Linear and Nonlinear Analysis
Volume 7, Number 3, 2021, 387–411

ISSN 2188-8167

TWO EXTRAGRADIENT METHODS FOR SOLVING VARIATIONAL INEQUALITIES IN REAL HILBERT SPACES

DUONG VIET THONG, QIAO-LI DONG, XIAO-HUAN LI, HOANG VAN THANG, PHAM VAN NGHIA, AND NGUYEN THI CAM VAN

This paper is dedicated to Prof. Wataru Takahashi in our memory.

Abstract. In this work, we introduce two new improved algorithms based on the subgradient extragradient method for solving *pseudomonotone* variational inequality in real Hilbert spaces. Our first algorithm requires only computing one projection onto the feasible set per iteration and the strong convergence is proved without the prior knowledge of the Lipschitz constant as well as the sequentially weakly continuity of the associated mapping by using the technique of choosing a new step size. The advantage of the second algorithm is that it does not require the Lipschitz continuous condition of the variational inequality mapping. Finally, some numerical experiments are presented.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset in H.

The purpose of this paper is to consider the classical *variational inequality problem* **(VI)** of Fichera [14, 15], which is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0 \quad \forall y \in C, \tag{1.1}$$

where $A : H \to H$ is a mapping. The solution set of (1.1), denoted by $\mathrm{VI(C, A)}$.

Many numerical methods have been constructed for solving variational inequalities and related optimization problems, see [4, 5, 6, 7, 16, 17, 23, 24, 26, 31, 29, 41, 44, 45, 47] and the references therein.

It is well known that, one of the most popular methods for solving the problem **(VI)** is the extragradient method **(EGM)** which was introduced in 1976 by Korpelevich [25] (also by Antipin [1] independently) as follows:

$$x \in C, \quad y_n = P_C(x_n - \lambda A x_n), \quad x_{n+1} = P_C(x_n - \lambda A y_n),$$

where $\lambda \in \left(0, \dfrac{1}{L}\right)$ and P_C denotes the metric projection from H onto C.

2020 *Mathematics Subject Classification.* 47H10, 47J25, 47H45, 65J15.

Key words and phrases. Subgradient extragradient method, viscosity method, variational inequality problem; pseudomonotone mapping, uniformly mapping.

This research is funded by National Economics University, Hanoi, Vietnam.

In recent years, the method **(EGM)** has received great attention from many authors, that is, there are many results that had been obtained by the extragradient method and its modifications when A is monotone and L-Lipschitz continuous in infinite-dimensional Hilbert spaces (see, for instance, [8, 28, 34, 39, 42, 43, 49]).

It is easy to observe that the extragradient method is the need to calculate two projections onto the closed convex set C in each iteration. So, in case the set C is not simple to calculate projection onto it, a minimum distance problem has to be solved twice in one iteration, which is a fact that might affect the efficiency and applicability of this method **(EGM)**.

To reduce the projection onto the feasible set C, Censor et al. [4] introduced the modified extragradient method which is called the subgradient extragradient method, in this new method, they replaced the second projection onto C by a projection onto a half-space:

$$x_0 \in H, \quad y_n = P_C(x_n - \lambda A x_n), \quad x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \tag{1.2}$$

where $T_n = \{w \in H : \langle x_n - \lambda A x_n - y_n, w - y_n\rangle \le 0\}$ and $\lambda \in \Big(0, \dfrac{1}{L}\Big)$.

In order to obtain the strong convergence of the method (1.2), Censor et al. [5] also introduced the following hybrid subgradient extragradient method:

$$\begin{aligned} x_0 \in H,\ y_n &= P_C(x_n - \mu A x_n), \\ z_n &= \alpha_n x_n + (1-\alpha_n) P_{T_n}(x_n - \lambda A y_n), \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \tag{1.3}$$

where $T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n\rangle \le 0\},\ C_n = \{w \in H : \|z_n - w\| \le \|x_n - w\|\},\ Q_n = \{w \in H : \langle x_n - w, x_0 - x_n\rangle \ge 0\}$. They proved under appropriate conditions the the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point $u^* = P_{VI(C,A)} x_0$.

The strong convergence of the sequence $\{x_n\}$ generated by (1.3) seems to be difficult to use in practical problems because the computation of the next iterate have to use a subproblem to find a point in the intersection of two additional half-spaces.

To overcome this barrier, Kraikaew and Saejung [26] combined the subgradient extragradient method and the Halpern method to propose an algorithm, which is called the Halpern subgradient extragradient method for solving the problem **(VI)** as follows:

$$x_0 \in H, \quad y_n = P_C(x_n - \lambda A x_n), \quad x_{n+1} = \alpha_n x_0 + (1-\alpha_n) P_{T_n}(x_n - \lambda A y_n), \tag{1.4}$$

where $T_n = \{x \in H : \langle x_n - \lambda A x_n - y_n, x - y_n\rangle \le 0\}$, $\lambda \in (0, \dfrac{1}{L})$, $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. They presented that the sequence $\{x_n\}$ generated by the method (1.4) converges strongly to a point $u^* = P_{VI(C,A)} x_0$.

In addition, in order to speed up the algorithm convergence rate, inertial algorithms for variational inequality and optimization problems has recently received considerable attention, see, for example [12, 13, 11, 41] and the references therein.

In [41], Thong et al. introduced an iterative algorithm for solving **VI** (1) when the variational inequality mapping is a *monotone* and Lipschitz continuous mapping. The obtained strong convergence result is based on the viscosity method and the inertial Tseng extragradient method.

The main drawback of the algorithm in [41] is that the step size requires the knowledge of the Lipschitz constant of the mapping, this can restrict the applications of the method because when this algorithm with a great value of the Lipschitz constant can lead to very small step size, which may lead to a slow convergent algorithm.

In [40], using the inertial technique Thong and Hieu proposed the inertial subgradient extragradient method as follows:

(1.5)

$$x_0, x_1 \in H, \ \vartheta_n = x_n + \alpha_n(x_n - x_{n-1}), \ y_n = P_C(\vartheta_n - \lambda A\vartheta_n), \ x_{n+1} = P_{T_n}(\vartheta_n - \lambda A y_n),$$

where $T_n = \{x \in H | \langle \vartheta_n - \lambda A\vartheta_n - y_n, x - y_n\rangle \le 0\}$. Under some suitable conditions, they showed that the above algorithm converges weakly to a solution of VI (1.1). Since in infinite dimensional spaces norm convergence is often more desirable, a natural question is raised:

Question: *Can we give strong convergence results for the algorithm (1.5) with a self adaptive step size without needing the information of the Lipschitz constant or non Lipschitz continuous condition of the mapping A?*

Motivated and inspired by the works of Attouch and Alvarez [2], Censor et al. [5], Moudafi [32]. In this paper, we give a positive answer to this question. We introduce two new improved subgradient extragradient type algorithms for finding a solution of the VI (1.1) in the setting of infinite-dimensional real Hilbert spaces. The advantage of the first algorithm has an additional inertial extrapolation term in the algorithm formulation which can be regarded as the procedure of speeding up the convergence properties of subgradient extragradient type iterative algorithms for variational inequality problems (see, for example, [2, 27, 35]). Moreover, it need only to compute one projection on the feasible set per iteration and the convergence of the first algorithm is proved without any requirement of the prior knowledge of the Lipschitz constant as well as the sequentially weak continuity of the variational inequality mapping. The advantage of the second algorithm is that it does not require the Lipschitz continuous condition of the associated mapping.

This paper is organized as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with analyzing the convergence of the proposed algorithms. Finally, in Sect. 4, we perform some numerical experiments to illustrate the behaviors of the proposed algorithms in comparison with other algorithms.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H.

- The weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$.
- The strong convergence of $\{x_n\}$ to x is written as $x_n \to x$ as $n \to \infty$.

For each $x, y \in H$, we have the following:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.1}$$

For all point $x \in H$, there exists the unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1 ([18]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have*

$$z = P_C x \iff \langle x - z, z - y\rangle \geq 0 \quad \forall y \in C.$$

Lemma 2.2 ([18]). *Let C be a closed convex subset in a real Hilbert space H and $x \in H$. Then*

(1) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ *for all* $y \in H$*;*
(2) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2$ *for all* $y \in C$*.*

Lemma 2.3 ([10]). *For $x \in H$ and $\alpha \geq \beta > 0$ the following inequalities hold.*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$

$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

For some more properties of the metric projection, refer to Section 3 in [18].

Definition 2.4. Let $T : H \to H$ be a mapping. Then

(1) The mapping T is called L-`Lipschitz continuous` with $L > 0$ if

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in H.$$

(2) T is called `monotone` if

$$\langle Tx - Ty, x - y\rangle \geq 0 \quad \forall x, y \in H.$$

(3) T is called `pseudomonotone` if

$$\langle Tx, y - x\rangle \geq 0 \Longrightarrow \langle Ty, y - x\rangle \geq 0 \quad \forall x, y \in H.$$

(4) The mapping T is called `sequentially weakly continuous` if for each sequence $\{x_n\}$ we have: x_n converges weakly to x implies $\{Tx_n\}$ converges weakly to Tx.

It is easy to see that every monotone operator is pseudomonotone but the converse is not true.

