollary 3.6. *Assume that $\mathcal{M} : \Delta \to \Xi$ is a pseudomonotone, weakly continuous and L-Lipschitz continuous operator and the solution set $Sol(\mathcal{M}, \Delta) \neq \emptyset$. Choose $\varkappa_0 > 0$, $u_0 \in \Xi$, $\mu \in (0, 1)$, $\theta \in (0, 2 - \sqrt{2})$. Choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$. First, we have to compute*

$$y_k = P_\Delta(u_k - \varkappa_k \mathcal{M}(u_k)).$$

Compute

$$u_{k+1} = P_\Delta(u_k - \varkappa_k\mathcal{M}(y_k)).$$

The stepsize must be modified as follows:

$$\varkappa_{k+1} = \begin{cases} \min\left\{\varkappa_k, \ \frac{(2-\sqrt{2}-\theta)\mu\|u_k-y_k\|^2 + (2-\sqrt{2}-\theta)\mu\|u_{k+1}-y_k\|^2}{2\left\langle \mathcal{M}(u_k)-\mathcal{M}(y_k), \ u_{k+1}-y_k\right\rangle}\right\} \\ \qquad\qquad if \qquad \left\langle \mathcal{M}(u_k) - \mathcal{M}(y_k), \ u_{k+1} - y_k\right\rangle > 0, \\ \varkappa_k, \qquad\qquad\qquad\qquad otherwise. \end{cases}$$

Then, the sequences $\{u_k\}$ *converge weakly to* $u^* \in Sol(\mathcal{M}, \Delta)$.

Corollary 3.7. *Assume that* $\mathcal{N} : \Delta \to \Xi$ *is a pseudomonotone, weakly continuous and L-Lipschitz continuous operator and the solution set* $Sol(\mathcal{N}, \Delta) \neq \emptyset$. *Choose* $\varkappa_0 > 0$, $u_0 \in \Xi$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$. *Choose a non-negative real sequence* $\{p_k\}$ *such that* $\sum_{k=1}^{+\infty} p_k < +\infty$. *First, we have to compute*

$$y_k = P_\Delta\big[u_k - \varkappa_k(u_k - \mathcal{N}(u_k))\big].$$

Given u_k, y_k, *and construct a half-space*

$$\Xi_k = \{z \in \mathcal{E} : \langle (1-\varkappa_k)u_k + \varkappa_k\mathcal{N}(u_k) - y_k, z - y_k\rangle \leq 0\}.$$

Compute

$$u_{k+1} = P_{\Xi_k}\big[u_k - \varkappa_k(y_k - \mathcal{N}(y_k))\big].$$

The stepsize rule for the following iteration is assessed as follows:

$$\varkappa_{k+1} = \begin{cases} \min\left\{\varkappa_k,\ \frac{(2-\sqrt{2}-\theta)\mu\|u_k-y_k\|^2+(2-\sqrt{2}-\theta)\mu\|u_{k+1}-y_k\|^2}{2\big\langle (u_k-y_k)-[\mathcal{N}(u_k)-\mathcal{N}(y_k)],\, u_{k+1}-y_k\big\rangle}\right\} \\ \quad \textit{if} \quad \big\langle (u_k-y_k)-[\mathcal{N}(u_k)-\mathcal{N}(y_k)],\, u_{k+1}-y_k\big\rangle > 0, \\ \varkappa_k \qquad \textit{otherwise.} \end{cases}$$

Then, the sequence $\{u_k\}$ *converges weakly to* $u^* \in Sol(\mathcal{N}, \Delta)$.

Corollary 3.8. *Assume that* $\mathcal{N} : \Delta \to \Xi$ *is a pseudomonotone, weakly continuous and L-Lipschitz continuous operator and the solution set* $Sol(\mathcal{N}, \Delta) \neq \emptyset$. *Choose* $\varkappa_0 > 0$, $u_0 \in \Xi$, $\mu \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$. *Choose a non-negative real sequence* $\{p_k\}$ *such that* $\sum_{k=1}^{+\infty} p_k < +\infty$. *First, we have to compute*

$$y_k = P_\Delta\big[u_k - \varkappa_k(u_k - \mathcal{N}(u_k))\big].$$

Compute

$$u_{k+1} = P_{\Xi_k}\big[u_k - \varkappa_k(y_k - \mathcal{N}(y_k))\big].$$

The stepsize rule for the following iteration is assessed as follows:

$$\varkappa_{k+1} = \begin{cases} \min\left\{\varkappa_k,\ \frac{(2-\sqrt{2}-\theta)\mu\|u_k-y_k\|^2+(2-\sqrt{2}-\theta)\mu\|u_{k+1}-y_k\|^2}{2\big\langle (u_k-y_k)-[\mathcal{N}(u_k)-\mathcal{N}(y_k)],\, u_{k+1}-y_k\big\rangle}\right\} \\ \quad \textit{if} \quad \big\langle (u_k-y_k)-[\mathcal{N}(u_k)-\mathcal{N}(y_k)],\, u_{k+1}-y_k\big\rangle > 0, \\ \varkappa_k \qquad \textit{otherwise.} \end{cases}$$

