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ON OPTIMALITY AND DUALITY FOR NONSMOOTH ROBUST MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING PROBLEM WITH MIXED CONSTRAINTS

PUSHKAR JAISAWAL* AND VIVEK LAHA†

Dedicated to the memory of Professor Wataru Takahashi

Abstract. In this paper, we deal with nonsmooth multiobjective semi-infinte programming problems with mixed constraints with uncertainty in the feasible region and in the objective functions. We derive necessary and sufficient optimality conditions for weak efficiency in terms of Clarke subdifferential using a robust approach, We also use Wolfe and Mond-Weir type robust dual models to establish duality results.

1. Introduction

In the real world problems, most of the time data is not to be evaluated exactly, in which some variations or errors are occurred. These variations or errors are comes out due to the measurement of equipments, suddenly demands or returns of any product in the future, and missing of some data in the given series are evaluated by some known numerical technique, etc. Such types of variations or errors are known as uncertain set. For deal with such type of problems in optimization theory, robust optimization is come. In 1973, one of the first researcher, who studied about it, is Soyster [21]. In robust optimization problems uncertain sets are always belonged in a bounded set and the set of feasible solution are comes under all realizations of data from the uncertainty set. When the uncertainty is present in the objective functions, then the robust optimization technique first find out the maximization of functions over the uncertainty and after that optimize the problem.

Many authors have been doing lots of work in robust optimization. Recently Lee and Son [17] gave the optimality conditions for nonsmooth robust multiobjective problems after that Lee and Lee [15] gave optimality conditions for the weakly and properly robust efficient solution. Choung [1] gave the optimality and duality results for robust multiobjective optimization problems in terms of limiting subdifferentials. In this year, Fakhar *et. al.,* [8] gave the sufficient optimality conditions

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with applications to portfolio optimization for nonsmooth robust multiobjective optimization problems and Lee and Lee, [16] gave the optimality and duality results for continuously differentiable semi-infinite multiobjective optimization problems.

In this paper, we consider a nonsmooth robust multiobjective semi-infinite programming problem with mixed constraints (RMOSIP). We establish KKT type necessary optimality conditions for nonsmooth multiobjective semi-infinite programming problem with mixed constraints in terms of Clarke subdifferantials under suitable constraint qualifications and with the help of it, we derive KKT type necessary optimality conditions for RMOSIP. Also, establish sufficient type optimality conditions for RMOSIP under convexity and generalized convexity assumptions. After that, formulate Wolfe type and Mond-Weir type robust dual models and establish weak and strong duality results for both the models under generalized convexity assumptions.

The outline of this paper is as follows: in Section 2, we give some well known definitions and theorems which will be used in the sequel. In Section 3, we derive KKT type necessary optimality conditions for nonsmooth multiobjective semi-infinite programming problem with mixed constraints and RMOSIP under suitable constraint qualification. Also, derive sufficient optimality conditions for RMOSIP under convexity and generalized convexity assumptions and lastly in this section give an example in the support of KKT type necessary optimality conditions. In Section 4, we give Wolfe and Mond-Weir type duality problems and derive weak and strong duality results for both the dual models under convexity and generalized convexity assumptions. And lastly, in Section 5, we conclude the results of this paper.

2. Preliminaries

Let X be a non empty subset of the Euclidean space $\mathbb{R}^n$. The cardinality of X (the number of elements in the set X, if X is finite), the closure of X, the convex cone generated by X containing the origin of $\mathbb{R}^n$, the convex hull of X, the interior of X, and the linear hull of X (the smallest subspace containing X) are denoted by $cardX$, clX, $coneX$, coX, $intX$, and $spanX$, respectively. The negative and strictly negative polar cones of X are denoted by $X^{\leq}$ and $X^{<}$, respectively, and are defined by

$$X^{\leq} := \{d \in \mathbb{R}^n \ : \ \langle x,\ d\rangle \leq 0,\ \forall\, x \in X\}$$

and

$$X^{<} := \{d \in \mathbb{R}^n \ : \ \langle x,\ d\rangle < 0,\ \forall\, x \in X\}.$$

We recall some basic results in the calculus of generalized gradient (for more details, see [3]-[6], [14]).

Definition 2.1 (Locally Lipschitz functions)**.** A function $f : X \to \mathbb{R}$ is said to be *locally Lipschitz* at $x \in X$, iff there exist an open neighbourhood N of x and a constant $L > 0$ such that

$$|f(y) - f(z)| \leq L||y - z||,\ \forall y,\ z \in N,$$

where L is called the *Lipschitz constant* and the function f is said to be a locally Lipschitz function on X, iff it is locally Lipschitz for all $x \in X$.

Definition 2.2 (One-sided directional derivative)**.** The *usual one-sided directional derivative* of f at $x \in X$ for each fixed $d \in \mathbb{R}^n$ is denoted by $f'(x;\ d)$ and is defined by

$$f'(x;\ d) = \lim_{t \downarrow 0} \frac{f(x+td) - f(x)}{t},$$

whenever this limit exists.

Definition 2.3 (Cone of feasible directions [12])**.** The *cone of all the feasible directions* of X at any point $x \in X$, denoted by $\Gamma(X,\ x)$ and is defined by

$$\Gamma(X,\ x) := \{d \in \mathbb{R}^n\ :\ \exists\ \epsilon \downarrow 0, \text{ such that } x + td \in X,\ \forall\ t \in]0,\ \epsilon[\}.$$

Definition 2.4 (Clarke's subdifferentials [6])**.** The *Clarke's directional derivative* of f at $x \in X$ in a direction d, denoted by $f^0(x;\ d)$ and is defined by

$$f^0(x;\ d) := \limsup_{h \to 0,\ t \downarrow 0} \frac{f(x+h+td) - f(x+h)}{t},$$

and the *Clarke's subdifferential* of f at $x \in X$, denoted by $\partial^0 f(x)$ and is defined by

$$\partial^0 f(x) := \{x^* \in \mathbb{R}^n\ :\ f^0(x;\ d) \geq \langle x^*,\ d\rangle,\ \forall\ d \in \mathbb{R}^n\}.$$

Some properties of the Clarke's subdifferentials are given as follows:

Lemma 2.5. *Let $f_1,\ f_2,\ \ldots,\ f_m$ be m real valued locally Lipschitz functions at $x \in X$ which are defined on X. Then,*

(1) $\partial^0 f_k(x)$ is nonempty, convex, weak* compact subset of $\mathbb{R}^n$ and $||x_k^*|| \leq L_k$ for every $x_k^* \in \partial^0 f_k(x)$ and $k \in M := \{1,\ 2,\ \ldots,\ m\}$, where L_k are Lipschitz constants for f_k at x.
(2) The function $d \mapsto f_k^0(x;\ d)$ is convex, positively homogeneous, finite and subadditive and satisfies

$$|f_k^0(x;\ d)| \leq L_k ||d||$$

for every $k \in M$.
(3) $\partial^0(\lambda f_k)(x) = \lambda \partial^0 f_k(x),\ \forall\ \lambda \in \mathbb{R},\ k \in M$.
(4) $f_k^0(x;\ d) = \max\{\langle x_k^*,\ d\rangle\ :\ x_k^* \in \partial^0 f_k(x)\},\ \forall\ d \in \mathbb{R}^n,\ \forall\ k \in M$.
(5) $\partial^0(f_1 + f_2 + \cdots + f_m)(x) \subset \partial^0 f_1(x) + \partial^0 f_2(x) + \cdots + \partial^0 f_m(x)$.

Let $\mathbb{B}$ be a sequentially compact topological space and let $g\ :\ \mathbb{R}^n \times \mathbb{B} \to \mathbb{R},\ (x,\ b) \mapsto g(x,\ b)$, satisfies the following hypothesis from [17]:

Hypothesis

($H1$) $g(x,\ b)$ is upper semicontinuous in $(x,\ b)$.
($H2$) g is a locally Lipschitz in x, uniformly for b in $\mathbb{B}$, that is, for each x in $\mathbb{R}^n$, there exist an open neighbourhood N of x and a constant $L_g > 0$ such that for all $y,\ z \in N$, and $b \in \mathbb{B}$,

$$|g(y,\ b) - g(z,\ b)| \leq L_g ||y - z||.$$

($H3$) $g_x^0(x,\ b;\ \cdot) = g_x'(x,\ b;\ \cdot)$, the derivatives being with respect to x.

($H4$) The generalized gradient $\partial_x^0 g(x,\ b)$ with respect to x is weak* upper semicontinuous in $(x,\ b)$.

Remark 2.6. A function $\psi\ :\ \mathbb{R}^n \to \mathbb{R}$ with

$$\psi(x) := \max\{g(x,\ b)\ :\ b \in \mathbb{B}\},$$

and we observe that our hypothesis $(H1) - (H4)$ holds, it implies that the functions ψ exists and finite (with the maximum defining ψ attained) on $\mathbb{R}^n$. Since each $g(\cdot,\ b)$ is locally Lipschitz, then the function ψ is also locally Lipschitz on $\mathbb{R}^n$. Let

$$\mathbb{B}(x) := \{b \in \mathbb{B}\ :\ g(x,\ b) = \psi(x)\},$$

then $\mathbb{B}(x)$ is nonempty and closed for each $x \in \mathbb{R}^n$.

The connection between the functions $\psi'(x;\ d)$ and $g_x^0(x,\ b;\ d)$ is given by the nonsmooth version of Danskin's theorem [7] for max-functions, which is given by the following lemma:

Lemma 2.7 ([7]). *Under the hypothesis $(H1) - (H4)$, the usual one sided directional derivative $\psi'(x;\ d)$ exists, and satisfies*

$$\begin{aligned}\psi'(x;\ d) = \psi^0(x;\ d) &= \max\{g_x^0(x,\ b;\ d)\ :\ b \in \mathbb{B}(x)\}\\ &= \max\{\langle x^*,\ d\rangle\ :\ x^* \in \partial_x^0 g(x,\ b),\ b \in \mathbb{B}(x)\}.\end{aligned}$$

Some important Lemma's which will be useful for remaining parts of the paper. This Lemma is given by Lee and Son [17]:

Lemma 2.8 ([17]). *Suppose that $\mathbb{B}$ is convex, and that $g(x,\ \cdot)$ is concave on $\mathbb{B}$, for each $x \in \mathbb{R}^n$, and the hypothesis $(H1) - (H4)$ hold. Then the following statements are satisfied:*

(1) *The set $\mathbb{B}(x)$ is convex and sequentially compact.*

(2) *The set*

$$\partial_x^0 g(x,\ \mathbb{B}(x)) := \{x^*\ :\ \exists\, b\ \in\ \mathbb{B}(x) \text{ such that } x^*\ \in\ \partial_x^0 g(x,\ b)\}$$

is convex and weak* compact.

