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EXISTENCE AND APPROXIMATE CONTROLLABILITY OF A CLASS OF IMPULSIVE NEUTRAL DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract. An impulsive neutral evolution differential inclusions with infinite delay is considered in a separable Banach space. Existence of mild solutions is studied using a fixed point principle for condensing maps. Then, a homotopic approach is employed to prove that the solution set is an R_δ-set and approximate controllability of the system is established by a fixed point principle for compositions of R_δ-maps due to L. Gorniewicz and M. Lassonde [24]. As an application, a Gurtin-Pipkin type system is solved in the end with the aid of our abstract results.

1. Introduction

Impulsive differential equations and inclusions have proved to be valuable tools in the modelling of many phenomena studied in various fields of science and engineering, such as control, viscoelasticity, chemical technology, population dynamics, biotechnology and economics and for this reason they have been the object of intensive investigations in recent years. For considerable developments in this topic, one may refer the monographs of Bainov and Simeonov [3], Benchohra et al. [11], Haddad et al. [27], Lakshmikantham et al. [36], and Samoilenko and Perestyuk [52]. The extension to neutral functional differential equations with impulsive effects has been done by Benchohra et. al. (see [7–10]) using a nonlinear alternative of Leray–Schauder type and Schaefer ' s theorem. A delay differential equation is a special type of functional differential equation with their evolution involves past value of the state variable. To study the abstract functional differential equations/inclusions with infinite delay, people usually employ an axiomatic definition of the phase space introduced first by Hale and Kato [28], and defined later by Hino, Murakami and Naito [32], and this has been the subject of many papers; see for example [29, 30], and many others. Partial neutral differential equation with unbounded delay lies in problems from various fields such as macroeconomic model including dynamics of income or value of capital stock (see the book of Chukwu [16]) and the theory of heat flow in materials with memory (see [26, 41, 42, 46]). With the above-mentioned works, it seems natually to extend all these known results to impulsive neutral

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functional differential equations with infinite delay, which are the main concerns of several papers [15, 17, 23, 31, 37, 49, 60].

Controllability is one of the most important properties of dynamic system. Nowadays, controllability theory for linear and nonlinear systems has already been well established, see the monographs [54]) for results in finite dimensional systems and [18] for infinite dimensional systems, respectively. In the study of controllability of nonlinear system, there are three frequently used approaches, namely, methods based on the stability theory of Lyapnov, method based on geometric theory and fixed-point method. The investigation of controllability problems for nonlinear systems by the methods of fixed-point theory go backs to 1960s. By using a fixed point theorem due to Bohnenblust-Karlin, Tarnove [55] first obtain in 1967 a sufficient condition for the controllability of a nonlinear system in a finite dimenaional space. Since then, controllability of abstract semilinear equations in finite and infinite dimensional spaces have been studied by many authors, and we cite [20, 44, 45, 56, 58, 61, 62, 64] for only a few of the huge amount of works in this field.

In recent years the corresponding parts of multivalued analysis are applied to obtain various controllability results for semilinear differential inclusions in infinite-dimensional Banach spaces (see [4–6, 12, 13, 39] and many others). To contribute the literaure of this topic, we are primarily concerened in the present article, with the approximate controllability for the impulsive neutral evolution differential inclusions with infinite delay in a Banach space X. More precisely, we consider the following class of evolution system:

$$
\begin{aligned}
\frac{d}{dt}[x(t)+g(t,x_t)] &\in A(t)x(t)+F(t,x_t)+Bu(t), && t\in[0,T], t\neq t_i\\
\Delta x(t_i) &= I_i(x_{t_i}), && i=1,2,\ldots,m,\\
x_0 &= \varphi\in\mathcal{B},
\end{aligned}
\tag{1.1}
$$

where $T>0$, $\{A(t)\}_{t\in[0,T]}$ is a family of linear operators in X generating an evolution operator; $0<t_1<\cdots<t_i<\cdots<t_m<T$ are pre-fixed numbers; the history $x_t:(-\infty,0]\to X, x_t(s)=x(t+s)$, belongs to some abstract phase space $\mathcal{B}$ which will be defined axiomatically later; $g:[0,T]\times\mathcal{B}\to X$ is some suitable function; F is a multimap from $[0,T]\times\mathcal{B}$ to the collection of all nonempty, compact and convex subset of X; for each $t\in[0,T]$; B is a bounded linear operator from a Banach space U into X and the control u takes value in U such that $B(.)u(.)\in L^p([0,T],X)$; $I_i:\mathcal{B}\to X, i=1,2,\ldots,m$ are suitable mappings satisfying some conditions which will be specified later and the symbol $\Delta x(t)$ represents the jump of the function x at t, which is defined by $\Delta x(t)=x(t^+)-x(t^-)$, where $x(t_i^+)$ and $x(t_i^-)$ represent the right and left limits of $x(t)$ at $t=t_i$. Throughout this work, 2^X denotes the family of nonempty subsets of X.

As mensioned above, the problem of controllability for various kinds of impulsive differential systems has been extensively studied by many authors in recent years using different approaches. There are several papers investigate the exact controllability for nonlinear systems using a method proposed in [51]. For example, the following autonomous version of system (1.1) is studied in [39]:

$$\begin{aligned} \frac{d}{dt}[x(t)+g(t,x_t)] &\in Ax(t)+F(t,x_t)+Bu(t), && t\in[0,T], t\neq t_i \\ \Delta x(t_i) &= I_i(x_{t_i}), && i=1,2,\ldots,m, \\ x_0 &= \varphi\in\mathcal{B}, \end{aligned}$$

assuming that the linear controllability operator

$$\mathcal{W}u=\int_0^T S(T-s)Bu(s)ds$$

is pseudo-invertible (i.e., $\mathcal{W}$ is surjective), where $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators generated by A. However, as it was pointed out in [57, 59], this assumption on $\mathcal{W}$ can not be satisfied in the cases when X is an infinite-dimensional space and/or B is a compact operator. Therefore, the concept of exact controllability is too strong. Regarding the lack of exact controllability in this situation, many authors consider a weaker concept of controllability, namely approximate controllability, see for instances [34, 45, 53, 61].

A dynamic system is said to be approximately controllable if it has a dense reachability set (see section 4 for the definition of reachability set). In this work, we use the technique of fixed point principle for condensing multivalued maps to develop an unified approach to the cases when the multivalued nonlinearity satisfies the Carathéodory condition. We first examine the mild solution of problem (1.1) and propose the solution multimap according to the formulation of mild solution. Then the concept of a measure of non-compactness is empolyed to show the existence of the fixed-ponit of the solution multimap. Thanks to theories from multivalued analysis, we are able to investigate the topological structure of the selution set, which allows us to show, by using similar arguments as in [53], that control system (1.1) and its corresponding linear problem have the same reachability set.

This article is organized in the following way. In Section 2, we prove the existence result for (1.1) under suitable assumptions on g and F. It is assumed that the range of g is contained in the common domain of the family $\{A(t)\}_{t\in[0,T]}$ and the multivalued nonlinearity F satisfies the Carathéodory condition and a regularity condition expressed in terms of a measure of non-compactness. These allow us to employ the fixed point principle for condensing maps to obtain the existence result. We investigate some properties related to the topological structure of the solution set in Section 3. In particular, we prove that the solution set is an R_δ-set by showing that the multivalued nonlinearity F is σ-Ca-selectionable and mLL-selectionable. This result is usd in section 4 to prove the invariance of reachability set for our problem under nonlinear perturbations, which in turn verifies that the controlled problem (1.1) is approximately controllable provided the corresponding linear problem is. In the last section, a Gurtin-Pipkin type system is provided as an illustrating example for our abstract theory.

2. Existence Result

In this paper, $T > 0$ is a fixed number and $(X, \|\cdot\|)$ is a Banach space. We denote by $L(X)$ the Banach space of bounded linear operators on X equipped with its natural topology. The linear part $A(t), t \in [0,T]$ of equation (1.1) are operators in X defined in a common domain $\mathscr{D}$ which is independent of t and dense in X. A family of linear operators $\{U(t,s)\}_{0\le s\le t\le T} \subset L(X)$ is called an evolution family of operators generated by $\{A(t) : t \in [0,T]\}$ if the following conditions hold (see [50]):

(i) $U(s,s) = I$,
(ii) $U(t,r)U(r,s) = U(t,s)$ for $0 \le s \le r \le t \le T$,
(iii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T$,
(iv) The function $t \to U(t,s)$ is differentiable in $(s,t]$ with

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s), \quad 0 \le s < t \le T.$$

Let $\{U(t,s)\}_{0\le s\le t\le T} \subset L(X)$ be the evolution family generated by $\{A(t) : t \in [0,T]\}$. We assume that the system

$$\begin{cases} u'(t) = A(t)u(t), & 0 \le s \le t \le T, \\ u(s) = x \in X, \end{cases}$$

has an associated evolution family of operators $\{U(t,s); 0 \le s \le t \le T\}$, which is uniformly bounded and put

$$M_0 = \sup_{(t,s)\in\Delta} \|U(t,s)\|_{L(X)},$$

where $\Delta := \{(t,s) : 0 \le s < t \le T\}$.

Let

$$\begin{aligned} \mathcal{PC}([0,T],X) = \{x : [0;T] \to X : \ & x(t) \text{ be continuous at } t \ne t_i, \\ & \text{left continuous at } t = t_i, \\ & \text{and the right limit } x(t_i^+) \text{ exists for } i = 1,2,\dots,m\}. \end{aligned}$$

Evidently $\mathcal{PC}(0,T;X)$ is a Banach space with the norm

$$\|x\|_{\mathcal{PC}} = \sup_{t\in[0,T]} \|x(t)\|.$$

We assume that the state space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a semi-norm linear space of functions mapping $(-\infty,0]$ into X, and satisfying the following axiom (see [28,32]).

: (A) If $T > 0$ and $x : (-\infty,T] \to X$ satisfies that $x_0 \in \mathcal{B}$ and $x|_{[0,T]} \in \mathcal{PC}(0,T;X)$, then for every t in $[0,T]$ the following conditions hold:
  (i): $x_t \in \mathcal{B}$,
  (ii): $|x(t)| \le H\|x_t\|_{\mathcal{B}}$,
  (iii): $\|x_t\|_{\mathcal{B}} \le K(t)\sup_{0\le s\le t} \|x(s)\| + M(t)\|x_0\|_{\mathcal{B}}$,

: where H is a constant, $K : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $M : \mathbb{R}^+ \to \mathbb{R}^+$ is locally bounded;

: H, K, M are independent of $x(.)$.

: (A1) For the function x in (A), x_t is a $\mathcal{B}$-valued continuous function for t in $[0,T]$.

: (B) The space $\mathcal{B}$ is complete.

