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RECOLLECTING BEST APPROXIMATION THEOREMS IN ABSTRACT CONVEX SPACES

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Abstract. In this paper we introduce some best approximation theorems for multimaps in abstract convex spaces. Our results extend a recent work of Mitrović, Hussain, de la Sen, and Radenović [8] on G-convex spaces. As applications, they derive results on the best approximations in hyperconvex and normed spaces. Consequently, our results generalize many known ones in the literature.

1. Introduction

A generalized convex space or G-convex space was first introduced by S. Park and H. Kim in 1996 and the monotonicity of its definition was removed later. Recently Z. D. Mitrović, A. Hussain, M. de la Sen, and S. Radenović [8] stated that, in G-convex spaces, many results were obtained in nonlinear analysis in 24 references. The aim of [8] was to obtain the best approximation theorem in G-convex spaces and to generalize theorems of A. Amini-Harandi and A. P. Farazadeh; W. A. Kirk, B. Sims and G. X. Z. Yuan; M. A. Khamsi; and S. Park. Also they obtained that almost quasi-convex and almost affine conditions is unnecessary in results of J. B. Prolla [25] and A. Carbone [1].

Recall that we initiated the KKM theory in 1992 and noticed that G-convex spaces and many others are contained in abstract convex spaces and (partial) KKM spaces in that theory [15]. In our previous work [20], we extended the contents of [8] on G-convex spaces to partial KKM spaces, and introduce new partial KKM metric spaces which can be applied our new results. The present work is a continuation of [20] to recall many related works on best approximations.

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained an intersection theorem which is known to be equivalent to the Brouwer fixed point theorem in 1912, the weak Sperner combinatorial lemma in 1928, and many other important theorems, see [18] . The KKM theory is first named by ourselves in 1992 as the study of applications of extensions or equivalents of the KKM theorem. Nowadays the theory is mainly concerned with abstract convex spaces and (partial) KKM spaces due to ourselves and the realm of the theory is very broad; see [18].

One of the topics in the KKM theory is related to generalized KKM maps initiated by Kassay-Kolumbán in 1990 [5] and Chang-Zhang in 1991 [2]. Since then many authors studied generalized KKM maps on various types of spaces and applied them

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to extend or refine well-known previous results. In fact, it has been followed by Chang-Ma in 1993, Yuan in 1995, Cheng in 1997, Tan in 1997, Lin-Chang in 1998, Lee-Cho-Yuan in 1999, Kirk-Sims-Yuan in 2000 for various classes of abstract convex spaces; see [17]. All of those authors applied their results on KKM type theorems and others to extend or refine well-known previous results in the KKM theory; for example, variational or quasi-variational inequalities, fixed point theorems, the Ky Fan type minimax inequalities, the von Neumann type minimax or saddle point theorems, Nash equilibrium problems, and others.

In our previous review [17], we gave a unified account for generalized KKM maps on abstract convex spaces in the works of Kim-Park [6], Lee [7], and Park-Lee [23]. We were mainly concerned with results closely related to the KKM type theorems and characterizations of generalized KKM maps on various types of abstract convex spaces. In short, we showed that generalized KKM maps can be reduced to the usual KKM maps in our abstract convex spaces. Some related topics were also added in [17].

Recently, Balaj and his colleagues extended generalized KKM maps w.r.t. a multimap to weak KKM maps and generalized weak KKM maps w.r.t. a multimap, and applied them to various problems in the KKM theory; see [21]. However, their works are mainly concerned within the realm of topological vector spaces.

Recall that we have recently established the Grand KKM Theory mainly on abstract convex spaces; see [19]. Since the recent results of Balaj and his colleagues in [21] are on topological vector spaces or G-convex spaces, it is better to extend them to abstract convex spaces. Our aim in this article is simply to try this task.

This article is organized as follows: Section 2 devotes preliminary for concepts on abstract convex spaces, partial KKM spaces and their subclasses in our previous works. In Section 3, we obtain generalizations of the main theorem and two corollaries of [8] to our partial KKM spaces. Finally, Section 4 deals with our studies related to best approximation theorems in chronological order such that some of them imply results of [8].