Lemma 2.5 ([21]). *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then, $A(M)$ is bounded.*

Lemma 2.6 ([9, Lemma 2.1]). *Consider the problem $VI(C, A)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudo-monotone and continuous. Then, x^* is a solution of $VI(C, A)$ if and only if*

$$\langle Ax, x - x^* \rangle \geq 0 \quad \forall x \in C.$$

Lemma 2.7 ([36]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, \quad \forall n \geq 1,$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$ then $\lim_{n\to\infty} a_n = 0$.

3. Main results

For the convergence of the methods, we assume the following conditions.

Condition 1. The feasible set C is nonempty, closed, and convex.

Condition 2. The mapping $A : H \to H$ is L-Lipschitz continuous, pseudomonotone on H.

Condition 3. The solution set $\mathrm{VI(C, A)}$ is nonempty.

Condition 4. Let $g : H \to H$ be a contraction mapping with contraction parameter $\kappa \in [0, 1)$. Moreover, we also assume $\{\tau_n\}$ and $\{\beta_n\}$ are two positive sequences such that $\tau_n = \circ(\beta_n)$, means $\lim_{n\to\infty} \dfrac{\tau_n}{\beta_n} = 0$, where $\{\beta_n\} \subset (0, 1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \beta_n = \infty.$$

We introduce the first algorithm:

Algorithm 1. Let $\lambda_1 > 0$, $\alpha > 0$, $\mu \in (0, 1)$ and $x_0, x_1 \in H$ be arbitrary and $\{\eta_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \eta_n < +\infty$.
Choose

$$\alpha_n = \begin{cases} \min\left\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{if otherwise.} \end{cases} \tag{3.1}$$

Compute

$$\begin{aligned} \vartheta_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ y_n &= P_C(\vartheta_n - \lambda_n A\vartheta_n) \\ x_{n+1} &= \beta_n g(x_n) + (1-\beta_n)P_{T_n}(\vartheta_n - \lambda_n Ay_n), \end{aligned}$$

where $T_n := \{z \in H : \langle \vartheta_n - \lambda_n A\vartheta_n - y_n, z - y_n\rangle \le 0\}$.
Update
(3.2)

$$\lambda_{n+1} = \begin{cases} \min\{\mu\dfrac{\|\vartheta_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A\vartheta_n - Ay_n, z_n - y_n\rangle}, \lambda_n + \eta_n\} & \text{if } \langle A\vartheta_n - Ay_n, z_n - y_n\rangle > 0, \\ \lambda_n + \eta_n & \text{otherwise,} \end{cases}$$

where$z_n := P_{T_n}(\vartheta_n - \lambda_n Ay_n)$.

Remark 3.1. As noted in [33] the sequence $\{\lambda_n\}$ generated by (3.2) is allowed to increase from iteration to iteration.

We start the analysis of the algorithm's convergence by proving the following lemmas

Lemma 3.2 ([33])**.** *Let $\{\lambda_n\}$ be a sequence generated by (3.2). Then*

$$\lim_{n\to\infty} \lambda_n = \lambda \text{ with } \lambda \in \left[\min\left\{\lambda_1, \frac{\mu}{L}\right\}, \lambda_1 + \eta\right],$$

where $\eta = \sum_{n=1}^{\infty} \eta_n$. Moreover, we also obtain

$$2\langle A\vartheta_n - Ay_n, z_n - y_n\rangle \le \frac{\mu}{\lambda_{n+1}}\|\vartheta_n - y_n\|^2 + \frac{\mu}{\lambda_{n+1}}\|z_n - y_n\|^2 \quad \forall n. \tag{3.3}$$

The following lemmas are quite helpful for analyzing the convergence of algorithm.

Lemma 3.3. *Assume that Conditions 1–3 hold. Let $\{z_n\}$ be a sequence generated by Algorithm 1. Then*

$$\begin{aligned} \|z_n - u^*\|^2 \le \|\vartheta_n - u^*\|^2 &- \Big(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|\vartheta_n - y_n\|^2 \\ &- \Big(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|z_n - y_n\|^2, \quad \forall u^* \in VI(C,A), \end{aligned} \tag{3.4}$$

where $z_n := P_{T_n}(\vartheta_n - \lambda_n Ay_n)$.

Proof. Using the inequality (3.3), Lemma 2.2 and $u^* \in VI(C,A) \subset C \subset T_n$ to prove the inequality (3.38). Indeed, we have

$$\begin{aligned} \|z_n - u^*\|^2 =& \|P_{T_n}(\vartheta_n - \lambda_n Ay_n) - P_{T_n}u^*\|^2 \le \langle z_n - u^*, \vartheta_n - \lambda_n Ay_n - u^*\rangle \\ =& \frac{1}{2}\|z_n - u^*\|^2 + \frac{1}{2}\|\vartheta_n - \lambda_n Ay_n - u^*\|^2 - \frac{1}{2}\|z_n - \vartheta_n + \lambda_n Ay_n\|^2 \end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}\|z_n-u^*\|^2+\frac{1}{2}\|\vartheta_n-u^*\|^2+\frac{1}{2}\lambda_n^2\|Ay_n\|^2-\langle\vartheta_n-u^*,\lambda_nAy_n\rangle\\
&-\frac{1}{2}\|z_n-\vartheta_n\|^2-\frac{1}{2}\lambda_n^2\|Ay_n\|^2-\langle z_n-\vartheta_n,\lambda_nAy_n\rangle\\
=&\frac{1}{2}\|z_n-u^*\|^2+\frac{1}{2}\|\vartheta_n-u^*\|^2-\frac{1}{2}\|z_n-\vartheta_n\|^2-\langle z_n-u^*,\lambda_nAy_n\rangle.
\end{aligned}
$$

This implies that

$$
\|z_n-u^*\|^2\le\|\vartheta_n-u^*\|^2-\|z_n-\vartheta_n\|^2-2\langle z_n-u^*,\lambda_nAy_n\rangle. \tag{3.5}
$$

Since u^* is the solution of VI, we have $\langle Au^*,x-u^*\rangle\ge 0$ for all $x\in C$. By the pseudomontonicity of A on C we have $\langle Ax,x-u^*\rangle\ge 0$ for all $x\in C$. Taking $x:=y_n\in C$ we get

$$
\langle Ay_n,u^*-y_n\rangle\le 0.
$$

Thus,

$$
\langle Ay_n,u^*-z_n\rangle=\langle Ay_n,u^*-y_n\rangle+\langle Ay_n,y_n-z_n\rangle\le\langle Ay_n,y_n-z_n\rangle. \tag{3.6}
$$

From (3.5) and (3.6) we obtain

$$
\begin{aligned}
\|z_n-u^*\|^2\le&\|\vartheta_n-u^*\|^2-\|z_n-\vartheta_n\|^2+2\lambda_n\langle Ay_n,y_n-z_n\rangle\\
=&\|\vartheta_n-u^*\|^2-\|z_n-y_n\|^2-\|y_n-\vartheta_n\|^2-2\langle z_n-y_n,y_n-\vartheta_n\rangle\\
&+2\lambda_n\langle Ay_n,y_n-z_n\rangle\\
=&\|\vartheta_n-u^*\|^2-\|z_n-y_n\|^2-\|y_n-\vartheta_n\|^2+2\langle\vartheta_n-\lambda_nAy_n-y_n,z_n-y_n\rangle. \qquad (3.7)
\end{aligned}
$$

Since $y_n=P_{T_n}(\vartheta_n-\lambda_nA\vartheta_n)$ and $z_n\in T_n$ we have

$$
\begin{aligned}
2\langle\vartheta_n-\lambda_nAy_n-y_n,&z_n-y_n\rangle\\
&=2\langle\vartheta_n-\lambda_nA\vartheta_n-y_n,z_n-y_n\rangle+2\lambda_n\langle A\vartheta_n-Ay_n,z_n-y_n\rangle\\
&\le 2\lambda_n\langle A\vartheta_n-Ay_n,z_n-y_n\rangle, \qquad (3.8)
\end{aligned}
$$

which, together with (3.3) implies that

$$
2\langle\vartheta_n-\lambda_nAy_n-y_n,z_n-y_n\rangle\le\mu\frac{\lambda_n}{\lambda_{n+1}}\|\vartheta_n-y_n\|^2+\mu\frac{\lambda_n}{\lambda_{n+1}}\|z_n-y_n\|^2.
$$

From (3.7) and (3.8) we get

$$
\|z_n-u^*\|^2\le\|\vartheta_n-u^*\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|y_n-\vartheta_n\|^2-(1-\mu\frac{\lambda_n}{\lambda_{n+1}})\|z_n-y_n\|^2.
$$

□

Modifying the technique in [43], we develop the following auxiliary result.

Lemma 3.4. *Assume that Conditions 1–3 hold and the mapping $A:H\to H$ satisfies the following condition*

$$
\textit{whenever } \{x_n\}\subset C, x_n\rightharpoonup z, \textit{ one has } \|Az\|\le\liminf_{n\to\infty}\|Ax_n\|.
$$

If there exists a subsequence $\{\vartheta_{n_k}\}$ of $\{\vartheta_n\}$ convergent weakly to $z\in H$ and $\lim_{k\to\infty}\|\vartheta_{n_k}-y_{n_k}\|=0$, then $z\in \mathrm{VI(C,A)}$.