We suppose that there a Banach space $(Y,\|.\|_Y)$ continuously imbedded in X such that $Y\subset\mathscr{D}$. Since Y is still a Banach space, it is nature to assume that the subspace

$$\mathscr{B}:=\{x\in\mathcal{B}:x(\sigma)\in Y\}$$

of $\mathcal{B}$ also satisfies the following conditions:

: (A') If $T>0$ and $x:(-\infty,T]\to Y$ satisfies that $x_0\in\mathscr{B}$ and $x|_{[0,T]}\in\mathcal{PC}(0,T;Y)$, then for every t in $[0,T]$ the following conditions hold:

(i): $x_t\in\mathscr{B}$,

(ii): $\|x(t)\|\le H\|x_t\|_{\mathscr{B}}$,

(iii): $\|x_t\|_{\mathscr{B}}\le K(t)\sup_{0\le s\le t}\|x(s)\|_Y+M(t)\|x_0\|_{\mathscr{B}}$,

: where H is a constant, $K:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and $M:\mathbb{R}^+\to\mathbb{R}^+$ is locally bounded;

: H,K,M are independent of $x(.)$.

: (A1') For the function x in (A'), x_t is a $\mathscr{B}$-valued continuous function for t in $[0,T]$.

: (B') The space $\mathscr{B}$ is complete.

Let

$$\begin{aligned}\mathcal{PC}([0,T],Y)=\{x:[0;T]\to Y:\ & x(t)\text{ be continuous at }t\neq t_i,\\ & \text{left continuous at }t=t_i,\\ & \text{and the right limit }x(t_i^+)\text{ exists for }i=1,2,\dots,m\}.\end{aligned}$$

Evidently $\mathcal{PC}(0,T;Y)$ is a Banach space with the norm

$$\|x\|_{\mathcal{PC}([0,T],Y)}=\sup_{t\in[0,T]}\|x(t)\|_Y.$$

Let $\mathcal{E}$ be the space

$$\{x:(-\infty,T]\to X\mid x_0\in\mathscr{B}\text{ and }x|_{[0,T]}\in\mathcal{PC}(0,T;Y)\}$$

with the semi-norm

$$\|x\|_{\mathcal{E}}=\|x_0\|_{\mathscr{B}}+\|x|_{[0,T]}\|_{\mathcal{PC}([0,T],Y)}.$$

Definition 2.1. *Let X be a Banach space, 2^X denote the collection of all nonempty subsets of X, and $(\mathcal{A},\ge)$ a partially ordered set. A function $\mu:2^X\to\mathcal{A}$ is called a measure of noncompactness in X if*

$$\mu(\overline{\text{co}}\ \Omega)=\mu(\Omega)\quad \Omega\in 2^X,$$

where $\overline{\text{co}}\ \Omega$ is the closure of convex hull of Ω. A measure of noncompactness μ is called

(i) *monotone, if for each $\Omega_1,\Omega_2\in 2^X$ such that $\Omega_1\subset\Omega_2$, we have $\mu(\Omega_1)\le\mu(\Omega_2)$;*

(ii) *nonsingular, if $\mu(\{a\}\cup\Omega)=\mu(\Omega)$ for any $a\in X,\Omega\in 2^X$;*

(iii) *invariant with respect to the union with a compact set, if* $\mu(K \cup \Omega) = \mu(\Omega)$ *for every relatively compact set* $K \subset X$ *and* $\Omega \in 2^X$*;*

Moreover, if $\mathcal{A}$ is a cone in a normed space, we say that μ is

(iv) algebraically semi-additive, if $\mu(\Omega_1 \cup \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$ for any $\Omega_1, \Omega_2 \in 2^X$;

(v) regular, if $\mu(\Omega) = 0$ is equivalent to the relative compactness of Ω.

The so-called Hausdorff measure of noncompactness, defined by

$$\chi(\Omega) := \inf\{\varepsilon : \Omega \text{ has a finite } \varepsilon \text{ net}\},$$

satisfies all the above properties. For the main result of the present paper, we introduce two measure of noncompactness on the space $\mathcal{PC}([0,T],X)$ of continuous functions on interval $[0,T]$ taking values in X.

(i) For each $\Omega \subset \mathcal{PC}([0,T],X)$, the *damped modulus of fiber non-compactness* of Ω is defined by

$$\eta(\Omega) = \sup_{t \in [0,T]} e^{-Lt} \chi(\Omega(t)),$$

where L is a nonnegative constant, χ is the Hausdorff measure of noncompactness on X and $\Omega(t) = \{\omega(t) : \omega \in \Omega\}$;

(ii) For each $\Omega \subset \mathcal{PC}([0,T],X)$, the *modulus of equicontinuity* of Ω is defined by

$$\mathrm{mod}\,_C(\Omega) = \lim_{\delta \to 0} \sup_{\omega \in \Omega} \max_{|t_1 - t_2| < \delta} \|\omega(t_1) - \omega(t_2)\|.$$

Now, consider the function $\nu : 2^{\mathcal{PC}([0,T],X)} \to [0,\infty] \times [0,\infty]$ given by

$$\nu(\Omega) = \max_{S \in D(\Omega)} (\eta(S), \mathrm{mod}_C(S)), \tag{2.1}$$

where $D(\Omega)$ is the collection of all denumerable subsets of Ω and the maximum is taken in the sense of the partial order in the cone $[0,\infty] \times [0,\infty]$. It is known that ν is a measure of noncompactness in the space $\mathcal{PC}([0,T],X)$, which satisfies all properties in Definition 2.1 and the maximum in (2.1) is attained in $D(\Omega)$ (see [33], Example 2.1.3 for details).

If V and W are topological spaces, a multimap $\mathcal{F} : V \multimap W$ has closed, bounded, compact or convex values if $\mathcal{F}(v)$ is closed, bounded, compact or convex, respectively, for all $v \in V$. We shall use the notations:

$$P_{cl}(V) := \{U \subset V : U \textit{ is closed}\}, \qquad P_{wcl}(V) := \{U \subset V : U \textit{ is wealkly closed}\},$$
$$P_b(V) := \{U \subset V : U \textit{ is bounded}\}, \qquad P_{wb}(V) := \{U \subset V : U \textit{ is wealkly bounded}\},$$
$$P_c(V) := \{U \subset V : U \textit{ is convex}\}, \qquad P_{wc}(V) := \{U \subset V : U \textit{ is wealkly concex}\},$$
$$P_k(V) := \{U \subset V : U \textit{ is compact}\}, \qquad P_{wk}(V) := \{U \subset V : U \textit{ is wealkly compact}\}.$$

Now, let $\mathcal{F} : [0,T] \multimap X$. $\mathcal{F}$ is said to be measurable, if it has compact values and $\mathcal{F}^{-1}(V)$ is measurable for every open subset V of X, where $\mathcal{F}^{-1}(V) := \{t \in I : \mathcal{F}(t) \cap V \neq \varnothing\}$. $\mathcal{F}$ is integrably bounded if and only if $\mathcal{F}$ is measurable and $\|\mathcal{F}(.)\| \in L^1([0,T],\mathbb{R})$, where $\|\mathcal{F}(t)\| = \sup_{x \in \mathcal{F}(t)} \|x\|$. A function $f : [0,T] \to X$ is called a selection of $\mathcal{F}$ provided $f(t) \in \mathcal{F}(t)$ for all $t \in [0, T]$. We denote by $S^p_{\mathcal{F}}$, $1 \leq p \leq \infty$, the set of all selections of $\mathcal{F}$, which belong to the Lebesgue-Bochner space $L^p([0, T], X)$, i.e.,

$$S^p_{\mathcal{F}} = \{f \in L^p([0, T], X) : f(t) \in \mathcal{F}(t) \text{ a.e. } t \in [0, T]\}.$$

If $S^p_{\mathcal{F}}$ is nonempty and $\mathcal{F}$ has closed valued, then it is known that $S^p_{\mathcal{F}}$ is a closed subset of $L^p([0, T], X)(1 \leq p \leq \infty)$. The following result, which is adapted from [47], is crucial to our main result.

Lemma 2.2. *If $\mathcal{F} : [0, T] \to P_{wk,wc}(X)$ is integrably bounded, then $S^1_{\mathcal{F}}$ is nonempty, convex and weakly compact in $L^1([0, T], X)$.*

Denote by $\mathcal{W} : L^1([0, T], X) \to \mathcal{PC}([0, T], X)$ the operator

$$(\mathcal{W}\psi)(t) = \int_0^t U(t, s)\psi(s)ds. \tag{2.2}$$

This is the so-called generalized Cauchy operator and it is known that it satisfies the following properties (see [33]):

($\mathcal{W}1$) there exists a constant $C > 0$ such that

$$\|(\mathcal{W}\psi_1)(t) - (\mathcal{W}\psi_2)(t)\| \leq C \int_0^t \|\psi_1(s) - \psi_2(s)\|ds,$$

for all $\psi_1, \psi_2 \in L^1([0, T], X), t \in [0, T]$;

($\mathcal{W}2$) for each compact set $K \subset X$ and sequence $\{\psi_n\} \subset L^1([0, T], X)$ such that $\{\psi_n(t)\} \subset K$ for a.e. $t \in [0, T]$, the weak convergence $\psi_n \to \psi_0$ implies $\mathcal{W}(\psi_n) \to \mathcal{W}(\psi_0)$ strongly in $\mathcal{PC}([0, T], X)$;

($\mathcal{W}3$) $\mathcal{W}$ sends each bounded set to equicontinuous one.

Definition 2.3. A countable set $\{f_n\}^{\infty}_{n=1} \subset L^1([0, T], X)$ is said to be semicompact if

(i) it is integrably bounded, i.e., if there exists $\psi \in L^1([0, T], \mathbb{R}^+)$ such that

$$\|f_n(t)\| \leq \psi(t) \quad \text{for a.e } t \in [0, T] \text{ and every } n \in \mathbb{N};$$

(ii) the set $\{f_n(t)\}^{\infty}_{n=1}$ is relatively compact in X for almost every $t \in [0, T]$.

Lemma 2.4. *Every semicompact sequence is weakly compact in $L^1([0, T], X)$.*

The following several results, which also come from [33], are useful in the proof of our main result.

Lemma 2.5. *For every semicompact set $\{f_n\}^{\infty}_{n=1} \subset L^1([0, T], X)$, the set $\{\mathcal{W}f_n\}^{\infty}_{n=1}$ is relatively compact in $C([0, T], X)$. Moreover, if $\{f_n\}^{\infty}_{n=1}$ converges weakly to f_0 in $L^1([0, T], X)$ then $\mathcal{W}f_n \to \mathcal{W}f_0$ in $C([0, T], X)$.*

Lemma 2.6. *Let $\{\psi_n\}$ be an integrably bounded sequence in $L^1([0, T], X)$, i.e.,*

$$\|\psi_n(t)\| \leq \eta(t) \text{ for a.e. } t \in [0, T],$$

where $\eta \in L^1([0,T])$*. If* Q *satisfies (*$\mathcal{W}$*1),(*$\mathcal{W}$*2) and there exists* $q \in L^1([0,T])$ *such that*

$$\chi(\{\psi_n(t)\}_{n=1}^{\infty}) \leq q(t) \text{ for a.e. } t \in [0,T],$$

then

$$\chi(\{Q(\psi_n)(t)\}_{n=1}^{\infty}) \leq 2C \int_0^t q(s)ds$$

for each $t \in [0,T]$*, where* $C > 0$ *is the constant given in condition (*$\mathcal{W}$*1).*

Definition 2.7. *A multimap* $\mathcal{F} : X \to P_k(X)$ *is said to be condensing with respect to a measure of noncompactness* μ *or* μ*-condensing if for every bounded set* $\Omega \subset X$*, the relation*

$$\mu(\mathcal{F}(\Omega)) \geq \mu(\Omega)$$

implies the relative compactness of Ω*.*

The following fixed point principle can be found in [33].