2. Preliminaries on abstract convex spaces

For the concepts on abstract convex spaces, partial KKM spaces and their subclasses, we follow [14, 15, 18] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D\rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D\rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D\rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

Definition 2.2. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map*.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement that the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

The (partial) KKM principle has a large number of equivalent formulations and, for example, the following are equivalent [15]:

(0) The partial KKM principle. *For any closed-valued [resp. open-valued] KKM map $G : D \multimap E$, the family $\{G(z) : z \in D\}$ has the finite intersection property.*

(I) The Fan matching property. *Let $S : D \multimap E$ be a map satisfying*

(i) *$S(z)$ is open [resp. closed] for each $z \in D$; and*

(ii) *$E = \bigcup_{z \in M} S(z)$ for some $M \in \langle D \rangle$.*

Then there exists an $N \in \langle M \rangle$ such that

$$\Gamma_N \cap \bigcap_{z \in N} S(z) \neq \emptyset.$$

There are plenty of examples of KKM spaces; see [18] and the references therein.

Now we have the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}$$
$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space}$$
$$\Longrightarrow \text{KKM space} \Longrightarrow \text{Partial KKM space}$$
$$\Longrightarrow \text{Abstract convex space.}$$

KKM maps on abstract convex spaces have at least two types of generalizations. The first one is as follows:

Definition 2.4. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map with respect to F*. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{KC}\text{-map}\}.$$

Similarly, $\mathfrak{KO}(E, Z)$ is defined.

The second and more earlier generalization of KKM maps are generalized KKM maps initiated by Kassay-Kolumbán in 1990 [5] and Chang-Zhang in 1991 [2].

Definition 2.5. Let $(X, D; \Gamma)$ be an abstract convex space and Y be a nonempty set such that, for each $A \in \langle Y\rangle$, there exists a function $\sigma_A : A \to D$. Then a new abstract convex space $(X, A; \Lambda_A)$ induced by Γ and A is defined by the following

$$\Lambda_A(J) := \Gamma(\sigma_A(J)) \text{ for each } J \subset A.$$

Moreover, a multimap $T : Y \multimap X$ (called a *generalized KKM map*) reduces to a KKM map on $(X, A; \Lambda_A)$ for each $A \in \langle Y\rangle$ satisfying $\Lambda_A(J) \subset T(J)$ for each $J \subset A$.

The following characterization of generalized KKM maps extends Theorem 2 of Park and Lee [24], which was stated for G-convex spaces:

The following characterization of generalized KKM maps given in [17] extends many other previously given versions by other authors:

Theorem 2.6. *Let $(X, D; \Gamma)$ be a partial KKM space [resp. KKM space], Y a nonempty set, and $T : Y \multimap X$ a map with closed [resp. open] values.*

(i) *If T is a generalized KKM map, then the family of its values has the finite intersection property.*
(ii) *The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.*

H. Kim and S. Park in [6] (Theorem 3) obtained the following extension of KKM theorems of Ky Fan [16]:

Corollary 2.7. *Let $(X, D; \Omega)$ be a G-convex space, S a nonempty set and $\Phi : S \multimap X$ a multifunction with closed (resp. open) values. If Φ is a generalized KKM map, then the family of its values has the finite intersection property (More precisely, for each $T \in \langle S\rangle$, there exists and $T' \in \langle D\rangle$ such that $\Omega_{T'} \cap \bigcap_{t\in T} \Phi(t) \neq \emptyset$.)*

In this paper we use the following Corollary of Theorem 2.6.

Theorem 2.8. *Let $(X; \Gamma)$ be a partial KKM space, S a nonempty set and $F : S \multimap X$ a generalized KKM map with closed values. If there exists a nonempty compact subset K of X such that $\bigcap_{t\in T} F(t) \subset K$ for some $T \in \langle S\rangle$ then $\bigcap_{s\in S} F(s) \neq \emptyset$.*

The following [8, Theorem 2] is used as a basis of [8]:

Corollary 2.9. *Let $(X; \Omega)$ be a G-convex space, S a nonempty set and $\Phi : S \multimap X$ a generalized KKM map with closed values. If there exists a nonempty compact subset L of X such that $\bigcap_{t\in T} \Phi(t) \subset L$ for some $T \in \langle S\rangle$ then $\bigcap_{s\in S} \Phi(s) \neq \emptyset$.*

3. Best approximation theorems

Let $r \in [0, +\infty)$, X be a metric space and $\emptyset \neq S \subset X$, we denote *r-parallel set* of S by

$$S + r = \bigcup \{B(s, r) : s \in S\},$$

where $B(s, r) = \{t \in X : d(s, t) \leq r\}$.

For nonempty subsets S and T of X, we define

$$d(S, T) = \inf\{d(s, t) : s \in S,\ t \in T\}.$$

We say that a set K is *metrically convex* if for any $x, y \in K$ and positive numbers p_i and p_j such that $d(x, y) \leq p_i + p_j$, there exists $z \in K$ such that $z \in B(x, p_i) \cap B(y, p_j)$.