(3) $\partial^0\psi(x) := \{x^*\ :\ \exists\, b\ \in\ \mathbb{B}(x)$ *such that* $x^*\ \in\ \partial_x^0 g(x,\ b)\}$.

This Lemma is another type of Carathéodory's theorem given by Rockafellar [20]:

Lemma 2.9. *Let $\{C_t\ :\ t \in T\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and let $K := cone(\cup_{t\in T} C_t)$. Then, every nonzero vectors of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to different C_t.*

The following lemma is given by Hiriart-Urruty and Lemaréchal [11]:

Lemma 2.10. *If X is a nonempty compact subset of $\mathbb{R}^n$, then the $co(X)$ is a compact set.*

The generalized Motzkin's alternative theorem given by Goberna and López [10] for the semi-infinite programming problems is as follows:

Lemma 2.11. *Let P, S and T be arbitrary (possibly infinite) index sets, $a_p := a(P) := (a_1(P), \ldots, a_k(P))$ maps P onto $\mathbb{R}^n$, and so does a_s and a_t. Suppose that the set $co\{a_p : p \in P\} + cone\{a_s : s \in S\} + span\{a_t : t \in T\}$ is closed. Then, the system*

$$\text{(i)} \begin{cases} \langle a_p,\ x\rangle < 0, & p \in P,\ P \neq \phi \\ \langle a_s,\ x\rangle \leq 0, & s \in S \\ \langle a_t,\ x\rangle = 0, & t \in T \end{cases} \qquad \textit{is consistent}$$

or

(ii) $0 \in co\{a_p : p \in P\} + cone\{a_s : s \in S\} + span\{a_t : t \in T\}$

but never both.

3. Optimality conditions

We consider the following multiobjective semi-infinite programming problem with mixed constraints:

$$\begin{array}{llll} & \text{(MOSIP)} & \text{Minimize} & f(x) := (f_1(x),\ f_2(x), \ldots,\ f_m(x)) \\ (3.1) & & \text{Subject to} & g_i(x) \leq 0,\ i \in I, \\ & & & h_j(x) = 0,\ j \in J, \end{array}$$

where I and J are arbitrary sets not necessarily finite and $f_k,\ g_i$ and h_j are real valued functions defined on $\mathbb{R}^n$, for all $k \in M,\ i \in I$ and $j \in J$, respectively. The above multiobjective semi-infinite programming problem with mixed constraints in the face of data uncertainty to the objective functions and the inequality constraints can be written by

$$\begin{array}{llll} & \text{(UMOSIP)} & \text{Minimize} & (f_1(x,\ a_1),\ f_2(x,\ a_2), \ldots,\ f_m(x,\ a_m)) \\ (3.2) & & \text{Subject to} & g_i(x,\ b_i) \leq 0,\ i \in I, \\ & & & h_j(x) = 0,\ j \in J, \end{array}$$

where a_k and b_i are uncertain parameters, $\mathbb{A}_k$ and $\mathbb{B}_i$ are sequentially compact topological spaces with $a_k \in \mathbb{A}_k$ and $b_i \in \mathbb{B}_i$ for $k \in M$ and $i \in I$, respectively. The functions are defined by $f_k : \mathbb{R}^n \times \mathbb{A}_k \to \mathbb{R}$, $g_i : \mathbb{R}^n \times \mathbb{B}_i \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$, for all $k \in M,\ i \in I$ and $j \in J$.

Goberna et al. [9] studied linear multiobjective semi-infinite programming problems after it Lee and Lee [16] studied multiobjective semi-infinite programming problems with data uncertainty in constraints only. The corresponding robust multiobjective semi-infinite programming problem with mixed constraints (RMOSIP) for the above (MOSIP) is:

$$\begin{array}{lll} \text{(RMOSIP)} & \text{Minimize} & \left(\max\limits_{a_1 \in \mathbb{A}_1} f_1(x,\ a_1),\ \max\limits_{a_2 \in \mathbb{A}_2} f_2(x,\ a_2), \ldots,\ \max\limits_{a_m \in \mathbb{A}_m} f_m(x,\ a_m)\right) \\ (3.3) & \text{Subject to} & g_i(x,\ b_i) \leq 0,\ \forall b_i \in \mathbb{B}_i,\ i \in I, \\ & & h_j(x) = 0,\ j \in J, \end{array}$$

where the uncertain objective and the constraint functions are embedded with the every possible value of the parameters within their prescribed uncertainty sets $\mathbb{A}_k,\ k \in M$ and $\mathbb{B}_i,\ i \in I$.

3.1. **Necessary optimality conditions.** Consider the MOSIP (3.1) and let

$$X := \{x \in \mathbb{R}^n\ :\ g_i(x) \leq 0,\ h_j(x) = 0,\ i \in I,\ j \in J\}$$

be the set of all feasible points for the MOSIP. A vector $\bar{x} \in X$ is said to be a *weakly efficient solutions* for MOSIP, iff

$$f(x) - f(\bar{x}) \notin -int\mathbb{R}^m_+,\ \forall\, x \in X.$$

Let $I(\bar{x}) := \{i \in I\ :\ g_i(\bar{x}) = 0\}$ for any $\bar{x} \in X$ and $F(\bar{x}) := \bigcup_{k=1}^{m} \partial^0 f_k(\bar{x})$. Also, suppose that

$$\Lambda(\bar{x}) := \Big(\bigcup_{i\in I(\bar{x})} \partial^0 g_i(\bar{x})\Big) \cup \Big(\bigcup_{j\in J} \partial^0 h_j(\bar{x})\Big) \cup \Big(-\bigcup_{j\in J} \partial^0 h_j(\bar{x})\Big).$$

The *negative polar cone* of $\Lambda(\bar{x})$ is given by

$$\big(\Lambda(\bar{x})\big)^{\leq} = \{d \in \mathbb{R}^n\ :\ g_i^0(\bar{x};\ d) \leq 0 \text{ for } i \in I(\bar{x}) \text{ and } h_j^0(\bar{x};\ d) = 0 \text{ for } j \in J\},$$

and the *convex cone* of $\Lambda(\bar{x})$ is given by

$$cone\big(\Lambda(\bar{x})\big) = cone\Big(\bigcup_{i\in I(\bar{x})} \partial^0 g_i(\bar{x})\Big) + span\Big(\bigcup_{j\in J} \partial^0 h_j(\bar{x})\Big).$$

Zangwill Constraint qualifications (ZCQ) was introduced by Zangwill [22] for inequality type optimization problems, which was extended by Kanzi and Nobakhtian [13] for mixed type semi-infinite optimization problems, which is as follows:

Definition 3.1. (Zangwill Constraint Qualifications) The *Zangwill Constraint Qualifications* (ZCQ) holds at $\bar{x} \in X$ for MOSIP, iff

$$\big(\Lambda(\bar{x})\big)^{\leq} \subseteq cl\Gamma(X,\ \bar{x}).$$

The following necessary condition for the MOSIP under ZCQ can be obtained.

Theorem 3.2. *Let $\bar{x} \in X$ be a weak efficient solution of MOSIP. Let f_k $(k \in M)$, g_i $(i \in I(\bar{x}))$ and h_j $(j \in J)$ are locally Lipschitz at $\bar{x} \in X$. Also, let ZCQ be satisfied at $\bar{x}$ and let the cone$(\Lambda(\bar{x}))$ be a closed cone in $\mathbb{R}^n$. Then, there exists $\bar{\lambda}_k \geq 0,\ k \in M$ with $\sum_{k\in M} \bar{\lambda}_k = 1$ and $\bar{\mu}_i \geq 0,\ i \in I,\ \bar{\gamma}_j \in \mathbb{R},\ j \in J$, with finitely many of them being nonzero, such that*

$$0 \in \sum_{k\in M} \bar{\lambda}_k \partial^0 f_k(\bar{x}) + \sum_{i\in I} \bar{\mu}_i \partial^0 g_i(\bar{x}) + \sum_{j\in J} \bar{\gamma}_j \partial^0 h_j(\bar{x}), \tag{3.4}$$

$$\textit{and} \qquad \bar{\mu}_i g_i(\bar{x}) = 0,\ \forall\, i \in I. \tag{3.5}$$

Proof. Let $\bar{x} \in X$ be a weak efficient solution of MOSIP. First of all, we are interested to show that

$$F(\bar{x})^{<} \cap \Gamma(X,\ \bar{x}) = \{\}. \tag{3.6}$$

Suppose that the above equality (3.6) is not satisfied, then there exists $d \in \mathbb{R}^n$ such that

$$d \in F(\bar{x})^{<} \cap \Gamma(X,\ \bar{x}),$$

i.e., $d \in F(\bar{x})^{<}$ and $d \in \Gamma(X,\ \bar{x})$. Hence,

$$\langle F(\bar{x}),\ d\rangle < 0,$$

i.e.,

$$\langle \bar{x}_k^*,\ d\rangle < 0,\ \forall\, \bar{x}_k^* \in \partial^0 f_k(\bar{x}),\ \forall\, k = 1,\ 2,\ \dots,\ m,$$

which gives,

$$f_k^0(\bar{x};\ d) = \max_{\bar{x}_k^* \in \partial^0 f_k(\bar{x})} \langle \bar{x}_k^*,\ d\rangle < 0,\ \forall\, k = 1,\ 2,\ \dots,\ m. \tag{3.7}$$