Proof. We have

$$\langle \vartheta_{n_k} - \lambda_{n_k} A\vartheta_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le 0 \quad \forall x \in C,$$

or equivalently

$$\frac{1}{\lambda_{n_k}}\langle \vartheta_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le \langle A\vartheta_{n_k}, x - y_{n_k}\rangle \quad \forall x \in C.$$

Consequently

$$\frac{1}{\lambda_{n_k}}\langle \vartheta_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle A\vartheta_{n_k}, y_{n_k} - \vartheta_{n_k}\rangle \le \langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle \quad \forall x \in C. \tag{3.9}$$

Being weakly convergent, $\{\vartheta_{n_k}\}$ is bounded. Then, by the Lipschitz continuity of A, $\{A\vartheta_{n_k}\}$ is bounded. As $\|\vartheta_{n_k} - y_{n_k}\| \to 0$, $\{y_{n_k}\}$ is also bounded and $\lambda_{n_k} \ge \min\{\lambda_1, \frac{\mu}{L}\}$. Passing (3.9) to limit as $k \to \infty$, we get

$$\liminf_{k\to\infty}\langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle \ge 0 \quad \forall x \in C. \tag{3.10}$$

Moreover, we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle + \langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle + \langle Ay_{n_k}, \vartheta_{n_k} - y_{n_k}\rangle. \tag{3.11}$$

Since $\lim_{k\to\infty}\|\vartheta_{n_k} - y_{n_k}\| = 0$ and A is L-Lipschitz continuous on H, we get

$$\lim_{k\to\infty}\|A\vartheta_{n_k} - Ay_{n_k}\| = 0$$

which, together with (3.10) and (3.11) implies that

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \ge 0.$$

Next, we show that $z \in \mathrm{VI(C, A)}$. We choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by N_k the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \epsilon_k \ge 0 \quad \forall j \ge N_k. \tag{3.12}$$

Since $\{\epsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each k, since $\{y_{N_k}\} \subset C$ we have $Ay_{N_k} \ne 0$ and, setting

$$v_{N_k} = \frac{Ay_{N_k}}{\|Ay_{N_k}\|^2},$$

we have $\langle Ay_{N_k}, v_{N_k}\rangle = 1$ for each k. Now, we can deduce from (3.12) that for each k

$$\langle Ay_{N_k}, x + \epsilon_k v_{N_k} - y_{N_k}\rangle \ge 0.$$

From A is pseudomonotone on H, we get

$$\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle \ge 0.$$

This implies that

$$\langle Ax, x - y_{N_k}\rangle \ge \langle Ax - A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle - \epsilon_k\langle Ax, v_{N_k}\rangle. \tag{3.13}$$

Now, we show that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Indeed, since $\vartheta_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|\vartheta_{n_k} - y_{n_k}\| = 0$, we obtain $y_{N_k} \rightharpoonup z$ as $k \to \infty$. By $\{y_n\} \subset C$, we obtain $z \in C$. Since the mapping A satisfies Condition 3, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

Note that, we can assume that $\|Az\| > 0$, if $Az = 0$ then stop and z is a solution of $VI(C, A)$. Since $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$.

Now, letting $k \to \infty$, then the right hand side of (3.13) tends to zero by A is uniformly continuous, $\{\vartheta_{N_k}\}, \{v_{N_k}\}$ are bounded and $\lim_{k\to\infty} \epsilon_k v_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

Hence, for all $x \in C$ we have

$$\langle Ax, x - z\rangle = \lim_{k\to\infty} \langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

By Lemma 2.6, $z \in$ VI(C, A) and the proof is complete. □

Theorem 3.5. *Assume that Conditions 1–4 hold and the mapping $A : H \to H$ satisfies the following condition*

$$\text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|. \tag{3.14}$$

Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to an element $u^* \in VI(C, A)$, where $u^* = P_{VI(C,A)} \circ g(u^*)$.

Proof. **Claim 1.** The sequence $\{x_n\}$ is bounded. Indeed, for $u^* = P_{VI(C,A)} \circ g(u^*)$, by Lemma 3.3 we have

$$\|z_n - u^*\|^2 \leq \|\vartheta_n - u^*\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|\vartheta_n - y_n\|^2 - (1 - \mu\frac{\lambda_n}{\lambda_{n+1}})\|z_n - y_n\|^2. \tag{3.15}$$

Since $\lim_{n\to\infty}(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0$, thus there exists $n_0 \in \mathbb{N}$ such that

$$(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}) > 0 \quad \forall n \geq n_0.$$

Therefore, we have

$$\|z_n - u^*\| \leq \|\vartheta_n - u^*\| \quad \forall n \geq n_0. \tag{3.16}$$

From the definition of ϑ_n, we get

$$\begin{aligned}\|\vartheta_n - u^*\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - u^*\| \\ &\leq \|x_n - u^*\| + \alpha_n\|x_n - x_{n-1}\| \\ &= \|x_n - u^*\| + \beta_n.\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{3.17}$$

Since (3.1), we have $\alpha_n\|x_n - x_{n-1}\| \le \tau_n$ for all n, which together with $\lim_{n\to\infty} \frac{\tau_n}{\beta_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le \lim_{n\to\infty} \frac{\tau_n}{\beta_n} = 0.$$

It follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \le M_1 \quad \forall n \ge 1. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we obtain

$$\|z_n - u^*\| \le \|\vartheta_n - u^*\| \le \|x_n - u^*\| + \beta_n M_1 \quad \forall n \ge n_0. \tag{3.19}$$

From the definition of $\{x_n\}$, we get

$$\begin{aligned}
\|x_{n+1} - u^*\| &= \|\beta_n g(x_n) + (1-\beta_n)z_n - u^*\| \\
&= \|\beta_n(g(x_n) - u^*) + (1-\beta_n)(z_n - u^*)\| \\
&\le \beta_n\|g(x_n) - u^*\| + (1-\beta_n)\|z_n - u^*\| \\
&\le \beta_n\|g(x_n) - g(u^*)\| + \beta_n\|g(u^*) - u^*\| + (1-\beta_n)\|z_n - u^*\| \\
&\le \beta_n\kappa\|x_n - u^*\| + \beta_n\|g(u^*) - u^*\| + (1-\beta_n)\|z_n - u^*\|. \qquad (3.20)
\end{aligned}$$

Substituting (3.19) into (3.20), we obtain

$$\begin{aligned}
\|x_{n+1} - u^*\| &\le (1-(1-\kappa)\beta_n)\|x_n - u^*\| + \beta_n M_1 + \beta_n\|g(u^*) - u^*\| \quad \forall n \ge n_0 \\
&= (1-(1-\kappa)\beta_n)\|x_n - u^*\| + (1-\kappa)\beta_n \frac{M_1 + \|g(u^*) - u^*\|}{1-\kappa} \quad \forall n \ge n_0 \\
&\le \max\Big\{\|x_n - u^*\|, \frac{M_1 + \|g(u^*) - u^*\|}{1-\kappa}\Big\} \quad \forall n \ge n_0 \\
&\le \cdots \\
&\le \max\Big\{\|x_{n_0} - u^*\|, \frac{M_1 + \|g(u^*) - u^*\|}{1-\kappa}\Big\}.
\end{aligned}$$

This implies $\{x_n\}$ is bounded. We also get $\{z_n\}, \{g(x_n)\}, \{\vartheta_n\}$ are bounded.

Claim 2.

$$\begin{aligned}
(1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|\vartheta_n - y_n\|^2 + (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|z_n - y_n\|^2 \\
\le \|x_n - u^*\|^2 - \|x_{n+1} - u^*\|^2 + \beta_n M_4,
\end{aligned}$$

for some $M_4 > 0$. Indeed, we get

$$\begin{aligned}
\|x_{n+1} - u^*\|^2 &\le \beta_n\|g(x_n) - u^*\|^2 + (1-\beta_n)\|z_n - u^*\|^2 \\
&\le \beta_n(\|g(x_n) - g(u^*)\| + \|g(u^*) - u^*\|)^2 + (1-\beta_n)\|z_n - u^*\|^2 \\
&\le \beta_n(\kappa\|x_n - u^*\| + \|g(u^*) - u^*\|)^2 + (1-\beta_n)\|z_n - u^*\|^2 \\
&\le \beta_n(\|x_n - u^*\| + \|g(u^*) - u^*\|)^2 + (1-\beta_n)\|z_n - u^*\|^2 \\
&= \beta_n\|x_n - u^*\|^2 + \beta_n(2\|x_n - u^*\| \cdot \|g(u^*) - u^*\| \\
&\quad + \|g(u^*) - u^*\|^2) + (1-\beta_n)\|z_n - u^*\|^2
\end{aligned}$$

$$\leq \beta_n\|x_n - u^*\|^2 + (1-\beta_n)\|z_n - u^*\|^2 + \beta_n M_2 \tag{3.21}$$

for some $M_2 > 0$. Substituting (3.15) into (3.21), we get

$$\begin{aligned}\|x_{n+1} - u^*\|^2 \leq \beta_n\|x_n - u^*\|^2 &+ (1-\beta_n)\|\vartheta_n - u^*\|^2 \\ &- (1-\beta_n)\Big(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|\vartheta_n \\ &- y_n\|^2 - (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|z_n - y_n\|^2 + \beta_n M_2,\end{aligned}$$

which implies from (3.19) that

$$\begin{aligned}\|\vartheta_n - u^*\|^2 &\leq (\|x_n - u^*\| + \beta_n M_1)^2 \\ &= \|x_n - u^*\|^2 + \beta_n(2M_1\|x_n - u^*\| + \beta_n M_1^2) \\ &\leq \|x_n - u^*\|^2 + \beta_n M_3, \end{aligned} \tag{3.22}$$

for some $M_3 > 0$. Combining (3.22) and (3.22), we obtain

$$\begin{aligned}\|x_{n+1} - u^*\|^2 &\leq \beta_n\|x_n - u^*\|^2 + (1-\beta_n)\|x_n - u^*\|^2 \\ &\quad + \beta_n M_3 - (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|\vartheta_n - y_n\|^2 \\ &\quad - (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|z_n - y_n\|^2 + \beta_n M_2 \\ &= \|x_n - u^*\|^2 + \beta_n M_3 - (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|\vartheta_n - y_n\|^2 \\ &\quad - (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|z_n - y_n\|^2 + \beta_n M_2.\end{aligned}$$