From Theorem 2.8 we obtain the following generalization of [8, Theorem 3] on best approximation theorem in abstract convex space.

Theorem 3.1. *Let a metric space $(X; \Gamma)$ be a partial KKM space with a metric d, S a nonempty Γ-convex subset of X, $F : S \multimap X$ is a continuous multimap with compact values such that*

$$F(x) + r \ is\ \Gamma - convex\ for\ all\ x \in S.\ r \geq 0 \tag{3.1}$$

and $g : S \to S$ is a continuous surjection. If there exists a nonempty compact subset K of X such that

$$\bigcap_{y \in M} \{x \in S : d(g(x), F(x)) \leq d(g(y), F(x))\} \subset K\ for\ some\ M \in \langle S \rangle, \tag{3.2}$$

then there exists $v_0 \in S$ such that

$$d(g(v_0), F(v_0)) = \inf_{x \in S} d(x, F(v_0)).$$

If S is metrically convex and $g(v_0) \notin F(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

Proof. Just follow the proof of [8, Theorem 3] and apply Theorem 2.8 instead of [8, Theorem 2]. □

Similarly, two Corollaries of [8, Theorem 3] can be improved as follows.

Corollary 3.2. *Let the metric space $(X; \Gamma)$ be a partial KKM space with metric d, S a nonempty Γ-convex subset contained in a compact subset of X, $F : S \multimap X$ is a continuous multimap with compact values such that condition (1) is satisfied and $g : S \to S$ is a continuous surjection. Then there exists $v_0 \in K$ such that*

$$d(g(v_0), F(v_0)) = \inf_{x \in S} d(x, F(v_0)).$$

If K is metrically convex and $g(v_0) \notin F(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

Corollary 3.3. *Let the metric space $(X; \Gamma)$ be a partial KKM space with metric d, S a nonempty Γ-convex subset contained in a compact subset of X, $F : S \multimap X$ is* *a continuous multimap with compact values such that condition (1) is satisfied and $g : S \to S$ is a continuous surjection. Then there exists $v_0 \in K$ such that*

$$d(g(v_0), F(v_0)) = \inf_{x \in S} d(x, F(v_0)).$$

If K is metrically convex and $v_0 \notin F(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

4. Some applications

In Section 3 of [8], as some applications of the authors' results, they gave the versions of Fan's best approximation theorems in hyperconvex and normed spaces.

Proposition PC. *Let X be a normed linear space, S a nonempty compact convex subset of X, $f : S \to X$ is a continuous map and $g : S \to S$ is a continuous surjection. Then there exists $v_0 \in S$ such that*

$$||g(v_0) - f(v_0)|| = \inf_{x \in S} d(x, f(v_0))$$

whenever

(P) *g is almost affine* (Prolla [25]) *or*

(C) *g is almost quasi-convex* (Carbone [1]).

Mitrović et al. [8] showed that their results imply that Proposition holds without assuming (P) and (C).

In the present section, we recall our study related to best approximation theorem in chronological order such that some of them imply Proposition PC without assuming (P) and (C).

(I) In 1987 [9], we give further generalizations of Ky Fan's theorem. In a normed vector space E, for any $X \subset E$, $\overline{I}_X(x)$ denotes the *weakly inward set* of X at $x \in E$, that is, the closure of the *inward set*

$$I_X(x) = \{x + r(u - x) \in E : \ u \in X, \ r > 0\}.$$

Theorem 4.1. *Let X be a nonempty convex subset of a normed vector space E, $f, g : X \to E$ continuous maps satisfying*

$$||gx - fx|| \leq ||x - fx|| \quad for\ all\ x \in X.$$

Let K be a nonempty compact subset of X and X_0 a nonempty subset of X contained in some precompact convex subset of X such that for each $x \in X \setminus K$, there exists a point $y \in X_0$ satisfying

$$||gx - fx|| > ||y - fx||.$$

Then there is an $x_0 \in K$ such that

$$||gx_0 - fx_0|| \leq ||z - fx_0|| \ for\ all\ z \in \overline{I}_X(gx_0).$$

More precisely, either

(a) *f and g have a coincidence point $x_0 \in K$, or*

(b) *there is an $x_0 \in K$ such that $gx_0 \in \mathrm{Bd}\, X$ and*

$$0 < ||gx_0 - fx_0|| \leq ||z - fx_0|| \ for\ all\ z \in \overline{I}_X(gx_0).$$

This implies Proposition PC without assuming (P) or (C).