Now, for any $k \in M$, one has

$$\begin{aligned}
\limsup_{t_k \downarrow 0} \frac{f_k(\bar{x} + t_k d) - f_k(\bar{x})}{t_k} &= \inf_{\alpha_k > 0,} \sup_{0 < t_k < \alpha_k} \frac{f_k(\bar{x} + t_k d) - f_k(\bar{x})}{t_k} \\
&\leqq \inf_{\substack{\beta_k > 0, \\ \alpha_k > 0,}} \sup_{\substack{\|h_k\| < \beta_k \\ 0 < t_k < \alpha_k}} \frac{f_k(\bar{x} + h_k + t_k d) - f_k(\bar{x} + h_k)}{t_k} \\
&= \limsup_{\substack{h_k \to 0, \\ t_k \downarrow 0}} \frac{f_k(\bar{x} + h_k + t_k d) - f_k(\bar{x} + h_k)}{t_k} \\
&= \limsup_{\substack{y = \bar{x} + h_k \to \bar{x} \\ t_k \downarrow 0}} \frac{f_k(y + t_k d) - f_k(y)}{t_k} \\
&= f_k^0(\bar{x};\ d)\ < 0,
\end{aligned}$$

which gives

$$\begin{aligned}
\limsup_{t_k \downarrow 0} \frac{f_k(\bar{x} + t_k d) - f_k(\bar{x})}{t_k} &= \inf_{\alpha_k > 0,} \sup_{0 < t_k < \alpha_k} \frac{f_k(\bar{x} + t_k d) - f_k(\bar{x})}{t_k} \\
&< 0,\ \forall\, k = 1,\ 2,\ \dots,\ m.
\end{aligned}$$

Hence, there exists $\alpha_k > 0$, such that for all $t_k \in]0,\ \alpha_k[$, one has

$$f_k(\bar{x} + t_k d) < f_k(\bar{x}),\ \forall\, k = 1,\ 2,\ \dots,\ m.$$

Let $\alpha = \min\{\alpha_1,\ \alpha_2,\ \dots,\ \alpha_k\}$, then for all $t \in]0,\ \alpha[$

$$f(\bar{x} + td) - f(\bar{x}) \in -int\mathbb{R}_+^m. \tag{3.8}$$

Also, $d \in \Gamma(X,\ \bar{x})$ gives the existence of $\beta > 0$ such that

$$\bar{x} + td \in X,\ \forall\, t \in]0,\ \beta[. \tag{3.9}$$

Taking $\epsilon = \min\{\alpha,\ \beta\}$, then (3.8) – (3.9) contradict that $\bar{x}$ is a weakly efficient solution for MOSIP. Hence our supposition is wrong and the equality (3.6) is satisfy. Then, the equality (3.6) gives

$$int(F(\bar{x}))^{<} \cap cl\Gamma(X,\ \bar{x}) = \{\}.$$

Since $(F(\bar{x}))^{<} = \{d \in \mathbb{R}^n \,:\, \langle \bar{x}_k^*,\ d\rangle < 0,\ \forall\, \bar{x}_k^* \in \partial^0 f_k(\bar{x}),\ \forall\, k = 1,\ 2,\ \ldots,\ m\}$ and $\langle \bar{x}_k^*,\ \cdot\rangle$ is continuous, we get that $(F(\bar{x}))^{<}$ is open, so

$$(F(\bar{x}))^{<} \cap cl\Gamma(X,\ \bar{x}) = \{\}.$$

Then, by the assumptions of the theorem, we get

$$(F(\bar{x}))^{<} \cap \big(\Lambda(\bar{x})\big)^{\leq} = \{\},$$

which indicates that there does not exist any $d \in \mathbb{R}^n$, for which the system

$$\begin{cases} \langle \bar{x}_k^*,\ d\rangle < 0,\ \ \forall\, \bar{x}_k^* \in \partial^0 f_k(\bar{x}),\ \forall\, k \in M,\\ \langle \bar{x}_i^*,\ d\rangle \leq 0,\ \ \forall\, \bar{x}_i^* \in \partial^0 g_i(\bar{x}),\ \forall\, i \in I(\bar{x}),\\ \langle \bar{x}_j^*,\ d\rangle = 0,\ \ \forall\, \bar{x}_j^* \in \partial^0 h_j(\bar{x}),\ \forall\, j \in J \end{cases}$$

is consistent, that is, the system

$$\begin{cases} \langle F(\bar{x}),\ d\rangle < 0,\\ \langle \partial^0 g_i(\bar{x}),\ d\rangle \leq 0,\ \ \forall\, i \in I(\bar{x}),\\ \langle \partial^0 h_j(\bar{x}),\ d\rangle = 0,\ \ \forall\, j \in J \end{cases}$$

is inconsistent.

Since a finite union of compact sets is compact in a finite dimensional space, which gives $F(\bar{x})$ is a compact set. By Lemma 2.10, $co(F(\bar{x}))$ is also a compact set. By assumptions of the theorem $cone\big(\Lambda(\bar{x})\big)$ is a closed set, then $co\big(F(\bar{x})\big) + cone\big(\bigcup_{i\in I(\bar{x})} \partial^0 g_i(\bar{x})\big) + span\big(\bigcup_{j\in J} \partial^0 h_j(\bar{x})\big)$ is a closed set. Then, by Lemma 2.11, we get

$$0 \in co\Big(F(\bar{x})\Big) + cone\Big(\bigcup_{i\in I(\bar{x})} \partial^0 g_i(\bar{x})\Big) + span\Big(\bigcup_{j\in J} \partial^0 h_j(\bar{x})\Big), \tag{3.10}$$

using the properties of the Clarke subdifferential and convex hull, there exists $\bar{\lambda}_k \geq 0$, for all $k \in M$ with $\sum_{k\in M} \bar{\lambda}_k = 1$, we get

$$0 \in \sum_{k\in M} \bar{\lambda}_k \partial^0 f_k(\bar{x}) + cone\Big(\Lambda(\bar{x})\Big). \tag{3.11}$$

By using Lemma 2.9, there exists $\bar{\mu}_i \geq 0,\ i \in I(\bar{x})$ and $\bar{\gamma}_j \in \mathbb{R},\ j \in J$ with finitely many of them being nonzero, we get

$$0 \in \sum_{k\in M} \bar{\lambda}_k \partial^0 f_k(\bar{x}) + \sum_{i\in I(\bar{x})} \bar{\mu}_i \partial^0 g_i(\bar{x}) + \sum_{j\in J} \bar{\gamma}_j \partial^0 h_j(\bar{x}). \tag{3.12}$$

Taking $\bar{\mu}_i = 0$ for $i \in I \setminus I(\bar{x})$, we get

$$0 \in \sum_{k\in M} \bar{\lambda}_k \partial^0 f_k(\bar{x}) + \sum_{i\in I} \bar{\mu}_i \partial^0 g_i(\bar{x}) + \sum_{j\in J} \bar{\gamma}_j \partial^0 h_j(\bar{x}). \tag{3.13}$$

Since $g_i(\bar{x}) = 0$ for all $i \in I(\bar{x})$ and by our choice $\bar{\mu}_i = 0$ for all $i \in I \setminus I(\bar{x})$. Thus the proof is complete. $\square$

3.2. **Robust necessary optimality conditions.** Consider the above RMOSIP (3.3), where $f_k : \mathbb{R}^n \times \mathbb{A}_k \to \mathbb{R}$, $g_i : \mathbb{R}^n \times \mathbb{B}_i \to \mathbb{R}$ are functions satisfying the above hypothesis $(H_1) - (H_4)$ with respect to sequentially compact topological spaces $\mathbb{A}_k$, $\mathbb{B}_i$ for all $k \in M$, $i \in I$, respectively and $h_j : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz functions for all $j \in J$. The robust feasible set of RMOSIP is

$$X_1 := \{x \in \mathbb{R}^n \ : \ \max_{b_i \in \mathbb{B}_i} g_i(x,\ b_i) \leq 0,\ i \in I,\ h_j(x) = 0,\ j \in J\}.$$

Let $\phi_k,\ \psi_i : \mathbb{R}^n \to \mathbb{R}$ for each $k \in M$, $i \in I$ and defined by

$$\phi_k(\bar{x}) := \max_{a_k \in \mathbb{A}_k} \{f_k(\bar{x},\ a_k)\} \text{ and } \psi_i(\bar{x}) := \max_{b_i \in \mathbb{B}_i} \{g_i(\bar{x},\ b_i)\}.$$

Let $\mathbb{A}_k(\bar{x}) := \{a_k \in \mathbb{A}_k \ : \ f_k(\bar{x},\ a_k) = \phi_k(\bar{x})\}$ and $\mathbb{B}_i(\bar{x}) := \{b_i \in \mathbb{B}_i \ : \ g_i(\bar{x},\ b_i) = \psi_i(\bar{x})\}$ for all $k \in M$ and $i \in I_1(\bar{x})$, where $I_1(\bar{x}) := \{i \in I \ : \ \psi_i(\bar{x}) = 0\}$.

Definition 3.3. (Weakly robust efficient solutions) Let $\bar{x} \in X_1$ be a *weakly robust efficient solution* of RMOSIP, if there does not exist any $x \in X_1$ of RMOSIP, such that

$$\max_{a_k \in \mathbb{A}_k} f_k(x,\ a_k) < \max_{a_k \in \mathbb{A}_k} f_k(\bar{x},\ a_k), \quad \forall\, k \in M.$$

Suppose that $\mathbb{A}_k$, $\mathbb{B}_i$ are convex, and that the functions $f_k(\bar{x},\ \cdot)$, $g_i(\bar{x},\ \cdot)$ are concave on $\mathbb{A}_k$, $\mathbb{B}_i$, respectively, for each $\bar{x} \in X$ and for each $k \in M$, $i \in I_1(\bar{x})$. Now, we define some sets with the help of Lemma 2.8,

$$\Lambda_1(\bar{x}) := \Big(\bigcup_{i \in I_1(\bar{x})} \big(\cup_{\bar{b}_i \in \mathbb{B}_i(\bar{x})} \partial_x^0 g_i(\bar{x},\ \bar{b}_i) \big) \Big) \cup \Big(\bigcup_{j \in J} \partial^0 h_j(\bar{x}) \Big) \cup \Big(- \bigcup_{j \in J} \partial^0 h_j(\bar{x}) \Big),$$

where $\partial_x^0 g_i(\bar{x},\ \bar{b}_i)$ is the Clarke subdifferential of g_i with respect to x. The negative polar cone of $\Lambda_1(\bar{x})$ is given by

$$\begin{aligned}&\big(\Lambda_1(\bar{x})\big)^{\leq} \\ &= \{d \in \mathbb{R}^n \ : \ \max_{\bar{b}_i \in \mathbb{B}_i(\bar{x})} g_{ix}^0(\bar{x},\ \bar{b}_i;\ d) \leq 0\ ,\ \forall\, i \in I_1(\bar{x}) \text{ and } h_j^0(\bar{x};\ d) = 0 \text{ for } j \in J\},\end{aligned}$$

where $g_{ix}^0(\bar{x},\ \bar{b}_i;\ d)$ is the Clarke directional derivative of g_i with respect to x and the convex cone of $\Lambda_1(\bar{x})$ is given by

$$cone\big(\Lambda_1(\bar{x})\big) = cone\Big(\bigcup_{i \in I_1(\bar{x})} \big(\cup_{\bar{b}_i \in \mathbb{B}_i(\bar{x})} \partial_x^0 g_i(\bar{x},\ \bar{b}_i) \big) \Big) + span\Big(\bigcup_{j \in J} \partial^0 h_j(\bar{x}) \Big).$$

Now, we extend the ZCQ in robust form and introduce robust Zangwill Constraint Qualifications.