Lemma 2.8. *If* U *is a closed convex subset of a Banach space* X *and* $\Phi : U \to P_{c,k}(X)$ *is a closed* μ*-condensing multimap, where* μ *is a nonsingular measure of noncompactness defined on the subsets of* U*. Then* Φ *has a fixed point.*

Lemma 2.9 (Proposition 3.5.1, [33])**.** *Let* K *be a closed subset of* X*,* $\mathcal{F} : K \to P_{c,k}(X)$ *a closed multimap, and* λ *a monotone measure of non-compactness defined on* X*. Suppose that* $\mathcal{F}$ *is* λ*-condensing on every bounded subset of* K*. Then* $\mathrm{Fix}(\mathcal{F}) := \{x \in K : x \in \mathcal{F}(x)\}$ *is compact, provided it is bounded.*

Definition 2.10. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be topological vector spaces. A multimap* $\mathcal{F} : \mathbb{X} \multimap \mathbb{Y}$ *is said to be upper semi-continuous if for any open subset* $U \subset \mathbb{Y}$*,* $\mathcal{F}^{-1}(U)$ *is an open subset of* $\mathbb{X}$*.*

Definition 2.11. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be metric space. A multi-valued map (multimap)* $\mathcal{F} : \mathbb{X} \to P_k(\mathbb{Y})$ *is said to be closed if its graph*

$$G_{\mathcal{F}} := \{(v,w) : w \in \mathcal{F}(v)\} \subset \mathbb{X} \times \mathbb{Y}$$

is a closed subset of $\mathbb{X} \times \mathbb{Y}$*.*

The following result can be found in [33].

Lemma 2.12. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be metric spaces and* $\mathcal{F} : \mathbb{X} \multimap \mathbb{Y}$ *a closed quasi-compact multimap with compact values. Then* $\mathcal{F}$ *is upper semi-continuous.*

To establish the existence result for system (1.1), we first introduce the notion of mild solution. The readers are referred to the works [14, 17, 38] for the formulation of this definition.

Definition 2.13. We say that a function $x \in \mathcal{E}$ is a *mild solution* of the system (1.1) if $x_0 = \varphi$, $x_t \in \mathcal{B}$ for every $t \in [0,T]$, $\Delta x(t_i) = I_i(x_{t_i})$, $i = 1, 2, \ldots, m,$ and the impulsive integral inclusion

$$x(t) \in U(t,0)[\varphi(0) + g(0,\varphi)] - g(t,x_t) - \int_0^t U(t,s)A(s)g(s,x_s)ds$$

$$+\int_0^t U(t,s)[F(s,x_s)+Bu(s)]ds+\sum_{i=1}^{m} U(t,t_i)I_i(x_{t_i}), \quad t\in[0,T].$$

is satisfied.

Let $p \geq 1$ be given and q the conjugate of p (i.e., $\frac{1}{p}+\frac{1}{q}=1$ and $q=\infty$ if $p=1$). In order to study the controllability of system (1.1), we impose the following hypotheses.

(H1) The function $g:[0,T]\times\mathcal{B}\to X$ satisfies
   (i) $\varphi\in\mathscr{B}\Longrightarrow g(t,\varphi)\in Y$ for all $t\in[0,T]$;
   (ii) $g:[0,T]\times\mathscr{B}\to Y$ is continuous;
   (iii) $g:[0,T]\times\mathscr{B}\to X$ is completely continuous;
   (iv) there exists a function $\alpha\in L^p([0,T],\mathbb{R}^+)$ such that

$$\|A(t)g(t,\varphi)\|\leq\alpha(t)(1+\|\varphi\|_{\mathscr{B}}) \quad \text{for a.e. } t\in[0,T] \text{ and } \varphi\in\mathscr{B}.$$

(H2) The multimap $F:[0,T]\times\mathcal{B}\multimap X$ satisfies the Carathéodory condition (see [1], p.298), i.e., for each $\varphi\in\mathcal{B}$, $F(.,\varphi)$ has a strongly measurable selection, and for a.e. $t\in[0,T]$, $F(t,.):\mathcal{B}\to P_{c,k}(X)$ is upper semi-continuous. Moreover, there exists a function $\beta\in L^p([0,T],\mathbb{R}^+)$ such that

$$\|F(t,\varphi)\|\leq\beta(t)(1+\|\varphi\|_{\mathcal{B}}) \quad \text{for a.e. } t\in[0,T].$$

where $\|F(t,\varphi)\|:=\sup_{f\in F(\cdot,\varphi)}\|f(t)\|$.

(H3) There exists a function $\gamma:[0,T]\in L^1([0,T],\mathbb{R}^+)$ such that for all bounded set $D\subset\mathcal{B}$

$$\chi\left(F(t,D)\right)\leq\gamma(t)\sup_{-\infty<\sigma\leq0}\chi(D(\sigma)) \quad \text{for a.e } t\in[0,t],$$

where $D(\sigma):=\{\phi(\sigma);\phi\in D\}$ and χ is the Hausdorff measure of noncompactness.

(H4) The function $I_i:\mathcal{B}\to X$ is continuous and there are positive constants $L_i, i=1,2,\ldots,m$, such that

$$\|I_i(\phi_1)-I_i(\phi_2)\|\leq L_i\|\phi_1-\phi_2\|_{\mathcal{B}},$$

for $\phi_1,\phi_2\in\mathcal{B}$ and $i=1,2,\ldots,m$.

Now, for each $u\in L^p([0,T],U)$, we introduce the multioperator $\Gamma_{u,\varphi}:\mathcal{PC}([0,T],Y)\multimap\mathcal{PC}([0,T],Y)$ by

$$(\Gamma_{u,\varphi}x)(t):=\Big\{U(t,0)[\varphi(0)+g(0,\varphi)]-g(t,\bar{x}_t)-\int_0^t U(t,s)A(s)g(s,\bar{x}_s)ds+\int_0^t U(t,s)[f(s)+Bu(s)]ds+\sum_{i=1}^{m}U(t,t_i)I_i(\bar{x}_{t_i});\ f\in S^1_{F(\cdot,\bar{x}_\cdot)},\ t\in[0,T]\Big\}, \tag{2.3}$$

where for every $x \in \mathcal{PC}([0,T],X)$, $\bar{x}$ denotes the extension of x given by

$$\bar{x}(t) = \begin{cases} x(t), & t \in [0,T] \\ \varphi(t), & t \leq 0. \end{cases}$$

It is ready to see that if $x \in \mathrm{Fix}(\Gamma_{u,\varphi}) := \{x \in \mathcal{PC}([0,T],Y) : x \in \Gamma_{u,\varphi}(x)\}$, then $\bar{x}$ is a mild solution of the system (1.1) and we therefore call $\mathrm{Fix}(\Gamma_{u,\varphi})$ the *solution set* corresponding to the control u.

Theorem 2.14. *Suppose that the hypotheses* (H1)-(H4) *are satisfied. Then the solution set of problem (1.1) corresponding to the control* u *is nonempty and compact provided*

$$M_0 \tilde{K} \sum_{i=1}^{m} L_i < 1 \tag{2.4}$$

where $\tilde{K} := \sup_{0 \leq t \leq T} |K(t)|$.

Proof. It suffices to show that the set $\mathrm{Fix}(\Gamma_{u,\varphi})$ is nonempty, and the proof is divided into several steps.

Step 1. It is already seen that $\Gamma_{u,\varphi}$ has convex values, using the hypotheses that the multimap F has convex values.

Step 2. To see that $\Gamma_{u,\varphi}$ has closed graph, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{PC}([0,T],Y)$ with $x_n \to x \in \mathcal{PC}([0,T],Y)$. Then by axiom (A)(iii),

$$\begin{aligned} \|(\bar{x}_n)_t - \bar{x}_t\|_{\mathscr{B}} &\leq K(t) \sup_{s \in [0,T]} \|x_n(s) - x(s)\| + M(t)\|(\bar{x}_n)_0 - \bar{x}_0\|_{\mathscr{B}} \\ &= K(t) \sup_{s \in [0,T]} \|x_n(s) - x(s)\| \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{2.5}$$

Now, for each $n \in \mathbb{N}$, choose $y_n \in \Gamma_{u,\varphi}(x_n)$. Then by (2.3), for each $n \in \mathbb{N}$, the mapping $t \mapsto F(t,(\bar{x}_n)_t)$ admits a selection f_n such that

$$\begin{aligned} y_n(t) \quad := \quad & U(t,0)[\varphi(0) + g(t,\varphi)] - g(t,(\bar{x}_n)_t) - \int_0^t U(t,s)A(s)g(s,(\bar{x}_n)_s)ds \\ & + \int_0^t U(t,s)[f_n(s) + Bu(s)]ds + \sum_{i=1}^{m} U(t,t_i)I_i((\bar{x}_n)_{t_i}), \quad t \in [0,T]. \end{aligned}$$

Let

$$W(t) = \overline{\mathrm{co}} \bigcup_{n \geq 1} F(t,(\bar{x}_n)_t)$$

Invoking Theorem 7.4.2 of [35] (p.90) and (H3), we have $W(t) \in P_{wkc}(X)$. Again by (H2), we have

$$\|W(t)\| = \sup_{w \in W(t)} \|w\| \leq \beta(t)(1+b)$$

yielding $W(\cdot)$ is integrably bounded, where $b = \sup_{n\geq 1, t\in[0,T]} \|(\bar{x}_n)_t\|_{\mathcal{B}}$, and hence by Lemma 2.2, we see that S^1_W is weakly compact in $L^1([0,T],X)$. We thus may assume, by passing to a subsequence if necessary, that

$$f_n \longrightarrow^w f \text{ in } L^1([0,T],X). \tag{2.6}$$

Moreover, it follows by Theorem 3.1 of [48] that

$$f(t) \in \overline{\text{co}}\left\{w - \overline{\lim}_n\{f_n(t)\}\right\} \subset \overline{\text{co}}\left\{w - \overline{\lim}_n F(t,(\bar{x}_n)_t)\right\} \subset F(t,\bar{x}_t) \text{ a.e. on } [0,T]. \tag{2.7}$$

where the last inclusion is guaranteed by (H2). Now,

$$\begin{aligned} y_n(t) &= U(t,0)[\varphi(0)+g(t,\varphi)] - g(t,(\bar{x}_n)_t) - \int_0^t U(t,s)A(s)g(s,(\bar{x}_n)_s)ds \\ &\quad + \int_0^t U(t,s)f_n(s)ds + \sum_{i=1}^m U(t,t_i)I_i((\bar{x}_n)_{t_i}), \quad t\in[0,T]. \\ &= U(t,0)[\varphi(0)+g(t,\varphi)] - g(t,(\bar{x}_n)_t) - \int_0^t U(t,s)A(s)g(s,(\bar{x}_n)_s)ds \\ &\quad + (\mathcal{W}(f_n+Bu))(t) + \sum_{i=1}^m U(t,t_i)I_i((\bar{x}_n)_{t_i}), \quad t\in[0,T]. \\ &\to U(t,0)[\varphi(0)+g(t,\varphi)] - g(t,\bar{x}_t) - \int_0^t U(t,s)A(s)g(s,\bar{x}_s)ds \\ &\quad + (\mathcal{W}(f+Bu))(t) + \sum_{i=1}^m U(t,t_i)I_i(\bar{x}_{t_i}), \quad t\in[0,T]. \end{aligned}$$

by (2.6), (2.7), ($\mathcal{W}$2), (H1), (H2) and the dominated convergence theorem. Set

$$\begin{aligned} y &:= U(t,0)[\varphi(0)+g(t,\varphi)] - g(t,\bar{x}_t) - \int_0^t U(t,s)A(s)g(s,\bar{x}_s)ds \\ &\quad + (\mathcal{W}(f+Bu))(t) + \sum_{i=1}^m U(t,t_i)I_i(\bar{x}_{t_i}) \end{aligned}$$

In view of (2.7), we see that

$$y \in \Gamma_{u,\varphi}(x),$$

and hence $\Gamma_{u,\varphi}$ has closed graph. By a similar argument, we obtain that $\Gamma_{u,\varphi}$ has compact values.