(II) In 1993 [22], we obtained two most general forms of Prolla's theorem

The following is a generalization of Prolla's theorem for two spaces and two maps related by a very mild coercivity condition:

Theorem 4.2. *Let X be a convex space, K a nonempty compact subset of X, $(E, ||\cdot||)$ a normed vector space, and $f, g : X \to E$ continuous maps. Suppose that*

(1) *g is almost quasiconvex; and*
(2) *for each $N \in \langle X \rangle$ there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \backslash K$ there exists a $y \in L_N$ such that $||gx - fx|| > ||gy - fx||$.*

Then there exists an $x_0 \in K$ such that

$$||gx_0 - fx_0|| = d(fx_0, g(X)).$$

Further, if $g(X)$ is convex, then

$$||gx_0 - fx_0|| = d(fx_0, \overline{I}_{g(X)}(gx_0));$$

more precisely, either

(a) *f and g have a coincidence point $x_0 \in K$; or*
(b) *there is an $x_0 \in K$ such that $gx_0 \in \mathrm{Bd}\ g(X)$ and*

$$0 < ||gx_0 - fx_0|| = d(fx_0, \overline{I}_{g(X)}(gx_0)).$$

This generalizes Carbone's theorem [1].

(III) Motivated by some best approximation or fixed point theorems for maps whose domains and ranges have different topologies, in 1994 [10], we generalize, improve, and unify known Fan or Prolla type best approximation theorems for single-valued maps.

The following is a general Fan type best approximation theorem and a variant of the Prolla type.

Theorem 4.3. *Let X be a convex space (in the sense of Lassonde), K a nonempty compact subset of X, (E, p) a seminormed vector space containing X as a subset, and $f, g : X \to (E, p)$ continuous maps. Suppose that*

(1) *$p(gx - fx) \leq p(x - fx)$ for all $x \in X$; and*
(2) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \setminus K$, there exists a $y \in L_N$ satisfying $p(gx - fx) > p(y - fx)$.*

Then there exists an $x_0 \in K$ such that

$$p(gx_0 - fx_0) \leq d_p(fx_0, X).$$

Moreover, if $gx_0 \in X$, then we have

$$p(gx_0 - fx_0) = d_p(fx_0, \overline{I}_X(gx_0)).$$

In this case, $gx_0 \in \mathrm{Bd}\, X$ and $fx_0 \notin \overline{I}_X(gx_0))$ whenever $p(gx_0 - fx_0) > 0$.

In this article, several applications to existence of fixed or coincidence points are also added.

(IV) In 1995 [11], we obtained best approximation theorems for admissible multimaps in the sense of Park. Let $(E;\tau)$ be a Hausdorff topological vector space and $(X;w)$ a weakly compact convex subset of E with the relative weak topology w. Let $S(E;w)$ the family of all continuous seminorms on $(E;w)$. Let $F \in \mathfrak{A}_c^\kappa(C,E)$, the admissible multimap in the sense of Park.

The following is [11. Theorem 2]:

Theorem 4.4. *Let C be a compact convex space, E a Hausdorff topological vector space containing C as a subset, and $F \in \mathfrak{A}_c^\kappa(C,E)$. Then, for each $p \in S(E,w)$ satisfying*

$$(x,y) \mapsto p(x-y) \text{ is continuous for } (x,y) \in C \times E,$$

there exists an $(x_0,y_0) \in F$ satisfying

$$p(x_0 - y_0) = d_p(y_0, \overline{I}_C(x_0)).$$

Moreover, $x_0 \in \operatorname{Bd} C$ whenever $p(x_0 - y_0) > 0$.

(V) In 1996 [12], the following is a main result:

Theorem 4.5. *Let (C,σ) be a convex space, $E = (E,\tau)$ a Hausdorff topological vector space containing C as a subset, $F : (C,\sigma) \to 2^E$, $p \in S(E,w)$, $Q_p : E \to 2^{(C,\tau)}$ the metric projection, and $g : (C,\tau) \to E$ a continuous map such that $C \subset g(C)$. Suppose that either*

(I) *E^* separates points of E, (C,σ) is compact, $g^{-1}Q_pF \in \mathfrak{A}_c^\kappa((C,\sigma),(C,\tau))$, and for each $q \in S(E,w)$, $(x,y) \mapsto q(x-y)$ is continuous on $(x,y) \in (C,\sigma) \times E$; or*