Definition 3.4. (Robust Zangwill Constraint Qualifications) The *Robust Zangwill Constraint Qualifications* (RZCQ) holds at $\bar{x} \in X_1$ for RMOSIP, iff

$$\big(\Lambda_1(\bar{x})\big)^{\leq} \subseteq cl\Gamma(X_1,\ \bar{x}).$$

Theorem 3.5. *Under the hypothesis $(H_1) - (H_4)$, suppose that $\mathbb{A}_k$, $\mathbb{B}_i$ are convex, and that the functions $f_k(\bar{x},\ \cdot)$ $(k \in M)$, $g_i(\bar{x},\ \cdot)$ $(i \in I_1(\bar{x}))$ are concave on $\mathbb{A}_k$ $(k \in M)$, $\mathbb{B}_i$ $(i \in I_1(\bar{x}))$, respectively, for each $\bar{x} \in X_1$. Let $\bar{x} \in X_1$ be a weakly* *robust efficient solution of RMOSIP. Let h_j $(j \in J)$ be locally Lipschitz at $\bar{x} \in X_1$. Also, let RZCQ be satisfied at $\bar{x}$ and $cone(\Lambda_1(\bar{x}))$ be a closed cone in $\mathbb{R}^n$. Then, there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$ $(k \in M)$, $\bar{b}_i \in \mathbb{B}_i(\bar{x})$ $(i \in I_1(\bar{x}))$ and $\tilde{\lambda}_k \geq 0$ $(k \in M)$ with $\sum_{k\in M} \tilde{\lambda}_k = 1$ and $\tilde{\mu}_i \geq 0$ $(i \in I)$, $\tilde{\gamma}_j \in \mathbb{R}$ $(j \in J)$, with finitely many of them being nonzero, such that*

$$0 \in \sum_{k\in M} \tilde{\lambda}_k \partial_x^0 f_k(\bar{x},\ \bar{a}_k) + \sum_{i\in I} \tilde{\mu}_i \partial_x^0 g_i(\bar{x},\ \bar{b}_i) + \sum_{j\in J} \tilde{\gamma}_j \partial^0 h_j(\bar{x}), \tag{3.14}$$

$$\textit{and} \qquad \tilde{\mu}_i g_i(\bar{x},\ \bar{b}_i) = 0,\ \forall\ i \in I. \tag{3.15}$$

Proof. Above defining ϕ_k $(k \in M)$ and ψ_i $(i \in I)$, the RMOSIP may be rewritten as:

$$\begin{array}{lll} \text{(MOSIP1)} & \text{Minimize} & (\phi_1(x),\ \phi_2(x), \ldots,\ \phi_m(x)) \\ & \text{Subject to} & \psi_i(x) \leq 0,\ i \in I, \\ & & h_j(x) = 0,\ j \in J. \end{array}$$

Since $\bar{x}$ is a robust weakly efficient solution of the RMOSIP. Therefore, $\bar{x}$ is a weakly efficient solution of the MOSIP1. Also, since RZCQ is satisfied at $\bar{x}$ and $cone(\lambda_1(\bar{x}))$ is closed in $\mathbb{R}^n$. Then, by Theorem 3.2, there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$ $(k \in M)$, $\bar{b}_i \in \mathbb{B}_i(\bar{x})$ $(i \in I_1(\bar{x}))$ and $\tilde{\lambda}_k \geq 0$ $(k \in M)$ with $\sum_{k\in M} \tilde{\lambda}_k = 1$ and $\tilde{\mu}_i \geq 0$ $(i \in I)$, $\tilde{\gamma}_j \in \mathbb{R}$ $(j \in J)$, with finitely many of them being nonzero, such that

$$0 \in \sum_{k\in M} \tilde{\lambda}_k \partial^0 \phi_k(\bar{x}) + \sum_{i\in I_1(\bar{x})} \tilde{\mu}_i \partial^0 \psi_i(\bar{x}) + \sum_{j\in J} \tilde{\gamma}_j \partial^0 h_j(\bar{x}),$$

$$\text{and} \qquad \tilde{\mu}_i \psi_i(\bar{x}) = 0,\ \forall\ i \in I_1(\bar{x}).$$

Under assumption of the theorem, it follows from Lemma 2.8 that

$$\partial^0 \phi_k(\bar{x}) = \{\bar{x}_k^* \ :\ \exists\ \bar{a}_k \in \mathbb{A}_k(\bar{x}) \text{ such that } \bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ \bar{a}_k)\},\ k = 1,\ 2,\ \ldots,\ m,$$

$$\text{and } \partial^0 \psi_i(\bar{x}) = \{\bar{x}_i^* \ :\ \exists\ \bar{b}_i \in \mathbb{B}_i(\bar{x}) \text{ such that } \bar{x}_i^* \in \partial_x^0 g_i(\bar{x},\ \bar{b}_i)\},\ i \in I_1(\bar{x}).$$

Then, there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$ $(k \in M)$ and $\bar{b}_i \in \mathbb{B}_i(\bar{x})$ $(i \in I_1(\bar{x}))$ satisfying the following conditions

$$0 \in \sum_{k\in M} \tilde{\lambda}_k \partial_x^0 f_k(\bar{x},\ \bar{a}_k) + \sum_{i\in I_1(\bar{x})} \tilde{\mu}_i \partial_x^0 g_i(\bar{x},\ \bar{b}_i) + \sum_{j\in J} \tilde{\gamma}_j \partial^0 h_j(\bar{x}),$$

$$\text{and} \qquad \tilde{\mu}_i g_i(\bar{x},\ \bar{b}_i) = 0,\ \forall\ i \in I_1(\bar{x}).$$

Hence, by setting $\tilde{\mu}_i = 0$ for $i \in I \setminus I_1(\bar{x})$, we get the required result. □

3.3. **Robust sufficient optimality conditions.** In this subsection, we derive sufficient optimality condition for RMOSIP, in which involved functions are either convex or generalized convex type.

Definition 3.6 (Convex function [2])**.** Let $f_k\ :\ \mathbb{R}^n \times \mathbb{A}_k \to \mathbb{R}$ be a real valued function for $k \in M$, then f_k $(k \in M)$ is said to be convex at $\bar{x} \in X_1$, iff for every $x \in X_1$, $\bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ a_k)$ and $a_k \in \mathbb{A}_k(\bar{x})$, one has

$$f_k(x,\ a_k) - f_k(\bar{x},\ a_k) \geq \langle \bar{x}_k^*,\ x - \bar{x}\rangle,\ \forall\ k \in M.$$

Definition 3.7. (Pseudo convex functions) Let $f_k : \mathbb{R}^n \times \mathbb{A}_k \to \mathbb{R}$ be a real valued function for $k \in M$, then f_k is said to be pseudo convex at $\bar{x} \in X_1$, iff for every $x \in X_1$, $\bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ a_k)$ and $a_k \in \mathbb{A}_k(\bar{x})$, one has

$$f_k(x,\ a_k) < f_k(\bar{x},\ a_k) \implies \langle \bar{x}_k^*,\ x - \bar{x}\rangle < 0,\ \forall\, k \in M.$$

Definition 3.8. (Quasi convex functions) Let $f_k : \mathbb{R}^n \times \mathbb{A}_k \to \mathbb{R}$ be a real valued function for $k \in M$, then f_k is said to be quasi convex at $\bar{x} \in X_1$, iff for every $x \in X_1$, $\bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ a_k)$ and $a_k \in \mathbb{A}_k(\bar{x})$, one has

$$f_k(x,\ a_k) \le f_k(\bar{x},\ a_k) \implies \langle \bar{x}_k^*,\ x - \bar{x}\rangle \le 0,\ \forall\, k \in M.$$

The following index sets will be used in the sequel:

$I^+ := \{i \in I\ :\ \tilde{\mu}_i > 0\}$, $J^+ := \{j \in J\ :\ \tilde{\gamma}_j > 0\}$ and $J^- := \{j \in J\ :\ \tilde{\gamma}_j < 0\}$.

Now, we are ready the prove a sufficient optimality condition.