This implies that

$$\begin{aligned}(1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|\vartheta_n - y_n\|^2 &+ (1-\beta_n)\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|z_n - y_n\|^2 \\ &\leq \|x_n - u^*\|^2 - \|x_{n+1} - u^*\|^2 + \beta_n M_4,\end{aligned}$$

where $M_4 := M_2 + M_3$.

Claim 3.

$$\begin{aligned}\|x_{n+1} - u^*\|^2 &\leq (1 - (1-\kappa)\beta_n)\|x_n - u^*\|^2 \\ &\quad + (1-\kappa)\beta_n \cdot \Big[\frac{2}{1-\kappa}\langle g(u^*) - u^*, x_{n+1} - u^*\rangle + \frac{3M}{1-\kappa}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n - x_{n-1}\|\Big]\end{aligned}$$

for some $M > 0$. Indeed, we have

$$\begin{aligned}\|\vartheta_n - u^*\|^2 &= \|x_n + \alpha_n(x_n - x_{n-1}) - u^*\|^2 \\ &= \|x_n - u^*\|^2 + 2\alpha_n\langle x_n - u^*, x_n - x_{n-1}\rangle + \alpha_n^2\|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - u^*\|^2 + 2\alpha_n\|x_n - u^*\|\|x_n - x_{n-1}\| + \alpha_n^2\|x_n - x_{n-1}\|^2.\end{aligned} \tag{3.23}$$

Using (2.1), we have

$$
\begin{aligned}
\|x_{n+1}-u^*\|^2 &= \|\beta_n g(x_n)+(1-\beta_n)z_n-u^*\|^2\\
&= \|\beta_n(g(x_n)-g(u^*))+(1-\beta_n)(z_n-u^*)+\beta_n(g(u^*)-u^*)\|^2\\
&\le \|\beta_n(g(x_n)-g(u^*))+(1-\beta_n)(z_n-u^*)\|^2\\
&\quad +2\beta_n\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\le \beta_n\|g(x_n)-g(u^*)\|^2+(1-\beta_n)\|z_n-u^*\|^2\\
&\quad +2\beta_n\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\le \beta_n\kappa^2\|x_n-u^*\|^2+(1-\beta_n)\|z_n-u^*\|^2\\
&\quad +2\beta_n\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\le \beta_n\kappa\|x_n-u^*\|^2+(1-\beta_n)\|z_n-u^*\|^2\\
&\quad +2\beta_n\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\le \beta_n\kappa\|x_n-u^*\|^2+(1-\beta_n)\|\vartheta_n-u^*\|^2\\
&\quad +2\beta_n\langle g(u^*)-u^*,x_{n+1}-u^*\rangle. \qquad (3.24)
\end{aligned}
$$

Substituting (3.23) into (3.24), we have

$$
\begin{aligned}
\|x_{n+1}-u^*\|^2 &\le (1-(1-\kappa)\beta_n)\|x_n-u^*\|^2+2\alpha_n\|x_n-u^*\|\|x_n-x_{n-1}\|\\
&\quad +\alpha_n^2\|x_n-x_{n-1}\|^2+2\beta_n\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&= (1-(1-\kappa)\beta_n)\|x_n-u^*\|^2\\
&\quad +(1-\kappa)\beta_n\cdot\frac{2}{1-\kappa}\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\quad +\alpha_n\|x_n-x_{n-1}\|(2\|x_n-u^*\|+\alpha_n\|x_n-x_{n-1}\|)\\
&\le (1-(1-\kappa)\beta_n)\|x_n-u^*\|^2\\
&\quad +(1-\kappa)\beta_n\cdot\frac{2}{1-\kappa}\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\quad +\alpha_n\|x_n-x_{n-1}\|(2\|x_n-u^*\|+\alpha\|x_n-x_{n-1}\|)\\
&\le (1-(1-\kappa)\beta_n)\|x_n-u^*\|^2\\
&\quad +(1-\kappa)\beta_n\cdot\frac{2}{1-\kappa}\langle g(u^*)-u^*,x_{n+1}-u^*\rangle+3M\alpha_n\|x_n-x_{n-1}\|\\
&\le (1-(1-\kappa)\beta_n)\|x_n-u^*\|^2\\
&\quad +(1-\kappa)\beta_n\cdot\Big[\frac{2}{1-\kappa}\langle g(u^*)-u^*,x_{n+1}-u^*\rangle\\
&\qquad\qquad +\frac{3M}{1-\kappa}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\Big],
\end{aligned}
$$

where $M:=\sup_{n\in\mathbb{N}}\{\|x_n-u^*\|,\alpha\|x_n-x_{n-1}\|\}>0$.

Claim 4. $\{\|x_n-u^*\|^2\}$ converges to zero. Indeed, by Lemma 2.7 it suffices to show that $\limsup_{k\to\infty}\langle g(u^*)-u^*,x_{n_k+1}-u^*\rangle\le 0$ for every subsequence $\{\|x_{n_k}-u^*\|\}$

of $\{\|x_n - u^*\|\}$ satisfying

$$\liminf_{k\to\infty}(\|x_{n_k+1} - u^*\| - \|x_{n_k} - u^*\|) \geq 0.$$

For this, suppose that $\{\|x_{n_k} - u^*\|\}$ is a subsequence of $\{\|x_n - u^*\|\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1} - u^*\| - \|x_{n_k} - u^*\|) \geq 0$. Then

$$\begin{aligned}
&\liminf_{k\to\infty}(\|x_{n_k+1} - u^*\|^2 - \|x_{n_k} - u^*\|^2) \\
&\quad = \liminf_{k\to\infty}[(\|x_{n_k+1} - u^*\| - \|x_{n_k} - u^*\|)(\|x_{n_k+1} - u^*\| + \|x_{n_k} - u^*\|)] \\
&\quad \geq 0.
\end{aligned}$$

By Claim 2 we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Big[(1-\beta_{n_k})\Big(1-\mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\Big)\|\vartheta_{n_k} - y_{n_k}\|^2 + (1-\beta_{n_k})\Big(1-\mu\frac{\lambda_{n_k}}{\lambda_{n_k+1}}\Big)\|z_{n_k} - y_{n_k}\|^2\Big] \\
&\qquad \leq \limsup_{k\to\infty}[\|x_{n_k} - u^*\|^2 - \|x_{n_k+1} - u^*\|^2 + \beta_{n_k}M_4] \\
&\qquad \leq \limsup_{k\to\infty}[\|x_{n_k} - u^*\|^2 - \|x_{n_k+1} - u^*\|^2] + \limsup_{k\to\infty}\beta_{n_k}M_4 \\
&\qquad = -\liminf_{k\to\infty}[\|x_{n_k+1} - u^*\|^2 - \|x_{n_k} - u^*\|^2] \\
&\qquad \leq 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\|y_{n_k} - \vartheta_{n_k}\| = 0 \text{ and } \lim_{k\to\infty}\|z_{n_k} - y_{n_k}\| = 0. \tag{3.25}$$

Thus

$$\lim_{k\to\infty}\|z_{n_k} - \vartheta_{n_k}\| = 0. \tag{3.26}$$

Now, we show that

$$\|x_{n_k+1} - x_{n_k}\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.27}$$

Indeed, we have

$$\|x_{n_k+1} - z_{n_k}\| = \beta_{n_k}\|z_{n_k} - g(x_{n_k})\| \to 0, \tag{3.28}$$

and

$$\|x_{n_k} - \vartheta_{n_k}\| = \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| = \beta_{n_k}.\frac{\alpha_{n_k}}{\beta_{n_k}}\|x_{n_k} - x_{n_k-1}\| \to 0. \tag{3.29}$$

From (3.26), (3.28) and (3.29), we get

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - \vartheta_{n_k}\| + \|\vartheta_{n_k} - x_{n_k}\| \to 0.$$

Since the sequence $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z \in H$, such that