Then, the sequence $\{u_k\}$ *converges weakly to* $u^* \in Sol(\mathcal{N}, \Delta)$.

4. Numerical illustrations

This section details a number of numerical experiments that were carried out to illustrate the efficacy of the suggested methodologies. Some of these numerical experiments give a comprehensive grasp of how to choose appropriate control settings. Some of them show how the suggested strategies outperform current ones in the literature.

Example 4.1. The first test problem here is drawn from the Nash-Cournot Oligopolistic Equilibrium model in [21]. In this example, the bifunction $\mathcal{F}$ can be defined as follows:

$$\mathcal{F}(u, y) = \langle Pu + Qy + c, y - u \rangle$$

where $c \in \mathbb{R}^M$ and P, Q matrices of order M. The matrix P is symmetric positive semi-definite and the matrix $Q - P$ is symmetric negative semi-definite with Lipschitz-like criteria $c_1 = c_2 = \frac{1}{2}\|P - Q\|$ (see [21] for more details).

Experiment 1: In first experiment, we look at Example 4.1 to examine how Algorithm 1 performs numerically when alternative control sequence μ options are used. This experiment assisted us in determining the best potential control parameter μ. The starting points for these numerical studies are $u_0 = (1, 1, \cdots, 1)$, $M = 5$ and error term $D_k = \|u_k - y_k\|$. Two matrices P, Q and vector c are defined by

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

The constraint set $\Delta \subset \mathbb{R}^M$ is considered as $\Delta := \{u \in \mathbb{R}^M : -2 \leq u_i \leq 5\}$. Figures 1-2 demonstrate a variety of outcomes for first 50 iterations. The following information about control settings should be considered: (i) Algorithm 1 (shortly, MEgA): $\varkappa_0 = \frac{1}{2c}, \theta = 0.050, p_k = \frac{100}{(k+1)^2}$.

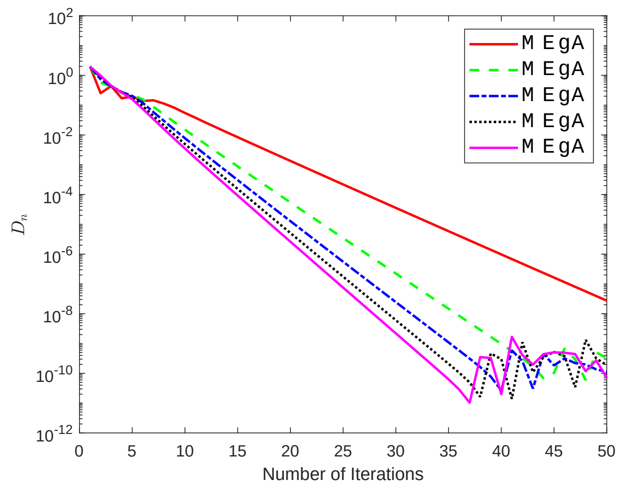

FIG. 1. Computational behavior of Algorithm 1 using different values of $\mu = 0.182, 0.393, 0.593, 0.754, 0.988$, respectively.

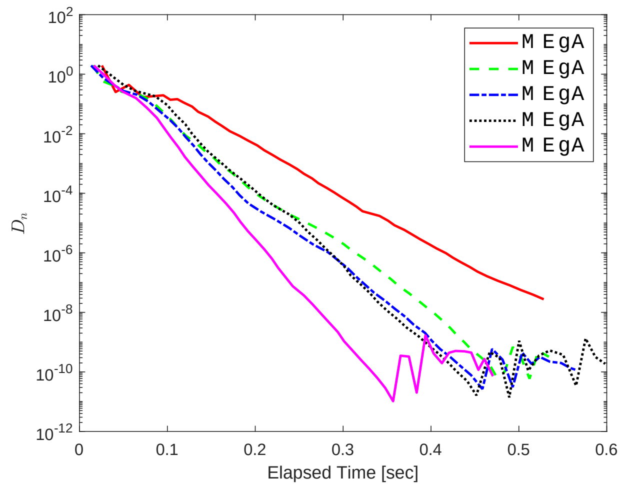

FIG. 2. Computational behavior of Algorithm 1 using different values of $\mu = 0.182, 0.393, 0.593, 0.754, 0.988$, respectively.