Theorem 3.9. *Let $\bar{x} \in X_1$. Suppose that there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$, $\tilde{\lambda}_k \ge 0$ $(k \in M)$, with $\sum_{k\in M} \tilde{\lambda}_k = 1$, and $\bar{b}_i \in \mathbb{B}_i(\bar{x})$, $\tilde{\mu}_i \ge 0$ $(i \in I)$, $\tilde{\gamma}_j \in \mathbb{R}$ $(j \in J)$, with finitely many of them being non zero (i.e. $(card(I^+) + card(J^+ \cup J^-))$ is finite), such that (3.14) – (3.15) are satisfied at $\bar{x}$. If $f_k(\cdot,\ \bar{a}_k)$ for each $\bar{a}_k \in \mathbb{A}_k$ $(k \in M)$ are pseudo convex and $g_i(\cdot,\ \bar{b}_i)$ for each $\bar{b}_i \in \mathbb{B}_i$ $(i \in I^+)$, $h_j(\cdot)$ $(j \in J^+)$ and $-h_j(\cdot)$ $(j \in J^-)$ are quasi convex at $\bar{x}$ over X_1. Then, $\bar{x}$ is a weakly robust efficient solution for RMOSIP.*

Proof. Let $\bar{x} \in X_1$ and suppose that there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$, $\tilde{\lambda}_k \ge 0$ $(k \in M)$, with $\sum_{k\in M} \tilde{\lambda}_k = 1$, and $\bar{b}_i \in \mathbb{B}_i(\bar{x})$, $\tilde{\mu}_i \ge 0$ $(i \in I)$, $\tilde{\gamma}_j \in \mathbb{R}$ $(j \in J)$, with finitely many of them being non zero (i.e. $(card(I^+) + card(J^+ \cup J^-))$ is finite), such that

$$\begin{aligned} &0 \in \sum_{k\in M} \tilde{\lambda}_k \partial_x^0 f_k(\bar{x},\ \bar{a}_k) + \sum_{i\in I^+} \tilde{\mu}_i \partial_x^0 g_i(\bar{x},\ \bar{b}_i) + \sum_{j\in J^+\cup J^-} \tilde{\gamma}_j \partial^0 h_j(\bar{x}),\\ &0 = \tilde{\mu}_i g_i(\bar{x},\ \bar{b}_i),\ i \in I. \end{aligned}$$

Then, there exist $\bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ \bar{a}_k)$, $k \in M$, $\bar{x}_i^* \in \partial_x^0 g_i(\bar{x},\ \bar{b}_i)$, $i \in I^+$, $\bar{x}_j^* \in \partial^0 h_j(\bar{x})$, $j \in J^+ \cup J^-$, such that

$$0 = \sum_{k\in M} \tilde{\lambda}_k \bar{x}_k^* + \sum_{i\in I^+} \tilde{\mu}_i \bar{x}_i^* + \sum_{j\in J^+\cup J^-} \tilde{\gamma}_j \bar{x}_j^*. \tag{3.16}$$

Suppose to the contrary that $\bar{x}$ is not a weakly robust efficient solution of RMOSIP. Then, there exist $\hat{x} \in X_1$, such that

$$\max_{a_k\in\mathbb{A}_k} f_k(\hat{x},\ a_k) < \max_{a_k\in\mathbb{A}_k} f_k(\bar{x},\ a_k),\ \forall\, k \in M,$$

which implies that for some $\hat{a}_k \in \mathbb{A}_k(\hat{x})$ and $\bar{a}_k \in \mathbb{A}_k(\bar{x})$, one has

$$f_k(\hat{x},\ \widehat{a}_k) < f_k(\bar{x},\ \bar{a}_k),\ \widehat{a}_k \in \mathbb{A}_k(\hat{x}),\ \bar{a}_k \in \mathbb{A}_k(\bar{x}),\ k \in M.$$

We have the following two cases:

(1) if $\bar{a}_k \in \mathbb{A}_k(\hat{x})$, then

$$f_k(\hat{x},\ \bar{a}_k) = f_k(\hat{x},\ \widehat{a}_k) < f_k(\bar{x},\ \bar{a}_k),\ \text{for some } k = 1, \ldots,\ m,$$

and

(2) if $\bar{a}_k \notin \mathbb{A}_k(\hat{x})$, then

$$f_k(\hat{x},\ \bar{a}_k) < f_k(\hat{x},\ \widehat{a}_k) < f_k(\bar{x},\ \bar{a}_k),\ \text{for some } k = 1, \ldots,\ m.$$

Under both the cases, we get

$$f_k(\hat{x},\ \bar{a}_k) \leq f_k(\hat{x},\ \widehat{a}_k) < f_k(\bar{x},\ \bar{a}_k),\ \forall\, k = 1, \ldots,\ m,$$

i.e.,

$$f_k(\hat{x},\ \bar{a}_k) < f_k(\bar{x},\ \bar{a}_k),\ \bar{a}_k \in \mathbb{A}_k(\bar{x}),\ \forall\, k = 1, \ldots,\ m. \tag{3.17}$$

Since, $\max_{b_i \in \mathbb{B}_i} g_i(\bar{x},\ b_i) = 0$ for any $i \in I^+$, there exists $\bar{b}_i \in \mathbb{B}_i(\bar{x})$ such that $g_i(\bar{x},\ \bar{b}_i) = 0$ for any $i \in I^+$ and $\hat{x} \in X_1$, one has

$$g_i(\hat{x},\ \bar{b}_i) \leq 0 = g_i(\bar{x},\ \bar{b}_i),\ \forall\, i \in I^+, \tag{3.18}$$

$$h_j(\hat{x}) = 0 = h_j(\bar{x}),\ \forall\, j \in J^+ \cup J^-. \tag{3.19}$$

By the pseudo convexity of f_k $(k \in M)$ and the quasi convexity of g_i $(i \in I^+)$, h_j $(j \in J^+)$, $-h_j$ $(j \in J^-)$ at $\bar{x}$ over X_1, we get

$$\begin{aligned} &\langle \bar{x}_k^*,\ \hat{x} - \bar{x} \rangle < 0,\ \forall\, k \in M, \\ &\langle \bar{x}_i^*,\ \hat{x} - \bar{x} \rangle \leq 0,\ \forall\, i \in I^+, \\ &\langle \bar{x}_j^*,\ \hat{x} - \bar{x} \rangle \leq 0,\ \forall\, j \in J^+, \\ &\langle \bar{x}_j^*,\ \hat{x} - \bar{x} \rangle \geq 0,\ \forall\, j \in J^-, \end{aligned}$$

for all $\bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ \bar{a}_k)$, $\bar{a}_k \in \mathbb{A}_k(\bar{x})$ $(k \in M)$, $\bar{x}_i^* \in \partial_x^0 g_i(\bar{x},\ \bar{b}_i)$, $\bar{b}_i \in \mathbb{B}_i(\bar{x})$ $(i \in I^+)$, $\bar{x}_j^* \in \partial^0 h_j(\bar{x})$ $(j \in J^+ \cup J^-)$, and $\hat{x} \in X_1$.

Therefore, for every $\bar{x}_k^* \in \partial_x^0 f_k(\bar{x},\ \bar{a}_k)$, $\bar{a}_k \in \mathbb{A}_k(\bar{x})$ $(k \in M)$, $\bar{x}_i^* \in \partial_x^0 g_i(\bar{x},\ \bar{b}_i)$, $\bar{b}_i \in \mathbb{B}_i(\bar{x})$ $(i \in I^+)$, $\bar{x}_j^* \in \partial^0 h_j(\bar{x})$ $(j \in J^+ \cup J^-)$, we get

$$\left\langle \sum_{k \in M} \tilde{\lambda}_k \bar{x}_k^* + \sum_{i \in I^+} \tilde{\mu}_i \bar{x}_i^* + \sum_{j \in J^+ \cup J^-} \tilde{\gamma}_j \bar{x}_j^*,\ \hat{x} - \bar{x} \right\rangle < 0,$$

which is a contradiction to equation (3.16) and hence, $\bar{x}$ is a weakly robust efficient solution of RMOSIP. □

Remark 3.10. Let $\bar{x} \in X_1$. Suppose that there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$, $\tilde{\lambda}_k \geq 0$ $(k \in M)$, with $\sum_{k \in M} \tilde{\lambda}_k = 1$, and $\bar{b}_i \in \mathbb{B}_i(\bar{x})$, $\tilde{\mu}_i \geq 0$ $(i \in I)$, $\tilde{\gamma}_j \in \mathbb{R}$ $(j \in J)$, with finitely many of them being non zero (i.e. $(card(I^+) + card(J^+ \cup J^-))$ is finite), such that $(3.14) - (3.15)$ are satisfied at $\bar{x}$. If $f_k(\cdot,\ \bar{a}_k)$ for each $\bar{a}_k \in \mathbb{A}_k$ $(k \in M)$, $g_i(\cdot,\ \bar{b}_i)$ for each $\bar{b}_i \in \mathbb{B}_i$ $(i \in I^+)$, $h_j(\cdot)$ $(j \in J^+)$ and $-h_j(\cdot)$ $(j \in J^-)$ are convex at $\bar{x}$ over X_1. Then $\bar{x}$ is a weakly robust efficient solution for RMOSIP.

The following example illustrate the above results in which we have taken only inequality constraints.

Example 3.11. Let $x := (x_1,\ x_2) \in \mathbb{R}^2$, consider a nonsmooth multiobjective semi-infinite programming problem with the data uncertainty:

$$\begin{aligned}
\text{(UMOSIP)}\quad \min\ \ & (f_1(x_1,\ x_2,\ a_1),\ f_2(x_1,\ x_2,\ a_2)) := (x_1^2 a_1,\ x_2^2 a_2 - 1),\\
\text{subject to}\ \ & g_i(x_1,\ x_2,\ b_i) := \frac{1}{i} - b_i - x_1,\ \forall\, b_i \in \mathbb{B}_i,\ \forall\, i \in \mathbb{N},\\
& g_i^{'}(x_1,\ x_2,\ b_i^{'}) := \frac{1}{i} - b_i^{'} - \frac{x_2}{2},\ \forall\, b_i^{'} \in \mathbb{B}_i^{'},\ \forall\, i \in \mathbb{N},
\end{aligned}$$

and $\mathbb{A}_1 \times \mathbb{A}_2 := [0,\ 1]^2,\ b_i \in \mathbb{B}_i := [0,\ 1],\ b_i^{'} \in \mathbb{B}_i^{'} := [1/2,\ 1],\ \forall i \in \mathbb{N}$. Let $X_1 := \{x \in \mathbb{R}^2\ :\ g_i(x,\ b_i) \leq 0,\ g_i^{'}(x,\ b_i^{'}) \leq 0,\ \forall\, b_i \in \mathbb{B}_i,\ \forall\, b_i^{'} \in \mathbb{B}_i^{'},\ \forall\, i \in \mathbb{N}\}$, then

$$X_1 := \{x \in \mathbb{R}^2\ :\ x_1 \geq 1,\ x_2 \geq 1\}.$$

and its robust counterpart is:

$$\begin{aligned}
\text{(RMOSIP1)}\quad \min\ \ & \left(\max_{(a_1,\ a_2)\in\mathbb{A}_1\times\mathbb{A}_2} f_1(x_1,\ x_2,\ a_1),\ \max_{(a_1,\ a_2)\in\mathbb{A}_1\times\mathbb{A}_2} f_2(x_1,\ x_2,\ a_2)\right)\\
\text{subject to}\ \ & g_i(x_1,\ x_2,\ b_i) \leq 0,\ \forall\, b_i \in \mathbb{B}_i,\ \forall\, i \in \mathbb{N},\\
& g_i^{'}(x_1,\ x_2,\ b_i^{'}) \leq 0,\ \forall\, b_i^{'} \in \mathbb{B}_i^{'},\ \forall\, i \in \mathbb{N}.
\end{aligned}$$