(3.30)

$$\limsup_{k\to\infty}\langle g(u^*) - u^*, x_{n_k} - u^*\rangle = \lim_{j\to\infty}\langle g(u^*) - u^*, x_{n_{k_j}} - u^*\rangle = \langle g(u^*) - u^*, z - u^*\rangle.$$

Using (3.29), we get

$$\vartheta_{n_k} \rightharpoonup z \text{ as } k \to \infty,$$

which, together with (3.25) and Lemma 3.4, we have $z \in VI(C, A)$. From (3.30) and the definition of $u^* = P_{VI(C,A)} \circ g(u^*)$, we have

$$\limsup_{k\to\infty} \langle g(u^*) - u^*, x_{n_k} - u^* \rangle = \langle g(u^*) - u^*, z - u^* \rangle \le 0. \tag{3.31}$$

Combining (3.27) and (3.31), we have

$$\begin{aligned} \limsup_{k\to\infty} \langle g(u^*) - u^*, x_{n_k+1} - u^* \rangle &\le \limsup_{k\to\infty} \langle g(u^*) - u^*, x_{n_k} - u^* \rangle \\ &= \langle g(u^*) - u^*, z - u^* \rangle \\ &\le 0. \end{aligned} \tag{3.32}$$

Hence, by (3.32), $\lim_{n\to\infty} \dfrac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| = 0$, Claim 3 and Lemma 2.7, we have $\lim_{n\to\infty} \|x_n - u^*\| = 0$. That is the desired result. □

Remark 3.6. (1) When the mapping A is monotone, we don't need to impose Condition (3.14), (see, [10, 42]).

(2) It should be emphasized here that, in our proof we need only to impose Condition (3.14) is weaker than the sequentially weakly continuity of A which used in recent articles [37, 42, 43].

(3) Our result generalizes some related results in the literature [38, 41, 46, 48, 47] and hence might be applied to a wider class of nonlinear mappings. For example, in our Theorem 3.5, we replaced the monotonicity by the pseudomonotonicity of A.

Next, we introduce the second algorithm, where the Lipschitz continuous condition of the variational inequality mapping is removed. The algorithm is of the form:

Algorithm 2. Let $\gamma > 0$, $\mu \in (0, 1)$, $l \in (0, 1)$, $\alpha > 0$ and $x_0, x_1 \in H$ be arbitrary. Choose

$$\alpha_n = \begin{cases} \min\left\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{if otherwise.} \end{cases}$$

Compute

$$\begin{aligned} \vartheta_n &= x_n + \alpha_n(x_n - x_{n-1}) \\ y_n &= P_C(\vartheta_n - \lambda_n A\vartheta_n), \\ x_{n+1} &= \beta_n g(x_n) + (1 - \beta_n) P_{T_n}(\vartheta_n - \lambda_n A y_n), \end{aligned}$$

where $T_n := \{z \in H : \langle \vartheta_n - \lambda_n A\vartheta_n - y_n, z - y_n \rangle \le 0\}$ and λ_n is chosen to be the largest $\lambda \in \{\gamma, \gamma l, \gamma l^2, ..., \gamma l^m, ...\}$ satisfying

$$\lambda \|A\vartheta_n - Ay_n\| \le \mu \|\vartheta_n - y_n\|. \tag{3.33}$$

In order to the convergence analysis of Algorithm 2, we use the following condition:

Condition $2'$**.** The mapping $A : H \to H$ is pseudomonotone on H and uniformly continuous on bounded subsets of H.

Lemma 3.7. *Assume that Conditions 1,* $2'$ *3 hold. Then, the Armijo linesearch rule (3.33) is well-defined and* $\lambda_n \le \gamma$ *for all* n*.*

Proof. If $\vartheta_n \in \mathrm{VI}(C, A)$ then $\vartheta_n = P_C(\vartheta_n - \gamma A\vartheta_n)$, therefore (3.33) holds with $m = 0$. For $\vartheta_n \notin \mathrm{VI}(C, A)$, suppose to the contrary that, for all m, we have

$$\gamma l^m \|A\vartheta_n - AP_C(\vartheta_n - \gamma l^m A\vartheta_n)\| > \mu\|\vartheta_n - P_C(\vartheta_n - \gamma l^m A\vartheta_n)\|. \tag{3.34}$$

Then,

$$\|A\vartheta_n - AP_C(\vartheta_n - \gamma l^m A\vartheta_n)\| > \mu\frac{\|\vartheta_n - P_C(\vartheta_n - \gamma l^m A\vartheta_n)\|}{\gamma l^m}. \tag{3.35}$$

If $\vartheta_n \in C$, invoking that P_C and A are continuous, one sees that $\lim_{m\to\infty} \|\vartheta_n - P_C(\vartheta_n - \gamma l^m A\vartheta_n)\| = 0$. From the uniform continuity of A on bounded subsets of H, we have

$$\lim_{m\to\infty} \|A\vartheta_n - AP_C(\vartheta_n - \gamma l^m A\vartheta_n)\| = 0.$$

Combining this and (3.35), we get

$$\lim_{m\to\infty} \frac{\|\vartheta_n - P_C(\vartheta_n - \gamma l^m A\vartheta_n)\|}{\gamma l^m} = 0. \tag{3.36}$$

For $t_m := P_C(\vartheta_n - \gamma l^m A\vartheta_n)$, we have

$$\langle t_m - \vartheta_n + \gamma l^m A\vartheta_n, x - t_m\rangle \ge 0 \quad \forall x \in C.$$

Hence,

$$\left\langle \frac{t_m - \vartheta_n}{\gamma l^m}, x - t_m \right\rangle + \langle A\vartheta_n, x - t_m\rangle \ge 0 \quad \forall x \in C.$$

Taking the limit $m \to \infty$ in this inequality and using (3.36), we obtain $\langle A\vartheta_n, x - \vartheta_n\rangle \ge 0 \ \ \forall x \in C$, which implies that $\vartheta_n \in \mathrm{VI}(C, A)$. This is a contradiction.

For $\vartheta_n \notin C$, we have

$$\begin{aligned} &\lim_{m\to\infty} \|\vartheta_n - P_C(\vartheta_n - \gamma l^m A\vartheta_n)\| = \|\vartheta_n - P_C\vartheta_n\| > 0, \\ &\lim_{m\to\infty} \gamma l^m \|A\vartheta_n - AP_C(\vartheta_n - \gamma l^m A\vartheta_n)\| = 0. \end{aligned} \tag{3.37}$$

From this, which together with (3.34), (3.37), we get another contradiction. □

Lemma 3.8. *Assume that Conditions 1,* $2'$ *and 3 hold. Let* $\{z_n\}$ *be a sequence generated by Algorithm 2. Then*

(3.38)
$$\|z_n - u^*\|^2 \le \|\vartheta_n - u^*\|^2 - (1-\mu)\|\vartheta_n - y_n\|^2 - (1-\mu)\|z_n - y_n\|^2, \quad \forall u^* \in VI(C, A),$$

where $z_n = P_{T_n}(\vartheta_n - \lambda_n Ay_n)$.

Proof. The proof is similar to that of Lemma 3.3, we leave for the reader. □

Lemma 3.9. *Assume that Conditions 1, $2'$ and 3 hold and the mapping $A : H \to H$ satisfies the following condition*

$$\text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|.$$

If there exists a subsequence $\{\vartheta_{n_k}\}$ of $\{\vartheta_n\}$ convergent weakly to $z \in H$ and $\lim_{k\to\infty} \|\vartheta_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI(C, A)}$.

Proof. Assume that, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in H$. From $y_{n_k} = P_C(\vartheta_{n_k} - \lambda_{n_k} A\vartheta_{n_k})$ we have

$$\langle \vartheta_{n_k} - \lambda_{n_k} A\vartheta_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \quad \forall x \in C.$$

or equivalently

$$\frac{1}{\lambda_{n_k}}\langle \vartheta_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle A\vartheta_{n_k}, x - y_{n_k}\rangle \quad \forall x \in C.$$

Consequently,

$$\frac{1}{\lambda_{n_k}}\langle \vartheta_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle A\vartheta_{n_k}, y_{n_k} - \vartheta_{n_k}\rangle \leq \langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle \quad \forall x \in C. \tag{3.39}$$

Now, we claim that

$$\liminf_{k\to\infty} \langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle \geq 0. \tag{3.40}$$

Indeed, suppose first that $\liminf_{k\to\infty} \lambda_{n_k} > 0$. By Lemma 2.5, $\{A\vartheta_{n_k}\}$ is bounded. Taking $k \to \infty$ in (3.39), since $\|\vartheta_{n_k} - y_{n_k}\| \to 0$, we get (3.40). Next, we assume that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Setting $t_{n_k} := P_C(\vartheta_{n_k} - \lambda_{n_k} l^{-1} A\vartheta_{n_k})$, as $\lambda_{n_k} l^{-1} > \lambda_{n_k}$, Lemma 2.3 yields

$$\|\vartheta_{n_k} - t_{n_k}\| \leq \frac{1}{l}\|\vartheta_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Hence, t_{n_k} weakly converges to $z \in C$. Because A is (uniformly) continuous on the bounded set $\{\vartheta_n\} \cup \{t_n\}$, we obtain

$$\|A\vartheta_{n_k} - At_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{3.41}$$