Step 3. We now prove that $\Gamma_{u,\varphi}$ is ν-condensing. For this, consider a bounded set $\Omega \subset \mathcal{PC}([0,T],X)$ such that

$$\nu(\Gamma_{u,\varphi}(\Omega)) \geq \nu(\Omega). \tag{2.8}$$

We will show that Ω is relatively compact in $\mathcal{PC}([0,T],X)$. In fact, there exists, by the definition of ν, a sequence $\{z_n\}_{n=1}^\infty$ which achieves the maximum, i,e;

$$\nu(\Gamma_{u,\varphi}(\Omega)) = \left(\eta(\{z_n\}_{n=1}^\infty), \text{mod}_C(\{z_n\}_{n=1}^\infty)\right).$$

Choose $\{x_n\}_{n=1}^{\infty} \subset \Omega$ so that for each $n \in \mathbb{N}$, $z_n \in \Gamma_{u,\varphi}(x_n)$. Then

$$\begin{aligned} z_n(t) &= U(t,0)[\varphi(0) + g(t,\varphi)] - g(t,(\bar{x}_n)_t) - \int_0^t U(t,s)A(s)g(s,(\bar{x}_n)_s)ds \\ &\quad + \int_0^t U(t,s)[f_n(s) + Bu(s)]ds + \sum_{i=1}^m U(t,t_i)I_i((\bar{x}_n)_{t_i}), \quad t \in [0,T]. \end{aligned}$$

where $f_n \in S^1_{F(\cdot,(\bar{x}_n)\cdot)}$. Since $g : [0,T] \times \mathcal{B} \to X$ is completely continuous by (H1), this implies that

$$\eta(\{z_n\}_{n=1}^{\infty}) = \eta(\{\mathcal{W}f_n\}_{n=1}^{\infty}),$$

Now, let $t \in [0,T]$ and it follows by (H3) that

$$\begin{aligned} \chi(\{f_n(s)\}_{n=1}^{\infty})) &\le \gamma(s) \sup_{\sigma \le 0} \chi(\{(\bar{x}_n)_s(\sigma)\}_{n=1}^{\infty}) \\ &\le \gamma(s) \sup_{\sigma \le 0} \chi\big(\{\bar{x}_n(s+\sigma)\}_{n=1}^{\infty}\big) \\ &\le \gamma(s) \sup_{\tau \le s} \chi\big(\{\bar{x}_n(\tau)\}_{n=1}^{\infty}\big) \\ &\le \gamma(s)e^{Ls} \sup_{0\le\tau\le T} \big[e^{-L\tau}\chi\big(\{x_n(\tau)\}_{n=1}^{\infty}\big)\big] \\ &= \gamma(s)e^{Ls}\eta(\{x_n\}_{n=1}^{\infty}), \end{aligned}$$

for all $s \in [0,t]$. Now, we apply Lemma 2.6 and obtain

$$\chi(\{\mathcal{W}g_n(t)\}_{n=1}^{\infty}) \le 2C\left(\int_0^t \gamma(s)e^{Ls}ds\right)\eta(\{x_n\}_{n=1}^{\infty})$$

which implies

$$e^{-Lt}\chi(\{\mathcal{W}g_n(t)\}_{n=1}^{\infty}) \le 2C\left(\int_0^t \gamma(s)e^{-L(t-s)}ds\right)\eta(\{x_n\}_{n=1}^{\infty}),$$

whence, in view of (2.8),

$$\eta(\{x_n\}_{n=1}^{\infty}) \le \eta(\{z_n\}_{n=1}^{\infty}) \le \zeta\eta(\{x_n\}_{n=1}^{\infty}). \tag{2.9}$$

where

$$\zeta := 2C \sup_{t\in[0,T]} \int_0^t e^{-L(t-s)}\gamma(s)ds$$

Now, choose the constant $L > 0$ in the definition of η so that

$$\zeta := 2C \sup_{t\in[0,T]} \int_0^t e^{-L(t-s)}\gamma(s)ds < 1, \tag{2.10}$$

and we therefore combine (2.9) and (2.10) to conclude

$$\eta(\{z_n\}_{n=1}^{\infty}) = 0.$$

On the other hand, it is evident from (H2) that $\{f_n\}$ is a bounded sequence in $L^1([0,T],X)$. Then the property ($\mathcal{W}$3) ensures that $\{\mathcal{W}f_n\}$ is equicontinuous in $\mathcal{PC}([0,T],X)$ and hence

$$\mathrm{mod}_C(\{z_n\}_{n=1}^{\infty}) = \mathrm{mod}_C(\{\mathcal{W}f_n\}_{n=1}^{\infty}) = 0.$$

Consequently,

$$\nu(\Omega) = (0,0)$$

and therefore, the regularity of ν guarantees the relative compactness of Ω. This shows that the multioperator $\Gamma_{u,\varphi}$ satisfies the conditions of Lemma 2.8, and therefore has a fixed point, which in turn proves that the system (1.1) has a nonempty solution set.

Step 4. It remains to show that the solution set belongs to a priori bounded set in $\mathcal{PC}([0,T],X)$. Indeed, if $x \in \mathrm{Fix}(\Gamma_{u,\varphi})$, then

$$x(t) = U(t,0)[\varphi(0) + g(0,\varphi)] - g(t,\bar{x}_t) - \int_0^t U(t,s)A(s)g(s,\bar{x}_s)ds + \int_0^t U(t,s)[f(s) + Bu(s)]ds + \sum_{i=1}^m U(t,t_i)I_i(\bar{x}_{t_i}),$$

for some $f \in F(\cdot,\bar{x}_\cdot)$ and thus by assumptions (H1)-(H4) and axioms (A)(iii), (A1), there follows the estimate:

$$\begin{aligned}\|x(t)\| \le\; & M_0\big[\|\varphi(0)\| + \|g(0,\varphi)\|\big] + \sup_{0\le t\le T}\|g(t,\bar{x}_t)\| \\ & + M_0\big(1+\tilde{M}\|\varphi\|_{\mathcal{B}}\big)\|\alpha\|_1 + M_0\tilde{K}\int_0^t \alpha(s)\sup_{0\le\tau\le s}\|x(\tau)\|ds \\ & + M_0\big(1+\tilde{M}\|\varphi\|_{\mathcal{B}}\big)\|\beta\|_1 + M_0\tilde{K}\int_0^t \beta(s)\sup_{0\le\tau\le s}\|x(\tau)\|d\tau \\ & + M_0\|Bu(\cdot)\|_{L^p([0,T],X)} \\ & + M_0\sum_{i=1}^m\Big(L_i\tilde{M}\|\varphi\|_{\mathcal{B}} + \|I_i(0)\|\Big) + M_0\tilde{K}\sup_{0\le\tau\le t}\|x(\tau)\|\sum_{i=1}^m L_i\end{aligned}$$

for all $t \in [0,T]$, or

$$\begin{aligned}\|x(t)\| \;\le\; & C_0 + \left(M_0\tilde{K}\sum_{i=1}^m L_i\right)\sup_{0\le\tau\le t}\|x(\tau)\| \\ & + M_0\tilde{K}\int_0^t(\alpha+\beta)(s)\sup_{0\le\tau\le s}\|x(\tau)\|ds, \qquad t\in[0,T],\end{aligned} \tag{2.11}$$

where $\tilde{M} := \sup_{0\le t\le T}\|M(t)\|$, $\tilde{K} := \sup_{0\le t\le T}\|K(t)\|$, and

$$\begin{aligned}C_0 =\; & M_0\big[\|\varphi(0)\| + \|g(0,\varphi)\|\big] + \sup_{0\le t\le T}\|g(t,x_t)\| \\ & + M_0\big(1+\tilde{M}\|\varphi\|_{\mathcal{B}}\big)(\|\alpha\|_1 + \|\beta\|_1) + M_0\|Bu(\cdot)\|_{L^p([0,T],X)} \\ & + M_0\sum_{i=1}^m\Big(L_i\tilde{M}\|\varphi\|_{\mathcal{B}} + \|I_i(0)\|\Big).\end{aligned}$$

Hence, if $\tilde{C} := M_0\tilde{K}\sum_{i=1}^m L_i < 1$, the inequality (2.11) becomes

$$\sup_{0\le\tau\le t}\|x(\tau)\| \le C_0(1-\tilde{C})^{-1} + M_0\tilde{K}(1-\tilde{C})^{-1}\int_0^t(\alpha+\beta)(s)\sup_{0\le\tau\le s}\|x(\tau)\|ds$$

yielding by Gronwall-Bellman inequality
(2.12)
$$\sup_{0\le\tau\le t}\|x(\tau)\|\le C_0(1-\tilde{C})^{-1}\exp\left(M_0\tilde{K}(1-\tilde{C})^{-1}\int_0^t(\alpha+\beta)(s)ds\right),\ \forall t\in[0,T].$$
The solution set is thus bounded and therefore, by Lemma 2.9, compact. This completes the proof. □

We set
$$C_1=M_0\big[\|\varphi(0)\|+\|g(0,\varphi)\|\big]+\sup_{0\le t\le T}\|g(t,x_t)\| \\ +M_0\big(1+\tilde{M}\|\varphi\|_{\mathcal{B}}\big)\big(\|\alpha\|_1+\|\beta\|_1\big)+\ M_0\sum_{i=1}^m\Big(L_i\tilde{M}\|\varphi\|_{\mathcal{B}}+\|I_i(0)\|\Big)$$
and then $C_0=C_1+M_0\|Bu(\cdot)\|_{L^p([0,T],X)}$, which in tern yields by (2.12)
$$(2.13)\qquad \begin{aligned}\|x\|_{C([0,T],X)}&\le C_0(1-\tilde{C})^{-1}\exp\Big(M_0\tilde{K}(1-\tilde{C})^{-1}(\|\alpha\|_1+\|\beta\|_1)\Big)\\&\le C_2\left(C_1+M_0\|Bu(\cdot)\|_{L^p([0,T],X)}\right)\end{aligned}$$
where $C_2=(1-\tilde{C})^{-1}\exp\Big(M_0\tilde{K}(1-\tilde{C})^{-1}(\|\alpha\|_1+\|\beta\|_1)\Big)$. This inequality will be used in the proof of the main result in section 4.