In second experiment, we look at Example 4.1 to examine how Algorithm 1 performs numerically when alternative control sequence θ options are used. This experiment assisted us in determining the best potential control parameter θ. The starting points for these numerical studies are $u_0 = (1, 1, \cdots, 1)$, $M = 5$ and error term $D_k = \|u_k - y_k\|$. Figures 3-4 shown a number of results by using a fixed number of iterations. Information concerning the control parameters shall be considered as follows: (i) Algorithm 1 (shortly, MEGA): $\varkappa_0 = \frac{1}{2c}, \mu = 0.55, p_k = \frac{100}{(k+1)^2}$.

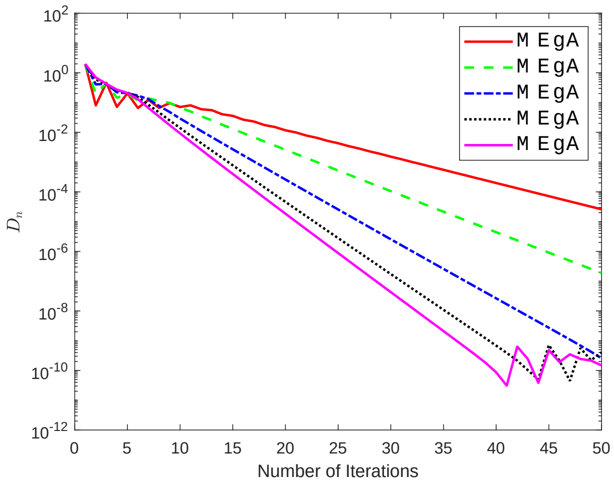

FIG. 3. Computational behavior of Algorithm 1 using different values of $\theta = 0.54, 0.46, 0.33, 0.18, 0.05$, respectively.

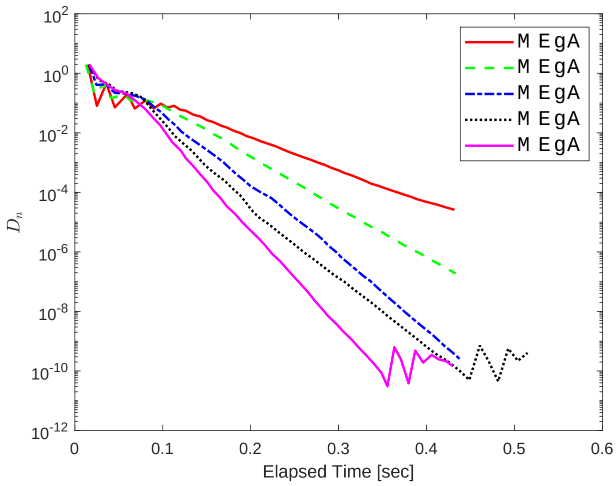

FIG. 4. Computational behavior of Algorithm 1 using different values of $\theta = 0.54, 0.46, 0.33, 0.18, 0.05$, respectively.

In the third experiment, we consider Example 4.1 to see the numerical comparison of Algorithm 1 with Algorithm 3.1 in [23]. For these numerical studies, starting points are $u_0 = (1, 1, \cdots, 1)$ and error term $D_k = \|u_k - y_k\|$. Figures 5-6 shown a number of results for first 50 iterations. Information regarding the control parameters shall be considered as follows: (i) Algorithm 1 (shortly, MEGA): $\varkappa_0 = 0.275, \mu = 0.55, \theta = 0.05, p_k = \frac{100}{(k+1)^2}$; (ii) Algorithm 1 in [12] (shortly, EGA): $\varkappa_0 = 0.275, \mu = 0.55$; (iii) Algorithm 2a in [21] (shortly, EEGA): $\alpha = 0.5; \theta = 0.5; \rho = 1$; (iv) Algorithm 1 in [1] (shortly, LEGA): $\varkappa_n = \frac{1}{n}, \alpha = 0.5; \theta = 0.5; \rho = 1$.