As $\lambda_{n_k} l^{-1} = \gamma l^{m_{n_k}} l^{-1} = \gamma l^{m_{n_k}-1}$, by the Armijo linesearch rule (3.33), we have

$$\lambda_{n_k} l^{-1}\|A\vartheta_{n_k} - AP_C(\vartheta_{n_k} - \lambda_{n_k} l^{-1} A\vartheta_{n_k})\| > \mu\|\vartheta_{n_k} - P_C(\vartheta_{n_k} - \lambda_{n_k} l^{-1} A\vartheta_{n_k})\|,$$

which is

$$\frac{1}{\mu}\|A\vartheta_{n_k} - At_{n_k}\| > \frac{\|\vartheta_{n_k} - t_{n_k}\|}{\lambda_{n_k} l^{-1}}.$$

Combining this and (3.41), we obtain

$$\lim_{k\to\infty} \frac{\|\vartheta_{n_k} - t_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0.$$

Furthermore, we have from the definition of t_{n_k} that

$$\langle \vartheta_{n_k} - \lambda_{n_k} l^{-1} A\vartheta_{n_k} - t_{n_k}, x - t_{n_k}\rangle \leq 0 \quad \forall x \in C.$$

Hence,

$$\frac{1}{\lambda_{n_k} l^{-1}}\langle \vartheta_{n_k} - t_{n_k}, x - t_{n_k}\rangle + \langle A\vartheta_{n_k}, t_{n_k} - \vartheta_{n_k}\rangle \le \langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle \quad \forall x \in C.$$

Taking the limit as $k \to \infty$, we get

$$\liminf_{k\to\infty}\langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle \ge 0.$$

Therefore, the claim (3.40) is proved.

Furthermore, we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle + \langle A\vartheta_{n_k}, x - \vartheta_{n_k}\rangle + \langle Ay_{n_k}, \vartheta_{n_k} - y_{n_k}\rangle. \tag{3.42}$$

As $\lim_{k\to\infty}\|\vartheta_{n_k} - y_{n_k}\| = 0$, by the uniform continuity of A on bounded subsets, we get $\lim_{k\to\infty}\|A\vartheta_{n_k} - Ay_{n_k}\| = 0$, which together with (3.40) and (3.42) implies that

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \ge 0. \tag{3.43}$$

Finally, we show that $z \in \mathrm{VI(C, A)}$. Take a sequence $\{\epsilon_k\}$ of positive numbers, decreasing and tending to 0. Choose an increasing sequence $\{N_k\}$ such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \epsilon_k \ge 0 \quad \forall j \ge N_k, \tag{3.44}$$

where the existence of N_k follows from (3.43). Moreover, for each k setting $v_{N_k} = Ay_{N_k}\|Ay_{N_k}\|^{-2}$, we have $\langle Ay_{N_k}, v_{N_k}\rangle = 1$. We deduce from (3.44) that, for each k,

$$\langle Ay_{N_k}, x + \epsilon_k v_{N_k} - y_{N_k}\rangle \ge 0.$$

In view of the pseudomonotonicity of A on H, we get

$$\langle A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle \ge 0.$$

This implies that

$$\langle Ax, x - y_{N_k}\rangle \ge \langle Ax - A(x + \epsilon_k v_{N_k}), x + \epsilon_k v_{N_k} - y_{N_k}\rangle - \epsilon_k\langle Ax, v_{N_k}\rangle. \tag{3.45}$$

We show that $\lim_{k\to\infty}\epsilon_k v_{N_k} = 0$. Indeed, since $x_{n_k} \rightharpoonup z$, $\lim_{k\to\infty}\|x_{n_k} - \vartheta_{n_k}\| = 0$, and $\lim_{k\to\infty}\|\vartheta_{n_k} - y_{n_k}\| = 0$, we obtain $y_{N_k} \rightharpoonup z$. Since the mapping A satisfies Condition 3, we have

$$0 < \|Az\| \le \liminf_{k\to\infty}\|Ay_{n_k}\|.$$

Since $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0$, we obtain

$$0 \le \limsup_{k\to\infty}\|\epsilon_k v_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\epsilon_k}{\|Ay_{n_k}\|}\right) \le \frac{\limsup_{k\to\infty}\epsilon_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty}\epsilon_k v_{N_k} = 0$.

Letting $k \to \infty$, the right-hand side of (3.45) tends to zero due to the uniform continuity of A. Thus, $\liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \ge 0$. Hence, we have, for all $x \in C$,

$$\langle Ax, x - z\rangle = \lim_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \ge 0.$$

By Lemma 2.6, $z \in \mathrm{VI(C, A)}$ and the proof is complete. □

Theorem 3.10. *Assume that Conditions 1, $2'$, 3 and 4 hold and the mapping $A : H \to H$ satisfies the following condition*

$$\text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|.$$

Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to an element $u^* \in VI(C, A)$, where $u^* = P_{VI(C,A)} \circ g(u^*)$.

Proof. The proof is similar to that of Theorem 3.5, and therefore is ommited. □

4. Numerical illustrations

In this section, we provide two numerical examples to show the practicability and the advantage of our proposed algorithm by comparing it with other algorithms. In this examples, we take $g(x) = \frac{x}{2}$.

First, we consider three examples in finite-dimensional Hilbert space.

Since the exact solution of example 4.1 and example 4.2 are not known, so, we use $\|x_{n+1} - x_n\|$ to measure the error of the n-th iteration, which also serves as the role of checking whether or not the proposed algorithm converges to the solution.

In examples 4.1 and 4.2, the initial point x_0 is randomly chosen in $\mathbb{R}^m$. And take $\mu = 0.3$, $\beta_n = \frac{1}{n+1}$, $\alpha = 0.1$ and

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\beta_n^2}{\|x_n - x_{n-1}\|}, \alpha\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

We also choose $\alpha_n = \frac{1}{n+1}$, $\mu = 0.5$ and $\rho = 0.3$ for Algorithm 3.1 in [38] and $\alpha_n = \frac{1}{n+1}$, $\mu = 0.8$ for Algorithm 3.1 in [46].

The feasible set $C \subset \mathbb{R}^m$ is a closed and convex subset defined by

$$C := \{x \in \mathbb{R}^m : Hx \leq d\},$$

where H is an $l \times m$ matrix and d is a nonnegative vector.

Next, we give two choices of operator A.

Example 4.1 ([20])**.** Consider the following fractional programming problem:

$$\min\left\{h(x) = \frac{x^T Qx + a^T x + a_0}{b^T x + b_0}\right\},$$
$$\text{subject to } x \in X := \{x \in \mathbb{R}^4 : b^T x + b_0 > 0\},$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite in $\mathbb{R}^4$ and consequently h is pseudo-convex on $X = \{x \in \mathbb{R}^4 : b^T x + b_0 > 0\}$. Then

$$A(x) := \nabla h(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0)}{(b^T x + b_0)^2}.$$

It is known that A is pseudo-monotone (see, e.g. [3, 22, 37] for details).

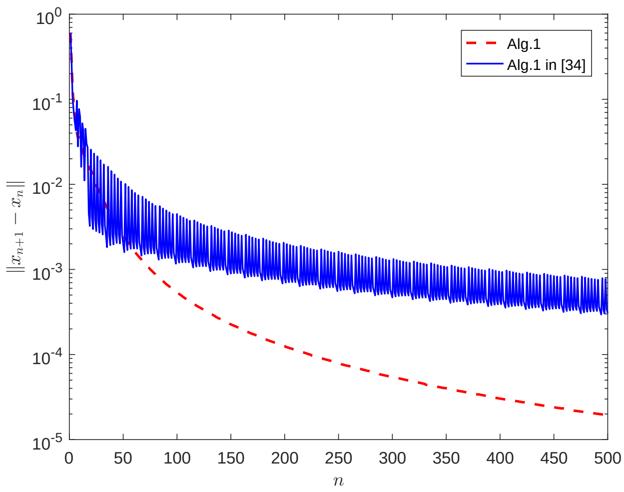

FIGURE 1. Comparison of Algorithm 1 and Algorithm 1 in [38].

We compared Algorithm 1 and Algorithm 3.1 in [38]. The numerical result is described in Figure 1, it is observed that Algorithm 3.1 behaves better than Algorithm 3.1 in [38].

Example 4.2. Next, let us define

$$A(x) = \begin{pmatrix} (x_1^2 + (x_2 - 1)^2)(1 + x_2) \\ -x_1^3 - x_1(x_2 - 1)^2 \end{pmatrix}$$

It is easy to see that A is not a monotone map. However, using the Monte Carlo approach (see [20]), it can be shown that A is pseudo-monotone.

For a given domain, the operator A is Lipschitz-continuous (see [37] for example).

We compared Algorithm 1 and Algorithm 1 in [46]. The numerical result is described in Figure 2, it illustrates that the performance of Algorithm 1 is better than that of Algorithm 1 in [46].

Example 4.3 ([13])**.** Take $A : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$A(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y)), \quad \forall x, y \in \mathbb{R}.$$

The authors [13] showed that A is $\sqrt{26}$-Lipschitz continuous and strongly monotone. Let $C = \{x \in \mathbb{R}^2 \mid -10e_1 \leq x \leq 10e_1\}$, where $e_1 = (1, 1)$. Therefore the variational inequality has a unique solution and $(0, 0)$ is its solution.