Then, the set of all feasible solutions of the problem is given by $\{(x_1,\ x_2) \in \mathbb{R}^2\ :\ x_1 \geq 1 \text{ and } x_2 \geq 1\}$. So, it is clear that $(\bar{x}_1,\ \bar{x}_2) = (1,\ 1) \in X_1$ is a weakly robust efficient solution of RMOSIP1. We have $\mathbb{A}_1(\bar{x}_1,\ \bar{x}_2) \times \mathbb{A}_2(\bar{x}_1,\ \bar{x}_2) = \{(1,\ 1)\}$ and $\mathbb{N}(\bar{x}_1,\ \bar{x}_2) = \{1\}$ are active indices for both g and g' functions and $\prod_{i\in\mathbb{N}} \mathbb{B}_i(\bar{x}_1,\ \bar{x}_2) = \{0,\ 0,\ \dots\}$ and $\prod_{i\in\mathbb{N}} \mathbb{B}_i^{'}(\bar{x}_1,\ \bar{x}_2) = \{1/2,\ 1/2,\ \dots\}$. Then, we can easily check that RZCQ holds at $(\bar{x}_1,\ \bar{x}_2)$ and if we choose $\tilde{\lambda}_1 = 1/2,\ \tilde{\lambda}_2 = 1/4,\ \tilde{\mu}_1 = 1,\ \tilde{\mu}_1^{'} = 1,\ (\bar{a}_1,\ \bar{a}_2) = (1,\ 1)$ and $(\bar{b}_1,\ \bar{b}_1^{'}) = (0,\ 1/2)$, then we set

$$\begin{aligned}
&\tilde{\lambda}_1 \partial_x^0 f_1(\bar{x}_1,\ \bar{x}_2,\ \bar{a}_1) + \tilde{\lambda}_2 \partial_x^0 f_2(\bar{x}_1,\ \bar{x}_2,\ \bar{a}_2)\\
&\qquad + \tilde{\mu}_1 \partial_x^0 g_1(\bar{x}_1,\ \bar{x}_2,\ \bar{b}_1) + \tilde{\mu}_1^{'} \partial_x^0 g_1^{'}(\bar{x}_1,\ \bar{x}_2,\ \bar{b}_1^{'}) = 0,\\
&\tilde{\mu}_1 g_1(\bar{x}_1,\ \bar{x}_2,\ \bar{b}_1) = 0,\ \tilde{\mu}_1^{'} g_1^{'}(\bar{x}_1,\ \bar{x}_2,\ \bar{b}_1^{'}) = 0.
\end{aligned}$$

Thus, Theorem 3.5 holds.

4. Robust duality

In this section, we formulate two types of dual problems for RMOSIP and derive weak and strong robust duality theorems under convexity and generalized convexity assumptions.

4.1. **Wolfe type robust dual.** Now, we formulate a Wolfe type robust dual problem for the RMOSIP denoted by RMOSIP-WD.

$$\text{(RMOSIP-WD)} \quad \max_{(y,\, a,\, b,\, \lambda,\, \mu,\, \gamma)} \Big(f_1(y,\ a_1) + \sum_{i\in I} \mu_i g_i(y,\ b_i) + \sum_{j\in J} \gamma_j h_j(y), \ldots, f_m(y,\ a_m) + \sum_{i\in I} \mu_i g_i(y,\ b_i) + \sum_{j\in J} \gamma_j h_j(y) \Big)$$

$$\text{subject to} \quad 0 \in \sum_{k=1}^{m} \lambda_k \partial_x^0 f_k(y,\ a_k) + \sum_{i\in I} \mu_i \partial_x^0 g_i(y,\ b_i) + \sum_{j\in J} \gamma_j \partial^0 h_j(y),$$

$$\lambda_k \geq 0,\ a_k \in \mathbb{A}_k,\ k = 1, \ldots,\ m,\ \sum_{k=1}^{m} \lambda_k = 1,$$

$$\mu_i \geq 0,\ b_i \in \mathbb{B}_i,\ i \in I,\ \gamma_j \in \mathbb{R},\ j \in J,$$

with $card(I^+) + card(J^+ \cup J^-)$, is finite and $I^+ := \{i \in I\ :\ \mu_i > 0\},\ J^+ := \{j \in J\ :\ \gamma_j > 0\},\ J^- := \{j \in J\ :\ \gamma_j < 0\}$. Let $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_m,\ \mathbb{B} = \prod_{i\in I} \mathbb{B}_i$ and $a \in \mathbb{A},\ b \in \mathbb{B}$.

The feasible set of RMOSIP-WD is defined by

$$W = \{(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma) : 0 \in \sum_{k=1}^{m} \lambda_k \partial_x^0 f_k(y,\ a_k) + \sum_{i\in I} \mu_i \partial_x^0 g_i(y,\ b_i) + \sum_{j\in J} \gamma_j \partial^0 h_j(y),$$

$$\lambda_k \geq 0,\ a_k \in \mathbb{A}_k,\ k \in M,\ a \in \prod_{k=1}^{m} \mathbb{A}_k = \mathbb{A},\ \sum_{k=1}^{m} \lambda_k = 1,$$

$$\mu_i \geq 0,\ b_i \in \mathbb{B}_i,\ i \in I,\ b \in \prod_{i\in I} \mathbb{B}_i = \mathbb{B},\ \gamma_j \in \mathbb{R},\ j \in J\},$$

and its projection on $\mathbb{R}^n$, denoted by X_W, is defined by

$$X_W := \{y \in \mathbb{R}^n\ :\ (y,\ a,\ b,\ \lambda,\ \mu,\ \gamma) \in W\}.$$

Let $(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma) \in W$. It is said to be weakly robust efficient solution of RMOSIP-WD, if there does not exist any $(y_0,\ a_0,\ b_0,\ \lambda_0,\ \mu_0,\ \gamma_0) \in W$, such that

$$f_k(y_0,\ a_{0k}) + \sum_{i\in I} \lambda_{0i} g_i(y_0,\ b_{0i}) + \sum_{j\in J} \gamma_{0j} h_j(y_0) > f_k(y,\ a_k) + \sum_{i\in I} \lambda_i g_i(y,\ b_i) + \sum_{j\in J} \gamma_j h_j(y),\ k = 1,\ \ldots,\ m.$$

Now, we derive weak and strong duality theorem between RMOSIP-WD and RMOSIP.

Theorem 4.1 (Weak Duality Theorem)**.** *Let $x \in X_1$ and $(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma) \in W$. Let $f_k(\cdot,\ a_k)$, for each $a_k \in \mathbb{A}_k$ $(k \in M)$, $g_i(\cdot,\ b_i)$, for each $b_i \in \mathbb{B}_i$ $(i \in I^+)$, $h_j(\cdot)$ $(j \in J^+)$ and $-h_j(\cdot)$ $(j \in J^-)$ are convex at y over $X_1 \cup X_W$, then the* *following inequalities cannot hold simultaneously:*

$$\max_{a_k \in \mathbb{A}_k(x)} f_k(x,\ a_k) < f_k(y,\ a_k) + \sum_{i \in I} \mu_i g_i(y,\ b_i) + \sum_{j \in J} \gamma_j h_j(y),\ k = 1,\ \ldots,\ m.$$

Proof. Let $x \in X_1$ and $(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma) \in W$, then

$$g_i(x,\ b_i) \leq 0,\ \forall\, b_i \in \mathbb{B}_i,\ \forall\, i \in I,\ h_j(x) = 0,\ \forall\, j \in J, \tag{4.1}$$

and

$$0 \in \sum_{k=1}^{m} \lambda_k \partial_x^0 f_k(y,\ a_k) + \sum_{i \in I} \mu_i \partial_x^0 g_i(y,\ b_i) + \sum_{j \in J} \gamma_j \partial^0 h_j(y)$$

i.e., there exist $y_k^* \in \partial_x^0 f_k(y,\ a_k)$ $(k \in M)$, $y_i^* \in \partial_x^0 g_i(y,\ b_i)$ $(i \in I^+)$, $\mu_i = 0,\ i \in I \setminus I^+$, and $y_j^* \in \partial^0 h_j(y)$ $(j \in J^+ \cup J^-)$, $\gamma_j = 0,\ j \in J \setminus (J^+ \cup J^-)$, then

$$0 = \sum_{k=1}^{m} \lambda_k y_k^* + \sum_{i \in I^+} \mu_i y_i^* + \sum_{j \in J^+ \cup J^-} \gamma_j y_j^*. \tag{4.2}$$

Suppose to the contrary that

$$\max_{a_k \in \mathbb{A}_k(x)} f_k(x,\ a_k) < f_k(y,\ a_k) + \sum_{i \in I} \mu_i g_i(y,\ b_i) + \sum_{j \in J} \gamma_j h_j(y),\ \forall\, k = 1,\ \ldots,\ m.$$

Using $\mu_i = 0,\ i \in I \setminus I^+$, and $\gamma_j = 0,\ j \in J \setminus (J^+ \cup J^-)$, we get

$$\max_{a_k \in \mathbb{A}_k(x)} f_k(x,\ a_k) < f_k(y,\ a_k) + \sum_{i \in I^+} \mu_i g_i(y,\ b_i) + \sum_{j \in J^+ \cup J^-} \gamma_j h_j(y), \quad \forall\, k = 1, \ldots,\ m. \tag{4.3}$$

Hence,

$$f_k(x,\ a_{kx}) < f_k(y,\ a_k) + \sum_{i \in I^+} \mu_i g_i(y,\ b_i) + \sum_{j \in J^+ \cup J^-} \gamma_j h_j(y),\ a_{kx} \in \mathbb{A}_k(x), \quad \text{for all } k = 1, \ldots,\ m.$$

We have the following two cases, first is $a_k \in \mathbb{A}_k(x)$ and other is $a_k \notin \mathbb{A}_k(x)$, so, under the first condition:

$$f_k(x,\ a_k) = f_k(x,\ a_{kx}) < f_k(y,\ a_k) + \sum_{i \in I^+} \mu_i g_i(y,\ b_i) + \sum_{j \in J^+ \cup J^-} \gamma_j h_j(y), \quad \text{for } k = 1, \ldots,\ m,$$

and under the second condition:

$$f_k(x,\ a_k) < f_k(x,\ a_{kx}) < f_k(y,\ a_k) + \sum_{i \in I^+} \mu_i g_i(y,\ b_i) + \sum_{j \in J^+ \cup J^-} \gamma_j h_j(y), \quad \text{for } k = 1, \ldots,\ m.$$