3. The solution multi-map

We first introduce the solution multi-map
$$(3.1)\qquad \begin{aligned}&\mathcal{S}_{F,\varphi}:L^p([0,T],U)\multimap\mathcal{PC}([0,T],X),\\&\mathcal{S}_{F,\varphi}(u):=\{\bar{x}:x\in\mathrm{Fix}(\Gamma_{u,\varphi})\},\end{aligned}$$
impose an additional assumption:

(H5) $U(t,s)$ is compact for all $t>s$,

and prove that the solution multimap is completely continuous.

Lemma 3.1. *Under assumptions (H1)-(H5), the solution multimap $\mathcal{S}_{F,\varphi}$ defined by (3.1) is completely continuous, i.e., it is upper-semicontinuous and sends each bounded set to a relatively compact one, provided the inequality (2.4) holds.*

Proof. For a bounded set $\mathcal{Q}\subset L^p([0,T],U)$, we shall show that $\mathcal{S}_{F,\varphi}(\mathcal{Q})$ is relatively compact in $C([0,T],X)$. Let $\{u_n\}$ be a sequence in $\mathcal{Q}$ and for each $n\in\mathbb{N}$, let x_n be a selection of $\mathcal{S}_{F,\varphi}(u_n)$. Then
(3.2)
$$x_n(t)\in U(t,0)[\varphi(0)+g(0,\varphi)]-g(t,(\bar{x}_n)_t)-\int_0^t U(t,s)A(s)g\left(s,(\bar{x}_n)_s\right)ds \\ +\int_0^t U(t,s)[F(s,(\bar{x}_n)_s)+Bu_n(s)]ds+\sum_{i=1}^m U(t,t_i)I_i((\bar{x}_n)_{t_i}),$$
Again, we use a similar argument as in the proof of theorem 2.14 to conclude that $\mathcal{S}_{F,\varphi}$ is closed and $\{x_n\}_{n=1}^\infty$ is a bounded sequence in $C([0,T],X)$ provided the inequality (2.4) holds. Hence, the set $\{(\bar{x}_n)_t\}_{n=1}^\infty$ is bounded in $\mathcal{B}$. It then follows by

(H2) that the sequence $\{F(., (\bar{x}_n).)\}_{n=1}^{\infty}$ is integrably bounded. On the other hand, since g is completely continuous by (H1) and $U(t,s)$ is compact for $0 \le s < t \le T$ by (H5), it follows by (3.2) that the set $\{x_n(t)\}_{n=1}^{\infty}$ is relatively compact for each $t \in [0,T]$. It then follows by (H3) that

$$\chi\left(\bigcup_{n=1}^{\infty} F(t,(\bar{x}_n)_t)\right) \le \gamma(t) \sup_{\sigma \le 0} \chi(\{(\bar{x}_n)_t(\sigma)\}_{n=1}^{\infty}) \le \gamma(t) \sup_{0 \le \tau \le T} \chi(\{x_n(\tau)\}_{n=1}^{\infty}) = 0,$$

i.e., $(\cup_{n=1}^{\infty} F(t,(\bar{x}_n)_t))$ is relatively compact, yielding the function

$$W(t) = \overline{\mathrm{co}} \bigcup_{n \ge 1} F(t,(\bar{x}_n)_t)$$

has compact and convex values, whence by Lemma 2.4, S_W^1 is weakly compact in $L^1([0,T],X)$. Thus if f_n are selections in $S^1_{F(\cdot,(\bar{x}_n).)}$ such that

$$\begin{aligned} x_n(t) = U(t,0)[\varphi(0) + g(0,\varphi)] - g(t,(\bar{x}_n)_t) + \int_0^t U(t,s)A(s)g\left(s,(\bar{x}_n)_s\right) ds \\ + \int_0^t U(t,s)[f_n(s) + B(s)u_n(s)]ds + \sum_{i=1}^{m} U(t,t_i)I_i((\bar{x}_n)_{t_i}), \end{aligned}$$

then by passing to a subsequence if necessary, we see that $\{f_n\}_{n=1}^{\infty}$ is weakly convergent to some function f in $L^1([0,T],X)$ which in turn yields by $(\mathcal{W}2)$ that $\mathcal{W}f_n \to \mathcal{W}f$ in $C([0,T],X)$. This shows that $\{x_n\}_{n=1}^{\infty}$ is relatively compact and the proof is hence completed. □

Next, we introduce the concept of σ-Ca-selectionable and mLL-selectionable maps (see [25], section 9.4).

Definition 3.2. *Let $(\mathbb{X}, d_{\mathbb{X}})$ be an metric space and $(Y, d_{\mathbb{Y}})$ a pseudo-metric space.*

(i) A single-valued map $f : [0,T] \times \mathbb{Y} \to \mathbb{X}$ is said to be measurable-locally Lipschitz (mLL), if $f(.,y)$ is measurable for every $y \in \mathbb{Y}$ and for every $y \in \mathbb{Y}$, there exists a neighborhood N_y of y and an integrable function $L_y : [0,T] \to [0,\infty)$ such that, for all $t \in [0,T]$ and $y_1, y_2 \in N_y$,

$$d_{\mathbb{X}}(f(t,y_1), f(t,y_2)) \le L_y(t) d_{\mathbb{Y}}(y_1,y_2)$$

(ii) A multivalued mapping $\mathcal{F} : [0,T] \times \mathbb{Y} \multimap \mathbb{X}$ is mLL-selectionable if it has an mLL-selection.

Definition 3.3. Let $(\mathbb{X}, d_{\mathbb{X}})$ be a metric space, $(Y, d_{\mathbb{Y}})$ a pseudo-metric space and $\mathcal{F} : [0,T] \times \mathbb{Y} \multimap \mathbb{X}$ a multivalued mapping. Then

(i) $\mathcal{F}$ is *Ca-selectionable* if there exists a selection f of $\mathcal{F}$ which satisfies the Carathéodory condition;

(ii) $\mathcal{F}$ is *σ-Ca-selectionable* if there is a sequence $\mathcal{F}_n : [0,T] \times \mathbb{Y} \multimap \mathbb{X}$, $n = 1, 2, ...$ such that:

 (a) $\mathcal{F}(t,\phi) \subset \cdots \subset \mathcal{F}_{n+1}(t,\phi) \subset \mathcal{F}_n(t,\phi) \subset \ldots,$

(b) $\mathcal{F}(t,\phi) = \bigcap_{n\geq 1} \mathcal{F}_n(t,\phi)$, and

(c) $\mathcal{F}_n$ is *Ca-selectionable* for all $n \in \mathbb{N}$.

To prove the main result of this section, we shall show that the nonlinear term $F(.,.)$ is σ-Ca-selectionable. For this, let us cover $\mathcal{B}$ by the open balls $\{B(\varphi, r_n) : \varphi \in \mathcal{B}, r_n = 3^{-n}, n \in \mathbb{N}\}$. Since $\mathcal{B}$ is a pseudometric space, then for each $n \in \mathbb{N}$, there exists by Stone theorem, a locally finite refinement $\{V_j^n\}_{j\in J_n}$ of the cover $\{B(\varphi, r_n) : \varphi \in \mathcal{B}, r_n = 3^{-n}, n \in \mathbb{N}\}$. Now, we can associate a locally Lipschitz partition of unity $\{p_j^n\}_{j\in J}$ subordinated to the open covering $\{V_j^n\}_{j\in J_n}$. For every $j \in J_n$ let φ_j^n be such that $V_j^n \subset B(\varphi_j^n, r_n)$ and define

$$F_n(t,\varphi) = \sum_{j\in J_n} p_j^n(\varphi) \cdot \overline{\mathrm{co}}\, F(t, B(\varphi_j^n, 2r_n)) \tag{3.3}$$

where

$$F(t, B(\varphi_j^n, 2r_n)) := \bigcup_{\varphi\in B(\varphi_j^n, 2r_n)} F(t,\varphi). \tag{3.4}$$

If $\varphi \in \mathcal{B}$, then $\varphi \in V_j^n \subset B(\varphi_j^n, r_n) \subset B(\varphi_j^n, 2r_n)$ for all $j \in J_n$ with $p_j^n(\varphi) \neq 0$, and it follows immediately by (3.3) and (3.4) that $F(t,\varphi) \subset F_n(t,\varphi)$, yielding

$$F(t,\varphi) \subset \bigcap_{n\geq 1} F(t,\varphi). \tag{3.5}$$

Conversely, let $\mathcal{O}$ be the collectopn of all open, convex set containing $F(t,\varphi)$. If $O \in \mathcal{O}$, then since $F(t,.)$ is upper semi-continuous, there is a $\delta > 0$ such that $F(t,\psi) \subset O$ for all $\psi \in \mathcal{B}$ with $\|\psi - \varphi\|_{\mathcal{B}} < \delta$. Thus, if $3r_n < \delta$, then $\overline{\mathrm{co}}\, F(t, B(\varphi, 3r_n)) \subset \overline{O}$. Let ν satisfy $3r_\nu < \delta$ and then

$$\bigcap_{n\geq 1} F_n(t,\varphi) \subset F_\nu(t,\varphi) \subset \overline{\mathrm{co}}\, F(t, B(\varphi, 3r_n)) \subset \overline{O}.$$

Since $O \in \mathcal{O}$ is arbitrary, this implies that

$$\bigcap_{n\geq 1} F_n(t,\varphi) \subset \bigcap_{O\in\mathcal{O}} O = F(t,\varphi). \tag{3.6}$$

Observe further that for each $\varphi_j^n, j \in J_n$ there is by (H2), a measurable selection h_j of $F(.,\varphi_j^n)$. Define $f_n : [0,T] \times \mathcal{B} \to X$ by

$$f_n(t,\varphi) = \sum_{j\in J_n} p_j^n(\varphi) \cdot h_j(t), \tag{3.7}$$

which shows that F_n is mLL-selectionable. Furthermore, each $F_n(t,.)$ is upper semicontinuous. In fact, if $V \subset X$ is an open set containing $F_n(t,\varphi)$, then by(3.3),

$$F(t, B(\varphi_j^n, 2r_n)) \subset \overline{\mathrm{co}}\, F((t, B(\varphi_j^n, 2r_n)) \subset V$$

for all $j \in J_n$ with $p_j^n(\varphi) \neq 0$. Let $\mathcal{J}_n := \{j \in J_n : p_j^n(\varphi) \neq 0\}$ and set

$$Z := \bigcap_{j\in\mathcal{J}_n} V_j^n$$

Since $\mathcal{J}_n$ is a finite set, we see that U is an open set containing φ, and

$$F(t,Z) \subset F(t,V_j^n) \subset F(t,B(\varphi_j^n,2r_n)) \subset \overline{\text{co}}\,F((t,B(\varphi_j^n,2r_n)) \subset V.$$

It follows again by(3.3) that

$$F_n(t,\psi) = \sum_{j\in J_n} p_j^n(\psi)\cdot \overline{\text{co}}\,F(t,B(\varphi_j^n,2r_n)) \subset V, \tag{3.8}$$

for all $\psi \in Z$. We summerize (3.3)-(3.8) in the following theorem:

Theorem 3.4. *The nonlinear term $F(.,.)$ of inclusion (1.1) is σ-Ca-selectionable. More precisely, there exists a sequence of multivalued maps $\{F_n\}_{n=1}^{\infty}$, $F_n : [0,T]\times \mathcal{B} \to P_{cl,c}(X)$ such that*

(i) *each multivalued map $F_n(t,\cdot) : \mathcal{B} \to P_{cl,c}(X), n \geq 1$ is continuous for a.e. $t \in [0,T]$,*
(ii) $F(t,\phi) \subset \cdots \subset F_{n+1}(t,\phi) \subset F_n(t,\phi) \subset \overline{\text{co}}\,F(t,B(\varphi,3r_n)), n \geq 1,$
(iii) $F(t,\phi) = \bigcap_{n\geq 1} F_n(t,\phi),$
(iv) *for $t \in [0,T]$ and $n \geq 1$, $F_n(t,.)$ is upper semicontinuous,*
(v) *for each $n \geq 1$ there exists a selection $f_n : [0,T]\times\mathcal{B} \to X$ of F_n, such that $f_n(.,\phi)$ is measurable and $f_n(t,.)$ is locally Lipschitz.*

We shall need the following results.