(II) *E is locally convex, $\sigma = \tau$ on C, and $g^{-1}Q_pF \in \mathfrak{A}_c^\sigma((C,\tau),(C,\tau))$ is compact.*

Then there exists an $(x_0,y_0) \in F$ such that

$$p(gx_0 - y_0) = d_p(y_0, C).$$

Moreover, if $gx_0 \in C$, then we have

$$p(gx_0 - y_0) = d_p(y_0, \overline{I}_C(gx_0)).$$

In this case, $gx_0 \in \operatorname{Bd} C$ and $y_0 \notin \overline{I}_C(gx_0)$ whenever $p(gx_0 - y_0) > 0$.

(VI) In 1997 [13], for a closed convex subset K of a Hilbert space X, the proximity map $p = p_K : X \to K$ is defined by

$$px = \{y \in K : \|x-y\| = d(x,K)\} \quad \text{for} \quad x \in X,$$

where $d(x,K) = \inf_{y \in K} \|x-y\|$. It is well-known that p is well-defined and nonexpansive.

We begin with the following best approximation theorem:

Theorem 4.6. *Let K be a closed convex subset of a Hilbert space X, $f : K \to X$ a nonexpansive map, and $p : X \to K$ the proximity map. Suppose that one of the following holds:*

(i) *K is bounded.*

(ii) *$f(K)$ is bounded.*

(iii) *There exists a $u \in K$ such that the set*

$$G(u, fu)^* := \{z \in K : \|z - fu\| \le \|z - u\| + d(fu, K)\}$$

is bounded.

(iv) *There exists a $u \in K$ such that the set*

$$LS(u, (pf)u) := \{z \in K : \mathrm{Re}\,\langle (pf)u - u,\ z - u\rangle > 0\}$$

is bounded.

(v) *There exists a $u \in K$ such that the set*

$$G(u, (pf)u) = \{z \in K : \|z - (pf)u\| \le \|z - u\|\}$$

is bounded.

(vi) *There exists a bounded set $A \subset K$ such that the set*

$$G(A) = \bigcap_{u \in A} G(u, (pf)u)$$

is either empty or bounded.

Then either f has a fixed point in K; or there exists an $x_0 \in \mathrm{Bd}\, K$ such that

$$0 < \|x_0 - fx_0\| = d(fx_0, K) = d(fx_0, \overline{I}_K(x_0)).$$

(VII) In 1999 [4], we begin with the following basis of best approximation or fixed point theorems for continuous multifunctions:

Theorem 4.7. *Let X be a convex space, $K \in k(X)$, L a c-compact subset of X, E a t.v.s. containing X as a subset such that the inclusion $i_X : X \hookrightarrow E$ is continuous, and $F : X \to kc(E)$. Let $p : X \times E \to [0, \infty)$ be a function such that, for each $x \in X$, $p(x, \cdot)$ is a continuous convex function on E. Suppose that*

(1) *for each $y \in X$,*

$$\{x \in X : \inf_{u \in Fx} p(x, x - u) \le \inf_{v \in Fx} p(x, y - v)\}$$

is compactly closed in X; and

(2) *for each $x \in X \backslash K$ with $\inf_{u \in Fx} p(x, x - u) > 0$, there exist a $z \in \overline{I}_L(x)$ and a $v \in Fx$ such that*

$$p(x, z - v) < \inf_{u \in Fx} p(x, x - u).$$

Then either

(a) *there exist an $\hat{x} \in X \backslash K$ and a $\hat{u} \in F\hat{x}$ such that $p(\hat{x}, \hat{x} - \hat{u}) = 0$; or*

(b) *there exist an $\hat{x} \in K$ and a $\hat{u} \in F\hat{x}$ such that*

$$p(\hat{x}, \hat{x} - \hat{u}) \le p(\hat{x}, y - v)$$

for all $y \in \overline{I}_X(\hat{x})$ and all $v \in F\hat{x}$.

(VIII) In 1997 [3], the following Browder type best approximation theorem is our main result.