Hence, we have

$$f_k(x,\ a_k) < f_k(y,\ a_k) + \sum_{i\in I^+} \mu_i g_i(y,\ b_i) + \sum_{j\in J^+\cup J^-} \gamma_j h_j(y),\ a_k \in \mathbb{A}_k,\quad \forall\ k = 1, \dots,\ m. \tag{4.4}$$

Multiplying (4.4) by $\lambda_k \geq 0$ for $k = 1,\ \dots,\ m$ with $\sum_{k=1}^{m} \lambda_k = 1$ and adding all of them, we get

$$\sum_{k=1}^{m} \lambda_k f_k(x,\ a_k) < \sum_{k=1}^{m} \lambda_k f_k(y,\ a_k) + \sum_{i\in I^+} \mu_i g_i(y,\ b_i) + \sum_{j\in J^+\cup J^-} \gamma_j h_j(y). \tag{4.5}$$

By (4.1), $\mu_i = 0,\ i \in I \setminus I^+,\ \mu_i > 0,\ i \in I^+,\ \gamma_j = 0,\ j \in J \setminus (J^+ \cup J^-)$ and $\gamma_j \neq 0,\ j \in J^+ \cup J^-$, we get

$$\sum_{i\in I^+} \mu_i g_i(x,\ b_i) + \sum_{j\in J^+\cup J^-} \gamma_j h_j(x) \leq 0,\ b_i \in \mathbb{B}_i. \tag{4.6}$$

By adding (4.5) and (4.6), we get

$$\begin{aligned} &\sum_{k=1}^{m} \lambda_k f_k(x,\ a_k) + \sum_{i\in I^+} \mu_i g_i(x,\ b_i) + \sum_{j\in J^+\cup J^-} \gamma_j h_j(x) \\ &< \sum_{k=1}^{m} \lambda_k f_k(y,\ a_k) + \sum_{i\in I^+} \mu_i g_i(y,\ b_i) + \sum_{j\in J^+\cup J^-} \gamma_j h_j(y),\ \text{for } a_k \in \mathbb{A}_k,\ b_i \in \mathbb{B}_i\ . \end{aligned} \tag{4.7}$$

Since $f_k(\cdot,\ a_k)$, for each $a_k \in \mathbb{A}_k$ $(k \in M)$, $g_i(\cdot,\ b_i)$, for each $b_i \in \mathbb{B}_i$ $(i \in I^+)$, $h_j(\cdot)$ $(j \in J^+)$ and $-h_j(\cdot)$ $(j \in J^-)$ are convex at y over $X_1 \cup X_W$, therefore

$$f_k(x,\ a_k) - f_k(y,\ a_k) \geq \langle y_k^*,\ x - y\rangle,\ \forall\ y_k^* \in \partial_x^0 f_k(y,\ a_k),\ \forall\ k \in M, \tag{4.8}$$

$$g_i(x,\ b_i) - g_i(y,\ b_i) \geq \langle y_i^*,\ x - y\rangle,\ \forall\ y_i^* \in \partial_x^0 g_i(y,\ b_i),\ \forall\ i \in I^+, \tag{4.9}$$

$$h_j(x) - h_j(y) \geq \langle y_j^*,\ x - y\rangle,\ \forall\ y_j^* \in \partial^0 h_j(y),\ \forall\ j \in J^+, \tag{4.10}$$

$$h_j(x) - h_j(y) \leq \langle y_j^*,\ x - y\rangle,\ \forall\ y_j^* \in \partial^0 h_j(y),\ \forall\ j \in J^-. \tag{4.11}$$

Since $\lambda_k \geq 0$ $(k \in M), \mu_i > 0$ $(i \in I^+)$, $\gamma_j > 0$ $(j \in J^+)$ and $\gamma_j < 0$ $(j \in J^-)$, then multiplying simultaneously in (4.8) – (4.11), respectively and adding all of them, we get

$$\sum_{k=1}^{m}\lambda_k f_k(x,\ a_k)+\sum_{i\in I^+}\mu_i g_i(x,\ b_i)+\sum_{j\in J^+\cup J^-}\gamma_j h_j(x)$$
$$-\sum_{k=1}^{m}\lambda_k f_k(y,\ a_k)-\sum_{i\in I^+}\mu_i g_i(y,\ b_i)-\sum_{j\in J^+\cup J^-}\gamma_j h_j(y)$$
$$\geq\Big\langle\sum_{k=1}^{m}\lambda_k y_k^*+\sum_{i\in I^+}\mu_i y_i^*+\sum_{j\in J^+\cup J^-}\gamma_j y_j^*,\ x-y\Big\rangle,$$
$$\forall\ y_k^*\in\partial_x^0 f_k(y,\ a_k),\ y_i^*\in\partial_x^0 g_i(y,\ b_i)\text{ and } y_j^*\in\partial^0 h_j(y).$$

Using (4.7), we get

$$\begin{aligned}0>&\Big\langle\sum_{k=1}^{m}\lambda_k y_k^*+\sum_{i\in I^+}\mu_i y_i^*+\sum_{j\in J^+\cup J^-}\gamma_j y_j^*,\ x-y\Big\rangle,\\ &\forall\ y_k^*\in\partial_x^0 f_k(y,\ a_k),\ y_i^*\in\partial_x^0 g_i(y,\ b_i)\text{ and } y_j^*\in\partial^0 h_j(y)\\ &\ =0\text{ (by (4.2)) for some } y_k^*\in\partial_x^0 f_k(y,\ a_k),\ y_i^*\in\partial_x^0 g_i(y,\ b_i)\text{ and } y_j^*\in\partial^0 h_j(y).\end{aligned}$$

Which is a contradiction. Hence, our supposition is wrong and we get the required result. □

Theorem 4.2 (Strong Duality Theorem)**.** *Let $\bar{x}\in X_1$ be a weakly robust efficient solution of RMOSIP such that the conditions of Theorem* 3.5 *are satisfied at $\bar{x}$. Then, there exists $\bar{a}_k\in\mathbb{A}_k(\bar{x})$, $\tilde{\lambda}_k\geq 0$ $(k\in M)$, with $\sum_{k\in M}\tilde{\lambda}_k=1$, and $\bar{b}_i\in\mathbb{B}_i(\bar{x})$, $\tilde{\mu}_i\geq 0$ $(i\in I)$, $\tilde{\gamma}_j\in\mathbb{R}$ $(j\in J)$, with finitely many of them being non zero (i.e. $(card(I^+)+card(J^+\cup J^-))$ is finite), such that $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma})\in W$.*

If $f_k(\cdot,\ \bar{a}_k)$, for each $\bar{a}_k\in\mathbb{A}_k(\bar{x})$ $(k\in M)$, $g_i(\cdot,\ \bar{b}_i)$, for each $\bar{b}_i\in\mathbb{B}_i(\bar{x})$ $(i\in I^+)$, $h_j(\cdot)$ $(j\in J^+)$ and $-h_j(\cdot)$ $(j\in J^-)$ are convex at $\bar{x}$ over $X_1\cup X_W$, then $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma})$ is a weakly robust efficient solution of RMOSIP-WD.

Proof. Since the conditions of Theorem 3.5 are satisfied at $\bar{x}$. Then, there exists $\bar{a}_k\in\mathbb{A}_k(\bar{x})$, $\tilde{\lambda}_k\geq 0$ $(k\in M)$, with $\sum_{k\in M}\tilde{\lambda}_k=1$, and $\bar{b}_i\in\mathbb{B}_i(\bar{x})$, $\tilde{\mu}_i\geq 0$ $(i\in I)$, $\tilde{\gamma}_j\in\mathbb{R}$ $(j\in J)$, with finitely many of them being non zero (i.e. $(card(I^+)+card(J^+\cup J^-))$ is finite), such that (3.14)−(3.15) are satisfied, which gives $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma})\in W$.

Now, we assume that $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma})$ is not a weakly robust efficient solution of RMOSIP-WD. Then, there exists a robust feasible point $(\bar{y},\ \hat{a},\ \hat{b},\ \hat{\lambda},\ \hat{\mu},\ \hat{\gamma})$ of

RMOSIP-WD such that

$$f_k(\bar{y},\ \hat{a}_k) + \sum_{i\in I}\hat{\mu}_i g_i(\bar{y},\ \hat{b}_i) + \sum_{j\in J}\hat{\gamma}_j h_j(\bar{y}) > f_k(\bar{x},\ \bar{a}_k) + \sum_{i\in I}\tilde{\mu}_i g_i(\bar{x},\ \bar{b}_i) + \sum_{j\in J}\tilde{\gamma}_j h_j(\bar{y}),\ k=1,\ 2,\ \ldots,\ m, \tag{4.12}$$

By (3.15) and the feasibility of $\bar{x}$, we get

$$\sum_{i\in I}\tilde{\mu}_i g_i(\bar{x},\ \bar{b}_i) + \sum_{j\in J}\tilde{\gamma}_j h_j(\bar{x}) = 0,$$

then (4.12) gives

$$f_k(\bar{y},\ \hat{a}_k) + \sum_{i\in I}\hat{\mu}_i g_i(\bar{y},\ \hat{b}_i) + \sum_{j\in J}\hat{\gamma}_j h_j(\bar{y}) > f_k(\bar{x},\ \bar{a}_k),\ k=1,\ 2,\ \ldots,\ m,$$

i.e.,

$$f_k(\bar{y},\ \hat{a}_k) + \sum_{i\in I}\hat{\mu}_i g_i(\bar{y},\ \hat{b}_i) + \sum_{j\in J}\hat{\gamma}_j h_j(\bar{y}) > \max_{\bar{a}_k\in\mathbb{A}_k(\bar{x})} f_k(\bar{x},\ \bar{a}_k),\ k=1,\ 2,\ \ldots,\ m,$$

Which contradicts weak duality. Thus, $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma})$ is a weakly robust efficient solution of RMOSIP-WD. $\square$

4.2. **Mond-Weir type robust dual.** Now, we formulate a Mond-Weir type robust dual problem for the RMOSIP denoted by RMOSIP-MWD.