Lemma 3.5 (Mazur's Lemma, see [43] Theorem 21.4)**.** *Let E be a normed space and $\{x_n\}_{n=1}^{\infty} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{n=1}^{m} \alpha_{mn}x_n$ with $\alpha_{mn} > 0$ for $n = 1,2,...,m$ and $\sum_{n=1}^{m}\alpha_{mn} = 1$ for every $m \in \mathbb{N}$, which converges strongly to x.*

Lemma 3.6 (see e.g. [2], Lemma 1.1.9)**.** *Let $\{K_n\}_{n=1}^{\infty} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{\text{co}}(\limsup_{n\to\infty} K_n) = \bigcap_{n\in\mathbb{N}} \overline{\text{co}}\left(\bigcup_{n\geq N} K_n\right),$$

where $\overline{\text{co}}(A)$ refers to the closure of the convex hull of A.

The following notions are essential for investigating the geometric structure of the solution set $\mathcal{S}_{F,\varphi}(u)$,

Definition 3.7. *Let $\mathbb{X}$ be a Banach space and $\mathbb{Y}$ a metric space.*

(i) A subset V of $\mathbb{Y}$ is said to be contractible in $\mathbb{Y}$ if the imbedding $i_V : V \to \mathbb{Y}$ is null-homotopic, i.e., there exists $y_0 \in \mathbb{Y}$ and a continuous map $h : V \times [0,1] \to \mathbb{Y}$ such that $h(y,0) = y$ and $h(y,1) = y_0$ for every $y \in V$.
(ii) A subset V of a metric space $\mathbb{Y}$ is called an R_δ-set if B can be represented as the intersection of decreasing sequence of compact contractible sets.
(iii) A multimap $\mathcal{F} : \mathbb{X} \to 2^{\mathbb{Y}}$ is said to be an R_δ-map if $\mathcal{F}$ is upper semicontinuous and for each $x \in \mathbb{X}$, $\mathcal{F}(x)$ is an R_δ-set in $\mathbb{Y}$.

We are then able to examine the geometric structure of the solution set $\mathcal{S}_{F,\varphi}(u)$

Theorem 3.8. *Assume the hypotheses of Theorem 2.14 and the inequality (2.4) holds. If in addition, X is separable and $F : [0,T] \times \mathcal{B} \multimap X$ takes closed values and is mLL-selectionable, then the solution set of problem (1.1) corresponding to the control u is contractible.*

Proof. Let $f \subset F$ be a measurable, locally Lipschitz selection and consider the single-valued problem

$$\begin{cases} \frac{d}{dt}[x(t) + g(t, x_t)] = A(t)x(t) + f(t, x_t) + Bu(t), & t \in [0,T], t \neq t_i \\ \Delta x(t_i) = I_i(x_{t_i}) \quad i = 1, 2, \ldots, m, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{3.9}$$

Using a similar argument as in [13], we can prove that problem (3.9) has exactly one solution for every $\varphi \in \mathcal{B}$. Define the homotopy $h : \mathcal{S}_{F,\varphi}(u) \times [0,1] \to \mathcal{S}_{F,\varphi}(u)$ by

$$h(x, \alpha)(t) = \begin{cases} x(t), & t \leq \alpha T, \\ \xi(t), & \alpha T < t \leq T, \end{cases}$$

where ξ is the unique solution of problem (3.9). In particular,

$$h(x, \alpha) = \begin{cases} x, & \text{for } \alpha = 1, \\ \xi, & \text{for } \alpha = 0. \end{cases}$$

Let $\{(x_n, \alpha_n)\}_{n=1}^{\infty} \subset \mathcal{S}_{F,\varphi}(u)$ be such that $(x_n, \alpha_n) \to (x, \alpha)$, as $n \to \infty$. Then

$$h(x_n, \alpha_n)(t) = \begin{cases} x_n(t), & t \leq \alpha_n T, \\ \xi(t), & \alpha_n T < t \leq T, \end{cases}$$

We shall prove that $h(x_n, \alpha_n) \to h(x, \alpha)$. In fact, for the case $\alpha = 0$ and $\lim_{n\to\infty} \alpha_n = 0$, since

$$h(x, 0)(t) = \begin{cases} \varphi(t), & t \leq 0, \\ \xi(t), & 0 < t \leq T, \end{cases}$$

and $x_n \equiv \varphi$ on $(\infty, 0]$, then

$$\|h(x_n, \alpha_n) - h(x, \alpha)\|_{\mathcal{E}} \leq \|x_n - \varphi\|_{\mathcal{B}} + \|x_n - \xi\|_{\mathcal{PC}([0,T],X)} \leq \|x_n - \xi\|_{\mathcal{PC}([0,\alpha_n T],X)} \to 0,$$

as $n \to \infty$. The case where $\alpha = 1$ and $\lim_{n\to\infty} \alpha_n = 1$ is treated similarly.

The case $0 < \lim_{n\to\infty} \alpha_n = \alpha < 1$ is divided into two subcases: $t \in (-\infty, \alpha T]$ and $t \in (\alpha T, T]$.

If $t \in (-\infty, \alpha T]$, then since $x_n(t) = \varphi(t)$ for all $t \in (-\infty, 0]$, it follows immediately by the definition of h that

$$h(x_n, \alpha_n)(t) = h(\varphi, \alpha_n)(t), \quad \forall t \leq 0.$$

For each $n \in \mathbb{N}$, let $y_n(.)$ be a selection of $F(., (\bar{x}_n).)$. Then for $t \in [0, \alpha_n T]$,

$$x_n(t) = U(t,0)[\varphi(0) + g(0,\varphi)] - g(t, x_n(t)) + \int_0^t A(s)U(t,s)g(s, x_n(s))\, ds$$
$$+ \int_0^t U(t,s)[y_n(s) + B(s)u(s)]ds + \sum_{i=1}^{m} U(t,t_i)I_i((\bar{x}_n)_{t_i}),$$

Now, x_n converges to x in $\mathcal{S}_{F,\varphi}(u) \subset \mathcal{E}$ implies

$$\sup_{n\in\mathbb{N}} \|(\bar{x}_n)_t\|_{\mathcal{B}} \leq \sup_{n\in\mathbb{N}} \|x_n\|_{\mathcal{E}} \leq R$$

for some $R > 0$. It follows that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is integrably bounded. Furthermore, since for each $t \in [0, T]$

$$\{y_n(t)\}_{n\in\mathbb{N}} \subset F(t, (\bar{x}_n)_t)$$

and by (H3)

$$\chi(F(t, (\bar{x}_n)_t)) \leq \gamma(t) \sup_{-\infty<\sigma\leq 0} \chi(x_n(t + \sigma)),$$

then $\{y_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in X. We may thus assume, in view of Lemma 2.4, that y_n converges weakly to y for some $y \in L^1([0, T], X)$ by passing to a subsequence if necessary. We shall show that $y \in F(., \bar{x}.)$ and

(3.10)

$$\begin{aligned} x(t) = U(t, 0)[\varphi(0) + g(0, \varphi)] - g(t, \bar{x}(t)) + \int_0^t A(s)U(t, s)g\,(s, \bar{x}(s))\,ds \\ + \int_0^t U(t, s)[y(s) + Bu(s)]ds + \sum_{i=1}^m U(t, t_i)I_i((\bar{x})_{t_i}), \quad t \in [0, T] \end{aligned}$$

Thanks to lemma 3.5 and dominated convergence theorem, there exist α_j^n for each $j = 1, 2, ..., k(n)$ such that $\sum_{j=1}^{k(n)} \alpha_j^n = 1$ such that the sequence of convex combinations

$$\eta_n(\cdot) = \sum_{j=1}^{k(n)} \alpha_j^n y_n(\cdot)$$

converges strongly to y in $L^1([0, T], X)$. Since F takes compact and convex values, we have

$$\begin{aligned} y(t) &\in \bigcap_{n\geq 1} \overline{\{\eta_k(t) : k \geq n\}} \\ &\subset \bigcap_{n\geq 1} \overline{\text{co}}\{y_k(t) : k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{\text{co}} \left\{ \bigcup_{k\geq n} F(t, (x_k)_t) \right\} \\ &= \overline{\text{co}} \left(\limsup_{k\to\infty} F(t, (x_k)_t) \right) \end{aligned}$$

where the last equality is guaranteed by Lemma 3.6, yielding

$$y(t) \in \overline{\text{co}}(F(t, x_t)) = F(t, x_t), \quad \text{a.e. } t \in [0, T],$$

since F is upper semi-continuous and takes closed and convex values. The validity of (3.10) follows by (H4) and the Lebesgue dominated convergence theorem.

Finally, if $t \in (\alpha T, T]$, then

$$h(x_n, \alpha_n)(t) = h(x, \alpha)(t) = \xi(t)$$

for n large enough. Consequently,

$$\|h(x_n, \alpha_n) - h(x, \alpha)\|_{\mathcal{E}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Therefore, h is a continuous function, proving that $\mathcal{S}_{F,\varphi}(u)$ is contractible to the point ξ which is the unique solution of problem (3.9). □

Now, we are in the position to state and prove the main result of this section.

Theorem 3.9. *Assume the hypotheses of Theorem 2.14. Then for each $u \in L^p([0,T], U)$, $\mathcal{S}_{F,\varphi}(u)$ is an R_δ-set, provided the inequality (2.4) holds and X is separable.*

Proof. By theorem 3.4, we see that the nonlinear term $F(.,.)$ of inclusion (1.1) is σ-Ca-selectionable, i.e., there exists a decreasing sequence of multivalued maps $\{F_n\}_{n=1}^{\infty}$, $F_n : [0,T] \times \mathcal{B} \to P_{cl,c}(X)$ such that each F_n satisfies the Carathéodory condition and is measurable-locally Lipschitz and

$$F(t, \phi) = \bigcap_{n \geq 1} F_n(t, \phi).$$

Then

$$\mathcal{S}_{F,\varphi}(u) = \bigcap_{n=1}^{\infty} \mathcal{S}_{F_n,\varphi}(u)$$

Since the set $\mathcal{S}_{F_n,\varphi}(u)$ is contractible for each $n \in \mathbb{N}$ by theorem 3.8, then $\mathcal{S}_{F,\varphi}(u)$ is an R_δ set. □

The next result follows immediately from Lemma 3.1 and theorem 3.9.