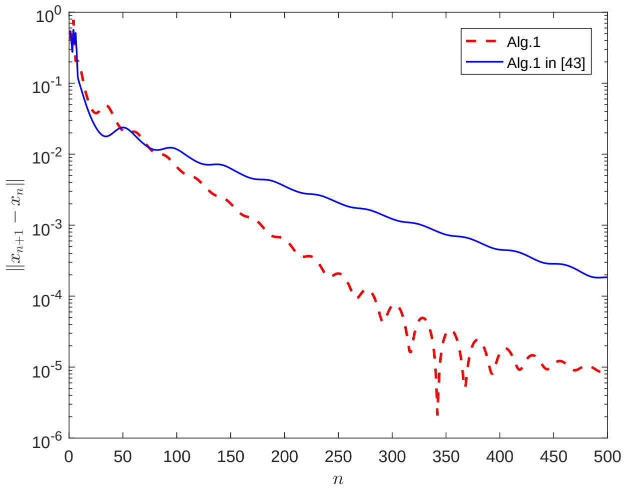

FIGURE 2. Comparison of Algorithm 1 and Algorithm 1 in [46].

We choose $x_0 = [2, -10]$ in this example. And take $\beta_n = \frac{1}{n}$, $\tau_n = \frac{1}{n^2}$, $\mu = 0.3$, $\alpha = 0.6$ and $\alpha_n = 0.7\bar{\alpha}$ in Algorithm 1, $\rho = 0, 1$ in Algorithm 3.1 [38], $\mu = 0.3$ and $\alpha_n = \frac{1}{n}$ in Algorithm 3.1 [38], Algorithm 1 in [46] and Algorithm 2 in [48].

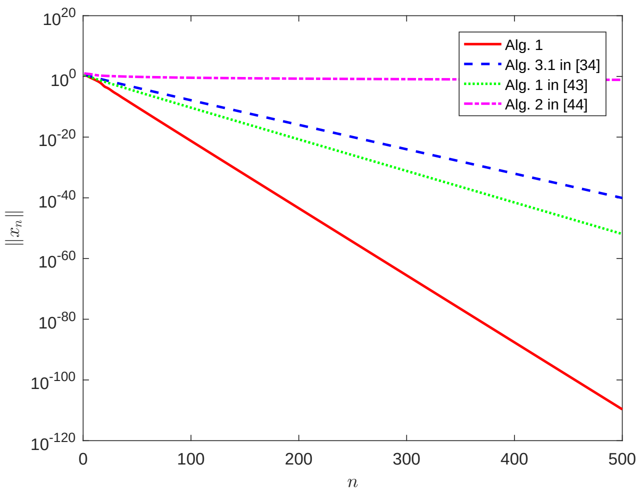

FIGURE 3. Comparison results of these algorithms in example 4.3.

From Figure 3, we know that the performance of Algorithm 1 is better than Algorithm 3.1 in [38], Algorithm 1 in [46] and Algorithm 2 in [48].

Next, we present an example in an infinite dimensional Hilbert space.

Example 4.4 ([37]). Suppose that $H = L^2([0,1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)dt,\ x, y \in H$. Let $C := \{x \in H : \|x\| \le 1\}$ be the unit ball. Define an operator $A : C \to H$ by

$$A(x)(t) = \int_0^1 (x(t) - F(t,s)v(x(s)))ds + h(t),\ x \in C, t \in [0,1],$$

where

$$F(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}},\ v(x) = \cos x,\ h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

It is known that A is monotone and L-Lipschitz-continuous with $L = 2$ and $\{0\}$ is the solution of the corresponding variational inequality problem.

Let $x_0 = cos(3\pi t)$, $x_1 = sin(3\pi t)$. Take $\beta_n = \frac{1}{n}$, $\tau_n = \frac{1}{n^2}$, $\mu = 0.3$, $\alpha = 0.6$ and $\alpha_n = 0.7\bar{\alpha}$ in Algorithm 1. And take $\tau = \frac{0.9}{L}$ and $\alpha_n = \frac{1}{n}$ in Algorithm (4) in [26], $\alpha_n = \frac{1}{n}$ in Algorithm (3.6) in [28] and $\lambda = \frac{0.9}{L}$, $\alpha_n = 0.1$ and $\beta_n = \frac{1}{n}$ in Algorithm 3.1 in [41], $\mu = 0.3$ and $\alpha_n = \frac{1}{n}$ in Algorithm 1 in [46].

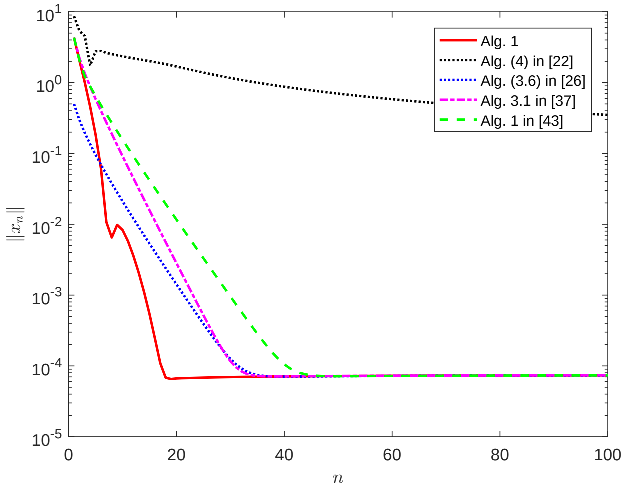

FIGURE 4. Comparison results of these algorithms in example 4.4.

From Figure 4, we get that although the value of $\|x_n\|$ of Algorithm 1, Algorithm (3.6) in [28], Algorithm 3.1 in [41] and Algorithm 1 in [46] are almost equal after 45 steps, Algorithm 1 converges faster. The iteration of Algorithm (4) in [26] is very slow.

Example 4.5. This example is taken from [19] and has been considered by many authors for numerical experiments. The operator A is defined by $A(x) := Mx + q$, where $M = BB^T + S + D$, with $B, S, D \in R^{m\times m}$ randomly generated matrices such that S is skew-symmetric (hence the operator does not arise from an optimization problem), D is a positive definite diagonal matrix (hence the variational inequality has a unique solution) and $q = 0$. The feasible set C is described by linear inequality constraints $Bx \leq b$ for some random matrix $B \in R^{m\times m}$ and a random vector $b \in R^k$ with nonnegative entries. It is easy to see that A is η-strongly pseudo-monotone with $\eta = \min(eig(M^TM))$ and Lipschitz continuous with $L = \|M\|$. Hence the zero vector is feasible and therefore the unique solution of the corresponding variational inequality.

These projections are computed using the MATLAB solver fmincon. Hence, for this class of problem, the evaluation of A is relatively inexpensive, whereas projections are costly. We present the corresponding numerical results (number of iterations and CPU times n seconds) using four different dimensions m and two different numbers of inequality constraints k.

TABLE 1. Comparison of Algorithm 1 and Algorithm 3.1 of [29] and Algorithm 1 of [30].

Problem	Size	Alg. 1		Alg. 3.1 of [29]		Alg. 1 of [30]	
k	m	Iter.	Sec.	Iter.	Sec.	Iter.	Sec.
30	10	111	0.104183	750	0.214737	2208	0.292347
	30	221	0.121658	6744	1.873812	6146	0.867736
	50	554	0.348353	15290	4.191198	16040	2.202233
	70	1002	0.58592	31033	8.651789	30828	4.319426
50	10	72	0.108587	778	0.474368	1388	0.307962
	30	210	0.162806	6315	2.72109	6980	1.49006
	50	522	0.441042	15684	6.8122369	17360	3.823579
	70	798	0.593785	23164	10.126965	33062	7.512002

We randomly choose the starting points $x_0 = (1, 1, ..., 1)$ and the stopping criterion as $\|x_n\| \leq 0.0001$. The size $k = 30, 50$ and $m = 10, 30, 50, 70$. The matrices B, S, D and the vector b are generated randomly. Set $\mu = 0.3$, $\beta_n = \frac{1}{n}$ and $\tau_n = \frac{1}{n^2}$ in Algorithm 1. Take $\lambda = 0.9\frac{\sqrt{2}-1}{L}$ in Algorithm 3.1 of [29] and $\varphi = \frac{\sqrt{5}+1}{2}$, $\phi = 0.9\varphi$ and $\bar{\lambda} = \lambda_0 = 1$ in Algorithm 1 of [30].

We can see from Table 1 and Figure 4.5 that Algorithm 1 performs better than Algorithm 3.1 in [29] and Algorithm 1 in [30].

5. Conclusions

In this paper, we introduce two variants of subgradient extragradient methods for solving *pseudomonotone* variational inequality in real Hilbert spaces. We present the strong convergence of algorithms under the Lipschitz continuous condition or non lipschitz continuous condition of the variational inequality mapping. The results obtain in this paper extend some recent results in the literature, see [38, 40, 41, 46, 48, 47]. The efficiency of the proposed algorithms has also been illustrated by several numerical experiments.