Theorem 4.8. *Let X be a convex space, K a nonempty compact subset of X, E a topological vector space, $p : X \times E \to \mathbb{R}$ a continuous map, and $f, g : X \to E$ continuous maps. Suppose that*

(1) *for each $x \in X, p(x, \cdot)$ is a convex function on E;*
(2) *g is almost p-quasiconvex; and*
(3) *for each $N \in \langle X \rangle$, there exists a compact convex subset L_N of X containing N such that, for each $x \in L_N \setminus K$, there exists a $y \in L_N$ such that*

$$p(x, fx - gx) > p(x, fy - gx).$$

Then there exists an $u \in K$ such that

$$p(u, fu - gu) = \inf\{p(u, fu - gu) : x \in X\}.$$

Further, if $g(X)$ is convex, then

$$p(u, fu - gu) = \inf\{p(u, fu - z) : z \in \mathrm{Cl}\, I_{g(X)}(gu)\}.$$

References

[1] A. Carbone, *An application of KKM-map principle*, Intern. J. Math. & Math. Sci. **15** (1992), 659–662.

[2] S.-S. Chang and Y. Zhang, *Generalized KKM theorem and variational inequalities*, J. Math. Anal. Appl. **159** (1991), 208–223.

[3] M.-P. Chen and S. Park, *On the Browder type best approximation theorems*, Indian J. Math. **39** (1997), 91–97.

[4] M.-P. Chen and S. Park, *Best approximations and fixed points of multimaps whose domains and ranges have different topologies*, Commun. Appl. Nonlinear Anal. **6** (1999), 77–101.

[5] G. Kassay and I. Kolumbán,*On the Knaster-Kuratowski-Mazurkiewicz and Ky Fan's theorem*, Babes-Bolyai Univ. Res. Seminars Preprint **7** (1990), 87–100.

[6] H. Kim and S. Park, *Generalized KKM maps, maximal elements and almost fixed points,* J. Korean Math. Soc. **44** (2007), 393–406.

[7] W. Lee, *A remark on generalized KKM maps on KKM spaces*, Nonlinear Anal. Forum **21** (2016), 1–5.

[8] Z. D. Mitrović, A. Hussain, M. de la Sen and S. Radenović, *On best approximations for set-valued mappings in G-convex spaces*, MDPI Mathematics 2020, 8, 347pp.

[9] S. Park, *On generalizations of Ky Fan's theorems on best approximations*, Numer. Funct. Anal. and Optim. **9** (1987), 619–628.

[10] S. Park, *Remarks on generalizations of best approximation theorems*, Honam Math. J. **16** (1994), 27–39.

[11] S. Park, *Best approximation theorems for composites of upper semicontinuous maps*, Bull. Austral. Math. Soc. **51** (1995), 263–272.

[12] S. Park, *Remarks on set-valued generalizations of best approximation theorems*, Kyungpook Math. J. **35** (1996), 771–782.

[13] S. Park, *Best approximations and fixed points of nonexpansive maps in Hilbert spaces*, Numer. Funct. Anal. Optim. **18** (1997), 649–657.

[14] S. Park, *Remarks on weakly KKM maps in abstract convex spaces*, Inter. J. Math. Math. Sci. **2008** (2008), Article ID 423596.

[15] S. Park, *The KKM principle in abstract con spaces: Equivalent formulations and applications*, Nonlinear Anal. **73** (2010), 1028–1042.
[16] S. Park, *Evolution of the 1984 KKM theorem of Ky Fan*, Fixed Point Theory Appl. **2012** (2012): 146. .
[17] S. Park, *A unified approach to generalized KKM maps*, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. **55** (2016), 1–20.
[18] S. Park, *Various examples of the KKM spaces*, J. Nonlinear Convex Anal. **21** (2020), 1–19.
[19] S. Park, *Extending KKM theory to a large scaled logical system*, J. Nonlinear Convex Anal. **22** (2021), 1045–1055.
[20] S. Park, *Best approximations for multimaps on abstract convex spaces*, Nonlinear Funct. Anal. Appl. **26** (2021), 165–175.
[21] S. Park, *Remarks on generalized weak KKM multimaps*, to appear
[22] S. Park and K. S. Jeong, *Generalizations of the Prolla type best approximations in normed vector spaces*, in: Proc. 2nd GARC Symp. Pure Appl. Math., Pt.II, RIM-GARC-SNU, 1993, pp. 13–25.
[23] S. Park and H. Kim, *Admissible classes of multifunction on generalized convex spaces*, Proc. Coll. Natur. Sci. Seoul National University **18** (1993), 1–21.
[24] S. Park and W. Lee, *A unified approach to generalized KKM maps in generalized convex spaces*, J. Nonlinear Convex Anal. **2** (2001), 157–166.
[25] J. B. Prolla, *Fixed point theorems for set-valued mappings and existence of best approximations,* Numer. Funct. Anal. and Optimiz. **5** (1983), 449–455.

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