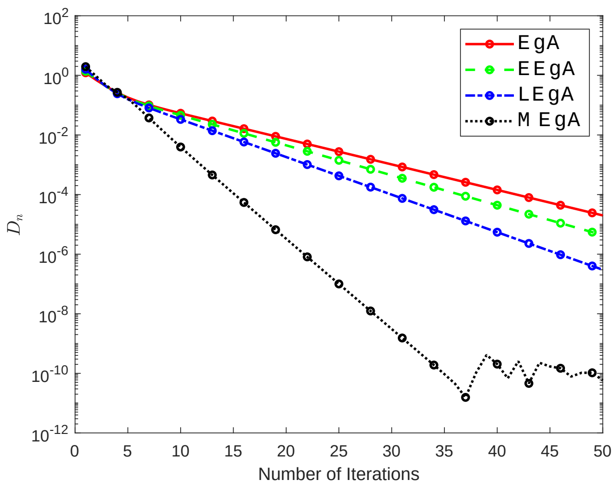

FIG. 5. Computational behavior of Algorithm 1 in comparison with other existing algorithms.

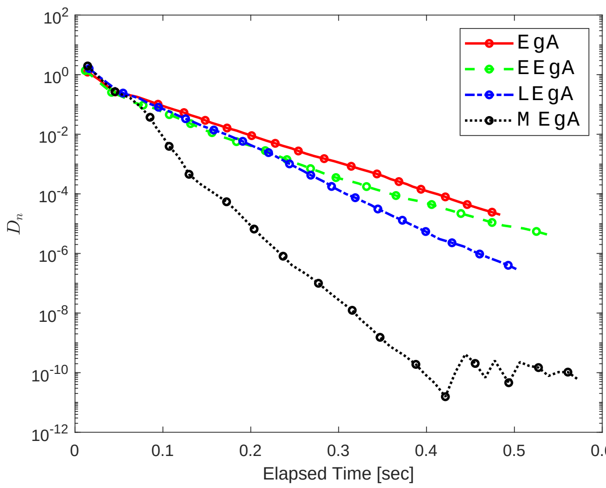

FIG. 6. Computational behavior of Algorithm 1 in comparison with other existing algorithms.

Example 4.2. Assume that an operator $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^2$ is specified as follows:

$$\mathcal{G}(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}$$

where

$$\Delta = \{u \in \mathbb{R}^2 : (u_1 - 2)^2 + (u_2 - 2)^2 \leq 1\}.$$

It is indeed clear that $\mathcal{G}$ is Lipschitz continuous with $L = 5$ and pseudomonotone. By defining the bifunction as $\mathcal{F}(u, y) = \langle \mathcal{G}(u), y - u\rangle$ and $c_1 = c_2 = \frac{5}{2}$. Figures

7-10 and Table 1 shown a number of results. Information regarding the control parameters shall be considered as follows: (i) Algorithm 1 (shortly, MEgA): $\varkappa_0 = 0.275, \mu = 0.55, \theta = 0.05, p_k = \frac{100}{(k+1)^2}$; (ii) Algorithm 1 in [12] (shortly, EgA): $\varkappa_0 = 0.275, \mu = 0.55$; (iii) Algorithm 2a in [21] (shortly, EEgA): $\alpha = 0.5; \theta = 0.5; \rho = 1$; (iv) Algorithm 1 in [1] (shortly, LEgA): $\varkappa_n = \frac{1}{n}, \alpha = 0.5; \theta = 0.5; \rho = 1$.

Table 1. Numerical results values for Figures 7–10.

	Number of Iterations			
u_0	Algorithm 1	Algorithm 3.1 in [23]	Algorithm 1	Algorithm 1
$(1.5; 1.7)^T$	25	20	13	8
$(2.0; 3.0)^T$	26	21	13	8
$(1.0; 2.0)^T$	26	22	14	8
$(2.7; 2.6)^T$	18	14	10	7

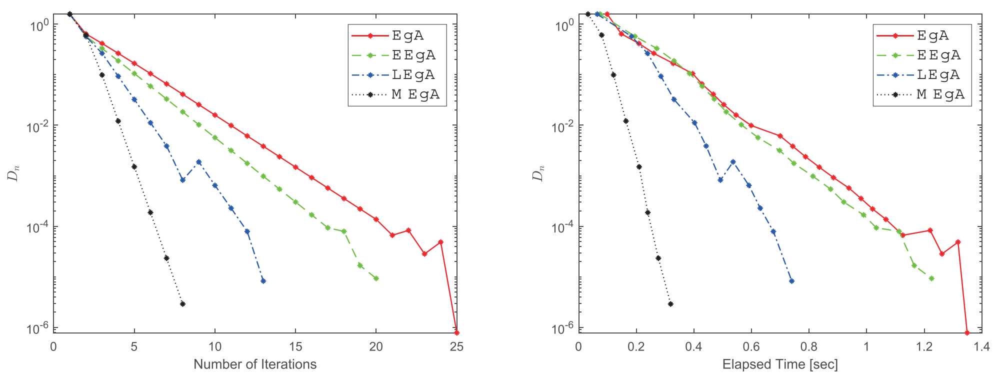

Fig. 7. Computational comparison of Algorithm 1 with $u_0 = (1.5; 1.7)^T$.