(RMOSIP-MWD) $\max\limits_{(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma)} \Big(f_1(y,\ a_1),\ldots,f_m(y,\ a_m)\Big)$

$$\begin{aligned}\text{subject to }\ & 0\in\sum_{k=1}^{m}\lambda_k\partial_x^0 f_k(y,\ a_k) + \sum_{i\in I}\mu_i\partial_x^0 g_i(y,\ b_i) + \sum_{j\in J}\gamma_j\partial^0 h_j(y),\\ & \lambda_k\ge 0,\ a_k\in\mathbb{A}_k,\ k=1,\ldots,\ m,\ \sum_{k=1}^{m}\lambda_k = 1,\\ & \mu_i\ge 0,\ \mu_i g_i(y,\ b_i)\ge 0,\ b_i\in\mathbb{B}_i,\ i\in I,\\ & \gamma_j\in\mathbb{R},\ \gamma_j h_j(y)=0,\ j\in J,\end{aligned}$$

with $card(I^+) + card(J^+\cup J^-)$, is finite and $I^+ := \{i\in I\ :\ \mu_i>0\},\ J^+ := \{j\in J\ :\ \gamma_j>0\},\ J^- := \{j\in J\ :\ \gamma_j<0\}$. Let $\mathbb{A} = \mathbb{A}_1\times\cdots\times\mathbb{A}_m,\ \mathbb{B} = \prod_{i\in I}\mathbb{B}_i$ and $a\in\mathbb{A},\ b\in\mathbb{B}$.

The feasible set of RMOSIP-MWD is defined by

$$\begin{aligned} MW=\{(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma): 0\in &\sum_{k=1}^{m}\lambda_k\partial_x^0 f_k(y,\ a_k)+\sum_{i\in I}\mu_i\partial_x^0 g_i(y,\ b_i)+\sum_{j\in J}\gamma_j\partial^0 h_j(y),\\ &\lambda_k\geq 0,\ a_k\in\mathbb{A}_k,\ k\in M,\ a\in\prod_{k=1}^{m}\mathbb{A}_k=\mathbb{A},\ \sum_{k=1}^{m}\lambda_k=1,\\ &\mu_i\geq 0,\ \mu_i g_i(y,\ b_i)\geq 0,\ b_i\in\mathbb{B}_i,\ i\in I,\ b\in\prod_{i\in I}\mathbb{B}_i=\mathbb{B},\\ &\gamma_j\in\mathbb{R},\ \gamma_j h_j(y)=0,\ j\in J\}, \end{aligned}$$

and its projection on $\mathbb{R}^n$, denoted by X_{MW}, is defined by

$$X_{MW}:=\{y\in\mathbb{R}^n\ :\ (y,\ a,\ b,\ \lambda,\ \mu,\ \gamma)\in MW\}.$$

Let $(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma)\in MW$. It is said to be weakly robust efficient solution of RMOSIP-MWD, if there does not exist any $(y_0,\ a_0,\ b_0,\ \lambda_0,\ \mu_0,\ \gamma_0)\in MW$, such that

$$f_k(y_0,\ a_{0k})>f_k(y,\ a_k),\ k=1,\ \ldots,\ m.$$

Now, we derive weak and strong duality theorem between RMOSIP-MWD and RMOSIP.

Theorem 4.3 (Weak Duality Theorem)**.** *Let $x\in X_1$ and $(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma)\in MW$. Let $f_k(\cdot,\ a_k)$, for each $a_k\in\mathbb{A}_k$ $(k\in M)$ is pseudo convex and $\mu_i g_i(\cdot,\ b_i)$, for each $b_i\in\mathbb{B}_i$ $(i\in I^+)$, $\gamma_j h_j(\cdot)$ $(j\in J^+\cup J^-)$ are quasi convex at y over $X_1\cup X_{MW}$, then the following inequalities cannot hold simultaneously:*

$$\max_{a_k\in\mathbb{A}_k(x)} f_k(x,\ a_k)<f_k(y,\ a_k),\ k=1,\ 2,\ \ldots,\ m. \tag{4.13}$$

Proof. Let $x\in X_1$, then

$$g_i(x,\ b_i)\leq 0,\ \forall\, b_i\in\mathbb{B}_i,\ i\in I \text{ and } h_j(x)=0,\ \forall\, j\in J. \tag{4.14}$$

Let $(y,\ a,\ b,\ \lambda,\ \mu,\ \gamma)\in MW$, then, there exists $y_k^*\in\partial_x^0 f(y,\ a_k)$ $(k\in M)$, $y_i^*\in\partial_x^0 g_i(y,\ b_i)$ $(i\in I^+)$, $y_j^*\in\partial^0 h_j(y)$ $(j\in J^+\cup J^-)$, $\mu_i=0,\ i\in I\setminus I^+$ and $\gamma_j=0,\ j\in J\setminus(J^+\cup J^-)$, one has

$$0=\sum_{k=1}^{m}\lambda_k y_k^*+\sum_{i\in I^+}\mu_i y_i^*+\sum_{j\in J^+\cup J^-}\gamma_j y_j^*, \tag{4.15}$$

$$\mu_i g_i(y,\ b_i)\geq 0,\ b_i\in\mathbb{B}_i,\ i\in I, \tag{4.16}$$

$$\gamma_j h_j(y)=0,\ j\in J. \tag{4.17}$$

Suppose to the contrary that (4.13) is not satisfied, then

$$\max_{a_k\in\mathbb{A}_k(x)} f_k(x,\ a_k)<f_k(y,\ a_k),\ \forall\, k=1,\ 2,\ \ldots,\ m.$$

Hence,

$$f_k(x,\ a_{kx})<f_k(y,\ a_k),\ a_{kx}\in\mathbb{A}_k(x),\ \forall\, k=1,\ 2,\ \ldots,\ m. \tag{4.18}$$

We have the following two cases: first is $a_k \in \mathbb{A}_k(x)$ and other is $a_k \notin \mathbb{A}_k(x)$, so, under the first condition:

$$f_k(x,\ a_k) = f_k(x,\ a_{kx}) < f_k(y,\ a_k),\ \text{for } k = 1,\ \ldots,\ m,$$

and under the second condition:

$$f_k(x,\ a_k) < f_k(x,\ a_{kx}) < f_k(y,\ a_k),\ \text{for } k = 1,\ \ldots,\ m.$$

Hence, we have

$$f_k(x,\ a_k) < f_k(y,\ a_k),\ a_k \in \mathbb{A}_k,\ \forall\ k = 1,\ \ldots,\ m. \tag{4.19}$$

Let $f_k(\cdot,\ a_k)$, for each $a_k \in \mathbb{A}_k$ $(k \in M)$ is pseudo convex and $\mu_i g_i(\cdot,\ b_i)$, for each $b_i \in \mathbb{B}_i$ $(i \in I^+)$, $\gamma_j h_j(\cdot)$ $(j \in J^+ \cup J^-)$ are quasi convex at y over $X_1 \cup X_{MW}$, then (4.19), (4.14), (4.16) and (4.17) gives

$$\langle y_k^*,\ x - y\rangle < 0,\ \forall\ y_k^* \in \partial_x^0 f_k(y,\ a_k),\ k \in M, \tag{4.20}$$

$$\Big\langle \sum_{i \in I^+} \mu_i y_i^*,\ x - y\Big\rangle \le 0,\ \forall\ y_i^* \in \partial_x^0 g_i(y,\ b_i), \tag{4.21}$$

$$\Big\langle \sum_{j \in J^+ \cup J^-} \gamma_j y_j^*,\ x - y\Big\rangle \le 0,\ \forall\ y_j^* \in \partial^0 h_j(y). \tag{4.22}$$

Since $\lambda_k \ge 0,\ k \in M$ with $\sum_{k \in M} \lambda_k = 1$, we get

$$0 > \Big\langle \sum_{k \in M} \lambda_k y_k^* + \sum_{i \in I^+} \mu_i y_i^* + \sum_{j \in J^+ \cup J^-} \gamma_j y_j^*,\ x - y\Big\rangle = 0\ (by\ (4.15)),$$

which is not possible, Hence, our supposition is wrong and we get the required result. □

Theorem 4.4 (Strong Duality Theorem). *Let $\bar{x} \in X_1$ be a weakly robust efficient solution of RMOSIP such that the conditions of Theorem 3.5 are satisfied at $\bar{x}$. Then, there exists $\bar{a}_k \in \mathbb{A}_k(\bar{x})$, $\tilde{\lambda}_k \ge 0$ $(k \in M)$, with $\sum_{k \in M} \tilde{\lambda}_k = 1$ and $\bar{b}_i \in \mathbb{B}_i(\bar{x})$, $\tilde{\mu}_i \ge 0$ $(i \in I)$, $\tilde{\gamma}_j \in \mathbb{R}$ $(j \in J)$, with finitely many of them being non zero (i.e. $(card(I^+) + card(J^+ \cup J^-))$ is finite), such that $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma}) \in MW$.*

If $f_k(\cdot,\ a_k)$, for each $a_k \in \mathbb{A}_k$ $(k \in M)$ is pseudo convex and $\tilde{\mu}_i g_i(\cdot,\ b_i)$, for each $b_i \in \mathbb{B}_i$ $(i \in I^+)$, $\tilde{\gamma}_j h_j(\cdot)$ $(j \in J^+ \cup J^-)$ are quasi convex at $\bar{x}$ over $X_1 \cup X_{MW}$, then $(\bar{x},\ \bar{a},\ \bar{b},\ \tilde{\lambda},\ \tilde{\mu},\ \tilde{\gamma})$ is a weakly robust efficient solution of RMOSIP-MWD.

Proof. Proof of this theorem is similar to the strong duality theorem for Wolfe type Robust duality. □

5. Conclusions

In this paper, we have taken nonsmooth robust multiobjective semi-infinite programming problem with mixed constraints. We have formulated necessary optimality conditions in terms of data uncertainty depends on the feasible region and in the objective functions. After that, we have established sufficient optimality conditions under convexity and generalized convexity assumptions and then gave an example for the support of KKT optimality conditions. Lastly, we derived two types of dual model for the above taken problem Wolfe and Mond-Weir and developed weak and strong duality results under convexity and generalized convexity assumptions between these dual and primal models.

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P. Jaisawal
Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India
E-mail address: pushkarjaisawal2@gmail.com

V. Laha
Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India
E-mail address: laha.vivek333@gmail.com