Corollary 3.10. *Under assumptions (H1)-(H5), the solution multimap $\mathcal{S}_{F,\varphi} : L^p([0,T], U) \multimap C([0,T], X)$ is an R_δ-map, provided the inequality (2.4) holds and X is separable.*

4. Approximate controllability

For the proof of our main results, we shall use the fixed point theorem developed in [24]. For this, we first recall some notions and known results of the so-called AR-space:

Definition 4.1. Let $\mathbb{Y}$ be a metric space.

(i) $\mathbb{Y}$ is called an absolute retract (AR-space) if for any metric space $\mathbb{X}$ and any closed $A \subset \mathbb{X}$, every continuous function $f : A \to \mathbb{Y}$ can be extended to a continuous function $f : \mathbb{X} \to \mathbb{Y}$.

(ii) $\mathbb{Y}$ is called an absolute neighborhood retract (ANR-space) if for any metric space $\mathbb{X}$, any closed $V \subset \mathbb{X}$, and continuous function $f : V \to \mathbb{Y}$, there exists a neighborhood N of V and a continuous extension $f : N \to Y$ of f .

The following result due to L. Gorniewicz and M. Lassonde [24], is crucial to the main result of the present paper.

Theorem 4.2. *Let Y be an AR-space. Assume that $\phi : Y \to 2^Y$ can be factorized as*

$$\phi = \phi_n \circ \phi_{n-1} \circ \dots \phi_1$$

where

$$\phi_i : Y_{i-1} \to 2^{Y_i}, i = 1, 2, \dots, n$$

are R_δ-maps and $Y_i, i = 1, 2, ..., n-1$ are ANR-spaces, $Y_0 = Y_n = Y$ are AR-spaces. If there is a compact set K such that $\phi(Y) \subset K \subset Y$ then ϕ has a fixed point.

We define the control operators $\mathcal{W}_T : L^p([0,T], X) \to X$:

$$\mathcal{W}_T\psi = \int_0^T U(T,s)\psi(s)ds,$$

and propose the following hypothesis which was introduced in [53].

(H6) for each $\psi \in L^p([0,T], X)$, there exists $v \in L^p([0,T], U)$ such that

$$\mathcal{W}_T B v = \mathcal{W}_T \psi.$$

It is obvious that condition (H6) is fulfilled if B is surjective. Now, since g is Y-valued, then $\xi_0 = U(T,0)g(0,\varphi) + g(T, x_T) \in Y$. Choose

$$\theta(s) = \frac{1}{T}(\xi_0 - sA(s)\xi_0) \quad \text{for } s \in [0,T],$$

and then $\theta(\cdot) \in C([0,T], X) \subset L^p([0,T], X)$ satisfying

$$\begin{aligned} & \frac{1}{T}\int_0^T U(T,s)(\xi - sA(s)\xi_0)ds \\ = \;& \frac{1}{T}\int_0^T U(T,s)\xi ds - \frac{1}{T}\int_0^T sU(T,s)A(s)\xi_0 ds \\ = \;& \frac{1}{T}\int_0^T U(T,s)\xi ds + \frac{1}{T}\int_0^T s\frac{dU(T,s)}{ds}\xi_0 ds \\ = \;& \xi_0, \end{aligned}$$

yielding

$$\mathcal{W}_T\theta = \xi_0. \tag{4.1}$$

The following lemma was stated in [53].

Lemma 4.3. *Let condition (H6) hold. Then there exists a continuous map $\mathbf{C} : L^p([0,T], X) \to L^p([0,T], U)$ such that for any $\psi \in L^p([0,T], X)$,*

$$\mathcal{W}_T B\mathbf{C}\psi + \mathcal{W}_T\psi = 0,$$
$$\|\mathbf{C}\psi\|_{L^p([0,T],U)} \le c\|\psi\|_{L^p([0,T],X)}$$

where c is a positive number.

Now, let us denote by $N_{\theta+Ag+F}$ the multivalued Nemytskii operator:

$$N_{\theta+Ag+F} : \mathcal{PC}([0,T], X) \multimap L^p([0,T], X)$$
$$N_{\theta+Ag+F}(x)(t) = \{\theta(t) + A(t)g(t, x_t) + f(t) : f \in F(\cdot, \bar{x}.)\}$$

Obviously, $N_{\theta+Ag+\mathcal{F}}$ is upper semicontinuous. Consider the multimap

$$\Phi : L^p([0,T],U) \multimap L^p([0,T],U)$$
$$\Phi(u) = \mathbf{C}N_{\theta+Ag+F}\mathcal{S}_{F,\varphi}(u_0+u)$$

where $\mathcal{S}_{F,\varphi}$ is the solution multimap and $\mathbf{C}$ is the operator in Lemma 4.3, $u_0 \in L^p([0,T],U)$ is given. The following result is the key point in this section.

Theorem 4.4. *Under assumptions* (H1)-(H6)*, there exists $\rho > 0$ such that Φ has a fixed point in $L^p([0,T],U)$ provided $\|B\| < \rho$.*

Proof. We first prove that there is a number $r > 0$ such that $\Phi(B(0,r)) \subset B(0,r)$, where $B(0,R)$ is the closed ball in $L^p([0,T],U)$ centered at origin with radius r. Let $v \in L^p([0,T],U)$ and $v \in \Phi(u) = \mathbf{C}N_{\theta+Ag+F}\mathcal{S}(u_0+u)$. Then we have

$$\|v\|_{L^p([0,T],U)} \leq c\|N_{\theta+Ag+F}(x)\|_{L^p([0,T],X)}$$

for some $x \in \mathcal{S}(u_0+u)$, where c is the constant given in Lemma 4.3. Hence from the definition of $N_{\theta+Ag+F}$ and assumptions (H1) and (H2), there is an $f \in F(.,x_.)$ such that

$$\begin{aligned}
&\|v\|_{L^p([0,T],U)} \\
\leq\ & c(\|\theta\|_{L^p([0,T],X)} + \|A(\cdot)g(\cdot,x_\cdot)\|_{L^p([0,T],X)} + \|f(\cdot)\|_{L^p([0,T],X)}) \\
\leq\ & c\big[\|\theta\|_{L^p([0,T],X)} + (\|\alpha\|_p + \|\beta\|_p)(1+\|x_s\|_{\mathcal{B}})\big] \\
\leq\ & c\big[\|\theta\|_{L^p([0,T],X)} + (\|\alpha\|_p + \|\beta\|_p)(1+\tilde{K}\|x\|_{C([0,T],X)} + \tilde{M}\|\varphi\|_{\mathcal{B}})\big] \\
\leq\ & c_1 + c_2\|x\|_{C([0,T],X)} \\
\leq\ & c_1 + c_2C_2\left(C_1 + M_0\|B(u_0+u)\|_{L^p([0,T],X)}\right)
\end{aligned}$$

where $c_1 = c\|\theta\|_{L^p([0,T],X)} + c\big[(\|\alpha\|_p + \|\beta\|_p)(1+\tilde{M}\|\varphi\|_{\mathcal{B}})\big]$, $c_2 = c\tilde{K}(\|\alpha\|_p + \|\beta\|_p)$, and the last inequality follows by (2.13). Thus,

$$\|v\|_{L^p([0,T],U)} \leq \rho_0 + \rho_1\|B\|\left(\|u_0\|_{L^p([0,T],U)} + \|u\|_{L^p([0,T],U)}\right)$$

where $\rho_0 = c_1 + c_2C_1C_2$ and $\rho_1 = c_2C_2M_0$. Therefore, if $\|B\| \leq \rho := \frac{1}{\rho_1}$, then there is a number $r > \rho_0 + \rho_1\|B\|\|u_0\|_{L^p([0,T],U)}$ such that

$$\|v\|_{L^p([0,T],U)} \leq r,$$

provided $\|u\|_{L^p([0,T],U)} \leq r$, yielding $\Phi(B(0,r)) \subset B(0,r)$.

Finally, by the compactness of $\mathcal{S}(u_0 + B(0,r))$, the existence of a fixed point for Φ follows from Theorem 4.2 due to the fact that

$$K := \Phi(B(0,r)) = \mathbf{C}N_{\theta+Ag+F}\mathcal{S}(u_0 + B(0,r)) \subset B(0,r)$$

is a compact set. The proof is completed. □

The reachability set of controlled problem (1.1) is defined by

$$\mathcal{R}_F := \{x(T)\,|\, x \in \mathcal{S}_{F,\varphi}(u) \text{ for some } u \in L^p([0,T],U)\}.$$

Similarly, $\mathcal{R}_0$ is the reachability set for the corresponding linear problem:

$$\frac{d}{dt}x(t) = A(t)x(t) + Bu(t), \qquad t \in [0,T], t \neq t_i$$

$$\Delta x(t_i) = I_i(x_{t_i}), \qquad i = 1, 2, \dots, m, \tag{4.2}$$
$$x(0) = \varphi(0).$$

In other words,

$$\mathcal{R}_0 = \Big\{ x(T) \,\Big|\, x(t) = U(t,0)\varphi(0) + \int_0^t U(t,s)Bu(s)ds + \sum_{i=1}^m U(t,t_i)I(x_{t_i})$$
$$\text{for all } t \in [0,T] \text{ and some control } u \in L^p([0,T],U) \Big\}$$

The readers are referred to [19] and [63] for the basic notions and facts of control problems. Now, it is time to introduce the notion of apprximate controllability of evolution system.

Definition 4.5. *Problem (1.1) is said to be exactly controllable if $\mathcal{R}_F = X$. It is called approximately controllable if $\overline{\mathcal{R}_F} = X$. The approximate controllability of (4.2) is defined likewise.*

It is easy to see taht (4.2) is approximately controllable if and only if $\overline{R(\mathcal{W}_T)} = X$. The follwing result can be shown using similar arguments as in [19, 61, 63].