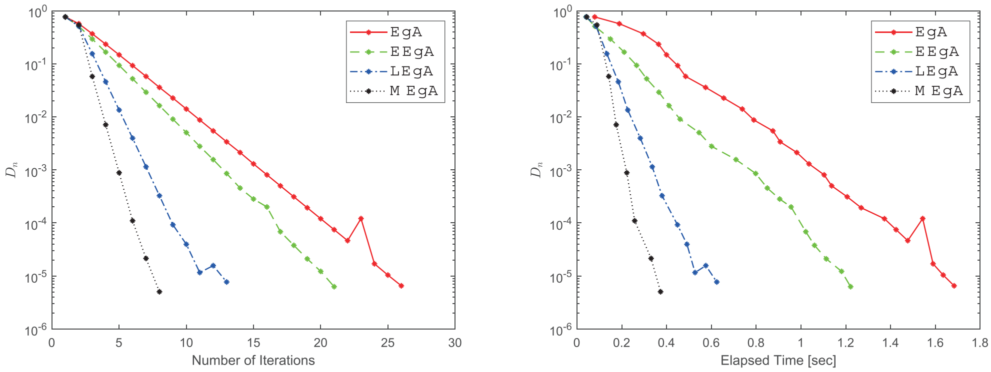

FIG. 8. Computational comparison of Algorithm 1 with $u_0 = (2.0; 3.0)^T$.

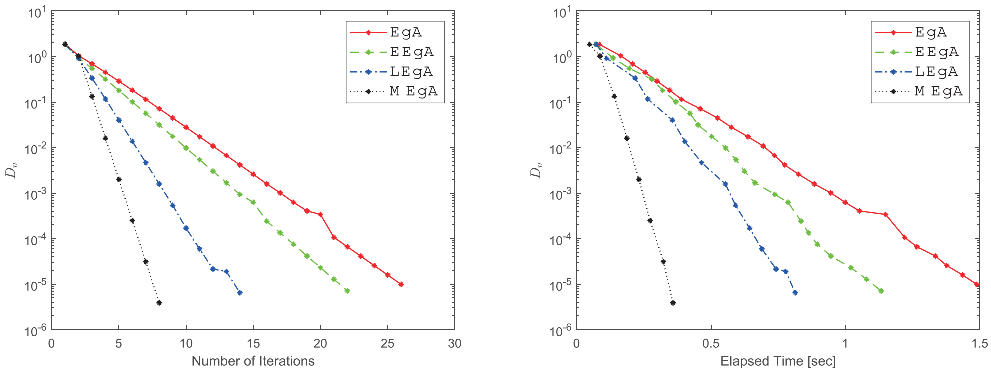

FIG. 9. Computational comparison of Algorithm 1 with $u_0 = (1.0; 2.0)^T$.

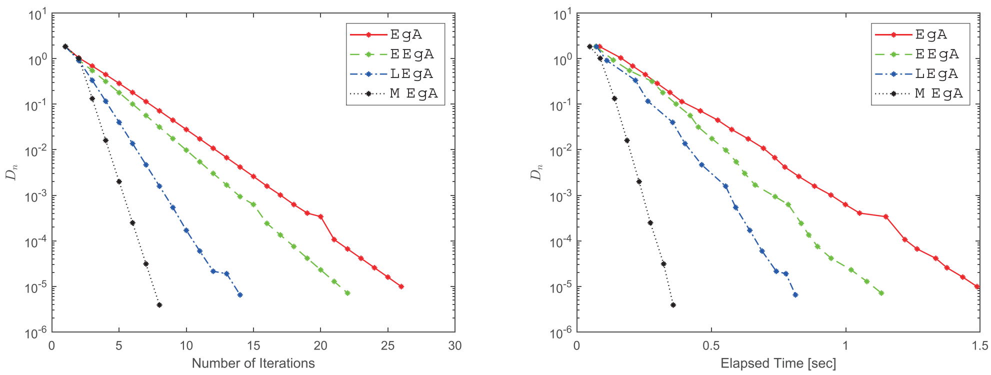

FIG. 10. Computational comparison of Algorithm 1 with $u_0 = (2.7; 2.6)^T$.

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