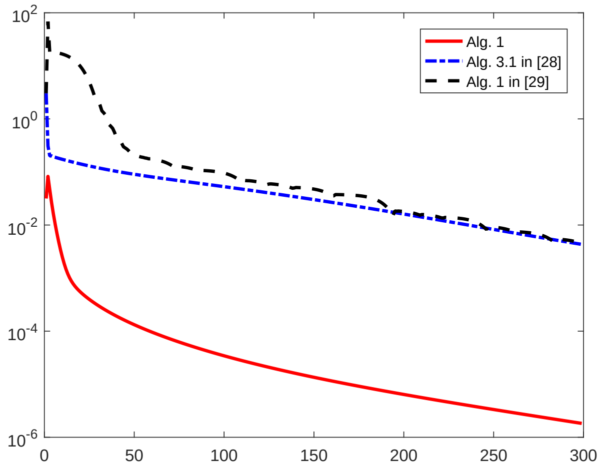

FIGURE 5. Comparison of Algorithm 1 and Algorithm 3.1 in [29] and Algorithm 1 in [30] in example 4.5 when $k = 30, m = 10$.

REFERENCES

[1] A. S. Antipin, *On a method for convex programs using a symmetrical modification of the Lagrange function*, Ekonomika i Mat. Metody **12** (1976), 1164–1173.

[2] F. Alvarez and H. Attouch, *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Set-Valued Anal. **9** (2001), 3–11.

[3] R. I. Boţ, E. R. Csetnek and P. T. Vuong, *The forward-backward-forward Method from continuous and discrete perspective for pseudo-monotone variational inequalities in Hilbert spaces*, arXiv:1808.08084

[4] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.

[5] Y. Censor, A. Gibali and S. Reich, *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optim. Meth. Softw. **26** (2011), 827–845.

[6] Y. Censor, A. Gibali and S. Reich, *Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space*, Optimization **61** (2012), 1119–1132.

[7] Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. Numer. Algorithms **59**, 301-323 (2012)

[8] L. C. Ceng, N. Hadjisavvas and N. C. Wong, *Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems*, J. Glob. Optim. **46** (2010), 635–646.

[9] R. W. Cottle and J. C. Yao, *Pseudo-monotone complementarity problems in Hilbert space*, J. Optim. Theory Appl. **75** (1992), 281–295.

[10] S. V. Denisov, V. V. Semenov and L. M. Chabak, *Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators*, Cybern. Syst. Anal. **51** (2015), 757–765.

[11] Q. L. Dong, Y. J. Cho, L. L. Zhong and Th. M. Rassias, *Inertial projection and contraction algorithms for variational inequalities*, J. Glob. Optim. **70** (2018), 687–704.

[12] Q. L. Dong, Y. Y. Lu and J. F. Yang, *The extragradient algorithm with inertial effects for solving the variational inequality*, Optimization **65** (2016), 2217–2226.

[13] Q. L. Dong, H. B. Yuan, Y. J. Cho and Th. M. Rassias, *Modified inertial Mann algorithm and inertial CQ-algorithm for nonexpansive mappings*, Optim. Lett. **12** (2018), 87–102.

[14] G. Fichera, *Sul problema elastostatico di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. **34** (1963), 138–142.

[15] G. Fichera, *Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno*, Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., Sez. I, VIII. Ser. **7** (1964), 91–140.

[16] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems. Volume I*, Springer Series in Operations Research, Springer, New York, 2003

[17] A. Gibali, S. Reich and R. Zalas, *Outer approximation methods for solving variational inequalities in Hilbert space*, Optimization **66** (2017), 417–437.

[18] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York; 1984

[19] P. T. Harker and J. S. Pang, *A damped-Newton method for the linear complementarity problem*, in: Computational Solution of Nonlinear Systems of Equations, Lectures in Appl. Math. 26, G. Allgower and K. Georg, eds., AMS, Providence, RI, 1990, pp. 265-284.

[20] X. Hu and J. Wang, *Solving pseudo-monotone variational inequalities and pseudo-convex optimization problems using the projection neural network*, IEEE Trans. Neural Netw. **17** (2006), 1487–1499.

[21] A. N. Iusem and M. Nasri, *Korpelevich's method for variational inequality problems in Banach spaces*, J. Global Optim. **50** (2011), 59–76.

[22] S. Karamardian and S. Schaible, *Seven kinds of monotone maps*, J. Optim. Theory Appl. **66** (1990), 37–46.

[23] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980

[24] I. V. Konnov, *Combined Relaxation Methods for Variational Inequalities*, Springer-Verlag, Berlin, 2001

[25] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekonomikai Matematicheskie Metody **12** (1976), 747–756.

[26] R. Kraikaew and S. Saejung, *Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces*, J. Optim. Theory Appl. **163** (2014), 399–412.

[27] P. E. Maingé, *Inertial iterative process for fixed points of certain quasi-nonexpansive mappings*, Set Valued Anal. **15** (2007), 67–79.

[28] P. E. Maingé, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim. **47** (2008), 1499–1515.

[29] Y. V. Malitsky, *Projected reflected gradient methods for monotone variational inequalities*, SIAM J. Optim. **25** (2015), 502–520.

[30] Y. V. Malitsky, *Golden ratio algorithms for variational inequalities*, Math. Program. **184** (2020), 383–410.

[31] Y. V. Malitsky and V. V. Semenov, *A hybrid method without extrapolation step for solving variational inequality problems*, J. Glob. Optim. **61** (2015), 193–202.

[32] A. Moudafi, *Viscosity approximation methods for fixed points problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[33] H. Liu and J. Yang, *Weak convergence of iterative methods for solving quasimonotone variational inequalities*, Comput. Optim. Appl. (2020) https://doi.org/10.1007/s10589-020-00217-8

[34] N. Nadezhkina and W. Takahashi, *Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **128** (2006), 191–201.

[35] B. T. Polyak, *Some methods of speeding up the convergence of iterarive methods*, Zh. Vychisl. Mat. Mat. Fiz. **4** (1964), 1–17.

[36] S. Saejung and P. Yotkaew, *Approximation of zeros of inverse strongly monotone operators in Banach spaces*, Nonlinear Anal. **75** (2012), 742–750.

[37] Y. Shehu, Q. L. Dong and D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optimization **68** (2019), 385–409.

[38] Y. Shehu and O. S. Iyiola, *Strong convergence result for monotone variational inequalities*, Numer. Algorithms **76** (2017), 259–282.

[39] Solodov, M.V., Svaiter, B.F.: A new projection method for variational inequality problems. SIAM J. Control Optim. **37** (1999), 765–776.

[40] D. V. Thong and D. V. Hieu, *Modified subgradient extragradient method for variational inequality problems*, Numer. Algorithms **79** (2018), 597–610.

[41] D. V. Thong, N. T. Vinh and Y. J. Cho, *A strong convergence theorem for Tseng's extragradient method for solving variational inequality problems*, Optim. Lett. **14** (2020), 1157–1175.

[42] P. T. Vuong, *On the weak convergence of the extragradient method for solving pseudo-monotone variational inequalities*, J. Optim. Theory Appl. **176** (2018), 399–409.

[43] P. T. Vuong and Y. Shehu, *Convergence of an extragradient-type method for variational inequality with applications to optimal control problems.* Numer. Algorithms **81** (2019), 269–291.

[44] Y. M. Wang, Y. B. Xiao, X. Wang and Y. J. Cho, *Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems*, J. Nonlinear Sci. Appl. **9** (2016), 1178–1192.

[45] Y. B. Xiao, N. J. Huang and Y. J. Cho, *A class of generalized evolution variational inequalities in Banach space*, Appl. Math. Lett. **25** (2012), 914–920.

[46] J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algorithms **80** (2019), 741–752.

[47] J. Yang and H. Liu, *A modified projected gradient method for monotone variational inequalities*, J. Optim. Theory Appl. **179** (2018), 197–211. (2018)

[48] J. Yang, H. Liu and Z. Liu, *Modified subgradient extragradient algorithms for solving monotone variational inequalities*, Optimization **67** (2018), 2247–2258.

[49] Y. Yao, G. Marino and L. Muglia, *A modified Korpelevich's method convergent to the minimum-norm solution of a variational inequality*, Optimization **63** (2014), 559–569.

Manuscript received 21 Feburary 2021
revised 11 January 2022

D. V. Thong
Faculty of Mathematical Economics, National Economics University, Hanoi City, Vietnam
E-mail address: thongduongviet@gmail.com

Q. L. Dong
Tianjin Key Laboratory for Advanced Signal Processing and College of Science, Civil Aviation University of China, Tianjin 300300, China
E-mail address: dongql@lsec.cc.ac.cn

X. H. Li
Tianjin Key Laboratory for Advanced Signal Processing and College of Science, Civil Aviation University of China, Tianjin 300300, China
E-mail address: xiaohuanlimath@163.com

H. V. Thang
Faculty of Mathematical Economics, National Economics University, Hanoi City, Vietnam
E-mail address: thanghv.ktqd@gmail.com

P. V. Nghia
Faculty of Mathematical Economics, National Economics University, Hanoi City, Vietnam
E-mail address: nghia25ktqd@gmail.com

N. T. C. Van
Faculty of Mathematical Economics, National Economics University, Hanoi City, Vietnam
E-mail address: camvanktqd@gmail.com