Theorem 4.6. *The linear impulsive system (4.2) is approximately controllable if $(\mathcal{W}_T)^*x^* = 0$ for all $x^* \in X^* \Longrightarrow x^* = 0$, i.e., if $(\mathcal{W}_T)^*$ is injective.*

We are now in the position to prove the main result of the present article:

Theorem 4.7. *Under the hypotheses of Throrem 4.6, Problem (1.1) is approximately controllable provided the correspoding linear problem (4.2) is approximately controllable, i.e., provided the conditions of Theorem 4.6 hold.*

Proof. Suppose that the corresponding linear problem (4.2) is approximately controllable. It suffices to show that $\mathcal{R}_0 \subset \mathcal{R}_F$. In fact, for each $\xi \in \mathcal{R}_0$, there exists $u_0 \in L^p([0,T],U)$ such that

$$\xi = U(T,0)\varphi(0) + \mathcal{W}_T Bu_0(\cdot)$$

Take a fixed point $\hat{u} \in \Phi(\cdot) = \mathbf{C}N_{\theta+Ag+F}\mathcal{S}(u_0 + \cdot)$ and set

$$u := u_0 + \hat{u}.$$

Choose $\hat{x} \in \mathcal{S}(u)$ and it is ready to see that

$$\hat{u} = \mathbf{C}N_{\theta+Ag+F}\mathcal{S}(\hat{x})$$

whence by (4.1) and Lemma 4.3

$$\begin{aligned}
\hat{x}(T) &= U(T,0)\varphi(0) + U(T,0)g(0,\varphi) + g(T,x_T) \\
&\quad + \mathcal{W}_T\big(A(\cdot)g(\cdot,\hat{x}.) + f(\cdot,\hat{x}.)\big) + \mathcal{W}_T Bu(\cdot) \\
&= U(T,0)\varphi(0) + \mathcal{W}_T\theta + \mathcal{W}_T\big(A(\cdot)g(\cdot,\hat{x}.) + f(\cdot,\hat{x}.)\big) + \mathcal{W}_T B\big(u_0(\cdot) + \hat{u}(\cdot)\big) \\
&= \big(U(T,0)\varphi(0) + \mathcal{W}_T Bu_0(\cdot)\big) + \mathcal{W}_T N_{\theta+Ag+F}(\hat{x}) + \mathcal{W}_T B\hat{u}(\cdot) \\
&= \xi + \mathcal{W}_T N_{\theta+Ag+F}(\hat{x}) + \mathcal{W}_T B\mathbf{C}N_{\theta+Ag+F}\mathcal{S}(\hat{x})
\end{aligned}$$

$$= \xi$$

for some integrable selection $f \in F$. Thus, $\mathcal{R}_0 \subset \mathcal{R}_F$ and the proof is completed. □

5. An example

Let Ω be a bounded domain in $\mathbb{R}^3$. In this section, we consider the following distributed control problem with a given profile in $L^2(\Omega)$ which arises in the study of heat flow in materials with memory:

$$\frac{\partial}{\partial t}\left[k_0 x(t,\omega) + \int_{-\infty}^{t} k_1(t-s)x(s,\omega)ds\right] + k_\infty(t,\omega)x(t,\omega)$$

$$= c_0\Delta x(t,\omega) + \int_{-\infty}^{t} k_2(t-s)Q(t,x(s,\omega))ds + b(\omega)u(t,\omega), \qquad \text{on } [0,T]\times\Omega, \tag{5.1}$$

$$x(t,\omega) = 0, \qquad \text{on } [0,T]\times\partial\Omega, \tag{5.2}$$

$$x(\sigma,\omega) = \varphi(\sigma,\omega) \in \mathcal{B}, \qquad \sigma \le 0,\ \xi \in \Omega, \tag{5.3}$$

$$\Delta x(t_i,\cdot) = x(t_i^+,\cdot) - x(t_i^-,\cdot) = \int_{\Omega} p_i(x(t_i,\omega),\cdot)d\omega, \tag{5.4}$$

where $x(t,\omega)$ represents the temperature of the point ω at time t, $k_0 > 0$ and $c_0 > 0$ are respectively the heat capacity and thermal conductivity, k_1 is nonnegative and measurable on $[0,\infty)$, $k_2 \in L^1(0,\infty)$ and nonincreasing,

$$k_\infty \in C^\alpha([0,T],X), \tag{5.5}$$

$Q : [0,T]\times\mathbb{R} \to P_{c,k}(\mathbb{R})$ is a multivalued function satisfying the Carathéodory condition, $b \in L^2(\Omega)$ is a given function and $u : [0,T]\times\Omega$ satisfies $u(t,\cdot) \in L^2(\Omega)$ for all $t \in [0,T]$.

Let $X = L^2(\Omega)$, $Y = W^{2,2}(\Omega)\cap W_0^{1,p}(\Omega)$ and consider the linear operator B defined on X by

$$(B\xi)(\omega) := b(\omega)\xi(\omega), \quad \xi \in X.$$

For simplicity, we choose $c_0 = k_0 = 1$. Define $A(t), t \in [0,T]$ in $L^2(\Omega)$ by

$$(A(t)\psi)(\omega) = (\Delta\psi)(\omega) - k_\infty(t,\omega)\psi(\omega), \quad \omega \in \Omega,$$

on the common domain $D(A(t)) := Y$. By Lunardi [40] (Theorem 3.1.3(ii), p.73), $A(t)$ are sectorial operators in $L^2(\Omega)$, and (5.5) implies that $A(\cdot) \in C^\alpha([0,T], L(Y,X))$. Therefore the following parabolic nonautonomous system

$$\begin{cases} x'(t) = A(t)x(t), & t \ge s, \\ x(s) = x_0 \in X. \end{cases}$$

has an associated evolution family $\{U(t,s)\}_{0\le s<t\le T}$ on X such that $U(t,s)X \subset Y$ and $\|U(t,s)\| \le M_0$ for all $0 \le s < t \le T$ and some $M_0 > 0$ (see [22,40]). Moreover, by the Rellick-Kondrachov theorem, the embedding $Y \hookrightarrow X$ is compact and it thus follows the compactness of $U(t,s)$. We take the phase space

$$\mathcal{B} = C^\infty = \{\phi \in C((-\infty,0],X) : \lim_{\sigma\to-\infty}\phi(\sigma) \text{ exists in } X\}$$

endowed with the norm

$$\|\phi\|_{\mathcal{B}} = \|\phi\|_\infty = \sup_{-\infty<\sigma\le 0} \|\phi(\sigma)\|, \quad \phi \in C^\infty.$$

It is known (see [32], p.14) that C^∞ satisfies the axioms (A), (A1) and (B) with

$$H = 1, \quad K(t) = M(t) = 1 \text{ for } t \ge 0. \tag{5.6}$$

To apply our abstract results, we set $\mathscr{B} := \{\varphi \in \mathcal{B} : \varphi(\sigma) \in Y \text{ for all } \sigma \le 0\}$, and for all $t \in [0, T]$, define respectively that

$$g(t, \phi) := \int_0^\infty k_1(s)\phi(-s, \cdot)ds, \quad \phi \in \mathcal{B}, \tag{5.7}$$

$$F(t, \phi) := \int_0^\infty k_2(s)Q(t, \phi(-s, \cdot))ds - A(t)g(t, \mathbf{1}_{\mathscr{B}}(\phi)), \quad \varphi \in \mathcal{B}, \tag{5.8}$$

where $\mathbf{1}_{\mathscr{B}}$ is the characteristic function of $\mathscr{B}$, and

$$I_i(\phi) = \int_\Omega p_i(\phi(0, \omega), \cdot)d\omega, \quad \phi \in \mathcal{B}. \tag{5.9}$$

Then it is already to see that the system (5.1-(5.4) is transformed into

$$\begin{aligned} \frac{d}{dt}[x(t) + g(t, x_t)] \in A(t)x(t) + F(t, x_t) + Bu(t), \qquad & t \in [0, T], t \ne t_i \\ \Delta x(t_i) = I_i(x_{t_i}), \qquad & i = 1, 2, \ldots, m, \\ x_0 = \varphi \in \mathcal{B}, \qquad & \end{aligned}$$

by letting $[x(t)](\omega) := x(t, \omega)$ for $t \in [0, T]$ and $[\varphi(\sigma)](\omega) := \varphi(\sigma, \omega)$ for $\sigma \le 0, \omega \in \Omega$. It is clear from (5.7) that $g(t, \varphi) \in Y$ whenever $\varphi \in \mathscr{B}$. To show that our abstract result can be applied to this system, we impose some assumptions:

(C1) $k_1, k_2 \in L^2(0, \infty)$.

(C2) $Q : [0, T] \times \mathbb{R} \to P_{c,k}(\mathbb{R})$ is a multimap satisfying the Carathéodory condition and there exist a function $m \in L^1([0, T])$ and a constant κ such that

$$\|Q(t, \tau)\| \le \kappa m(t)|\tau|, \quad 0 \le t \le T, \tau \in \mathbb{R}.$$

(C3) The functions $p_i : \mathbb{R} \times \Omega \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are continuous, and there are positive constants l_i such that

$$|p_i(\tau_1, \omega) - p_i(\tau_2, \omega)| \le l_i|\tau_1 - \tau_2|, \quad \omega \in \Omega, \ \tau_1, \tau_2 \in \mathbb{R}.$$

We shall prove that g and F satisfies assumptions (H1)-(H3). In fact

$$\|A(t)g(t, \varphi)\| = \left\| \int_0^\infty k_1(s)A(t)\varphi(-s)ds \right\| \le \alpha(t)(1 + \|\varphi\|_{\mathscr{B}})$$

where $\alpha(t) := \|k_1\|_2 \|A(t)\|_{L(Y,X)}$. Choose a selection $q(\cdot, \cdot)$ in $Q(\cdot, \cdot)$, and $\varphi \in \mathcal{B}$ and then

$$\|F(t, \varphi)\|^2 = \int_\Omega \left(\int_{-\infty}^0 k_2(-s)q(t, \varphi(s, \omega))ds \right)^2 d\omega$$

$$\leq \kappa \int_{\Omega} \left(\int_{-\infty}^{0} k_2(-s)m(t)|\varphi(s,\omega)|ds \right)^2 d\omega$$
$$\leq (\kappa\|k_2\|_2 m(t)\|\phi\|_{\mathcal{B}})^2 ,$$

yielding

$$\|F(t,\varphi)\| \leq \beta(t)(1+\|\varphi\|_{\mathcal{B}}),$$

where $\beta(t) := \kappa\|k_2\|_2 m(t)$. This shows that F satisfies (H3). Moreover, it follows by (5.9), condition (C3) and axiom (A)(ii), we have

$$\begin{aligned}\|I_1(\phi) - I_i(\psi)\| &\leq \int_{\Omega}\int_{\Omega} |p_i(\phi(0,\xi),\omega) - p_i(\psi(0,\xi),\omega)|d\xi d\omega \\ &\leq l_i \int_{\Omega}\int_{\Omega} |\phi(0,\xi) - \psi(0,\xi)|d\xi d\omega \\ &\leq |\Omega| l_1 \|\phi(0) - \psi(0)\| \\ &\leq |\Omega| l_1 \|\phi - \psi\|_{\mathcal{B}},\end{aligned}$$

where $|\Omega|$ is the volume of Ω.

Let

$$\mathbf{B}u := \int_0^T U(T,s)Bu(s)ds$$

and $\mathbf{B}^*$ be the adjoint operator of $\mathbf{B}$. The next result is a consequence of Theorem 4.6 and 4.7.

Theorem 5.1. *Suppose that the conditions (C1)-(C3) are satisfied, $\|b\|_{L^2(\Omega)}$ is sufficiently small, and*

$$M_0|\Omega|\sum_{i=1}^{m} l_i < 1.$$

If the implication

$$\mathbf{B}^* x^* = 0 \ for \ x^* \in X^* \Longrightarrow x^* = 0$$

holds, then the system (5.1)-(5.4) is approximately controllable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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