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AN ITERATIVE METHOD FOR SOLVING THE FIXED POINT PROBLEM FOR A SET-VALUED MAPPING

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Abstract. In 1989 N. Mizoguchi and W. Takahash established the existence of a fixed point for a set-valued mapping satisfying certain assumptions. In this paper we study this mapping an analyze an iterative scheme which allows us, for any initial state, to construct a sequence of iterates which converge to a fixed point of the mapping.

1. Introduction

During more than fifty-five years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 5, 11, 13, 14, 17, 18, 19, 21, 24, 25, 26, 27, 28, 29, 34, 35] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [2, 4, 6, 7, 8, 9, 10, 12, 15, 16, 22, 31, 32, 34, 35].

In [20] N. Mizoguchi and W. Takahash answered a question posed by S. Reich in [23] and established the existence of a fixed point for a set-valued nonexpansive mapping satisfying certain assumptions. In this paper we study this mapping an analyze an iterative scheme which allows us, for any initial state, to construct a sequence of iterates which converge to a fixed point of the mapping. Note that in [30] it was given an example showing that the Mizoguchi-Takahashi fixed point theorem does not reduce to the Nadler theorem.

Our paper contains two results which are obtained in Section 2.

2. Main results

Let (X, ρ) be a complete metric space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}.$$

For each $x \in X$ and each nonempty set $D \subset X$ set

$$\rho(x, D) = \inf\{\rho(x, y) : \ y \in D\}.$$

Denote by $S(X)$ the collection of all nonempty closed bounded subsets of X.

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For each $A, B \in S(X)$ set

$$H(A,B) = \max\{\sup\{\rho(x,B) :\ x \in A\},\ \sup\{\rho(x,A) :\ x \in B\}\}.$$

Clearly, $(S(X), H)$ is a metric space.

Assume that a function $\phi : (0,\infty) \to [0,1)$ satisfies for each $t \in [0,\infty)$,

$$\limsup_{r \to t^+} \phi(r) < 1. \tag{2.1}$$

Let a mapping $A : X \to S(X)$ satisfy

$$H(A(x), A(y)) \le \phi(\rho(x,y))\rho(x,y) \tag{2.2}$$

for all $x, y \in X$ such that $x \neq y$. This mapping was studied in [20] where it was shown that the mapping A has a fixed point. Put

$$\phi(0) = 0.$$

Then (2.2) holds for all $x, y \in X$ and the domain of ϕ is $[0,\infty)$.

In this paper we study this mapping A an analyze an iterative scheme which allows us, for any initial state, to construct a sequence of iterates which converge to a fixed point of the mapping.

We consider the following algorithm.

Initialization: select an arbitrary

$$x_0 \in X.$$

Iterative step: given a current iteration vector $x_t \in X$ calculate $x_{t+1} \in A(x_t)$ as an approximate solution of the minimization problem

$$\rho(x_t, x) \to \min,\ x \in A(x_t).$$

Note that the existence of an exact solution of the minimization problem above is not guaranteed if $A(x_t)$ is not a compact. Moreover, even if it exists, this exact solution usually cannot be calculated because of computational errors produced by our computer system.

In this paper we prove the following result.

Theorem 2.1. *Let $x_0 \in X$, a sequence $\{\Delta_i\}_{i=0}^{\infty} \subset (0,\infty)$ satisfy*

$$\lim_{i \to \infty} \Delta_i = 0 \tag{2.3}$$

and let a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy for each integer $i \ge 0$,

$$x_{i+1} \in A(x_i), \tag{2.4}$$

$$\text{if } x_i \in A(x_i),\ \text{then } x_{i+1} = x_i, \tag{2.5}$$

$$\rho(x_i, x_{i+1}) \le \rho(x_i, A(x_i)) + \Delta_i \tag{2.6}$$

and have the following property:

(P1) *if $i \ge 1$ and*

$$\rho(x_i, A(x_i)) < \rho(x_{i-1}, x_i),$$

then

$$\rho(x_i, x_{i+1}) < \rho(x_{i-1}, x_i).$$

Then $\lim_{i\to\infty} \rho(x_i, x_{i+1}) = 0$. *Moreover, if*

$$\sum_{i=0}^{\infty} \Delta_i < \infty,$$

then the sequence $\{x_i\}_{i=0}^{\infty}$ *converges to a fixed point of* A.

Note that for every $x_0 \in X$ and every sequence $\{\Delta_i\}_{i=0}^{\infty} \subset (0,1)$ satisfying (2.3) there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies (2.4)-(2.6) and has property (P1) for all integers $i \geq 0$. The iterative scheme used in Theorem 2.1 is related to allowable ranges in the sense of [33] .

Proof of Theorem 2.1. We may assume without loss of generality that for all integers $i \geq 0$,

$$x_i \notin A(x_i). \tag{2.7}$$

This implies that for all integers $i \geq 0$,

$$x_i \neq x_{i+1}. \tag{2.8}$$

In view of (2.6),

$$\rho(x_0, x_1) \leq \rho(x_0, A(x_0)) + \Delta_0. \tag{2.9}$$

By (2.2), (2.4) and (2.6), for each integer $i \geq 1$,

$$\begin{aligned} \rho(x_i, x_{i+1}) &\leq \rho(x_i, A(x_i)) + \Delta_i \\ &\leq H(A(x_{i-1}), A(x_i)) + \Delta_i \\ &\leq \phi(\rho(x_{i-1}, x_i))\rho(x_{i-1}, x_i) + \Delta_i. \end{aligned} \tag{2.10}$$

By (2.2), (2.4), (2.8), for each integer $i \geq 1$,

$$\begin{aligned} \rho(x_i, A(x_i)) &\leq H(A(x_{i-1}), A(x_i)) \\ &\leq \phi(\rho(x_{i-1}, x_i))\rho(x_{i-1}, x_i) < \rho(x_{i-1}, x_i). \end{aligned} \tag{2.11}$$

It follows from (2.11) and property (P1) that for each integer $i \geq 1$,

$$\rho(x_i, x_{i+1}) < \rho(x_{i-1}, x_i). \tag{2.12}$$

In view of (2.12) there exists

$$\begin{aligned} \tau &= \lim_{i\to\infty} \rho(x_i, x_{i+1}) \\ &= \inf\{\rho(x_i, x_{i+1}) :\ i \geq 0 \text{ is an integer}\}. \end{aligned} \tag{2.13}$$

Set

$$\lambda = \limsup_{r\to\tau^+} \phi(r). \tag{2.14}$$

In view of (2.1) and (2.14),

$$0 \leq \lambda < 1. \tag{2.15}$$

By (2.12), (2.13) and (2.14), there exists a natural number p_0 such that for each integer $i \geq p_0$,

$$\phi(\rho(x_i, x_{i+1})) \leq 2^{-1}(1+\lambda). \tag{2.16}$$

It follows from (2.10) and (2.16) that for each integer $i \geq p_0 + 1$,

$$\begin{aligned} \rho(x_i, x_{i+1}) &\leq \phi(\rho(x_{i-1}, x_i))\rho(x_{i-1}, x_i) + \Delta_i \\ &\leq 2^{-1}(1+\lambda)\rho(x_{i-1}, x_i) + \Delta_i. \end{aligned} \tag{2.17}$$

Equations (2.3), (2.13) and (2.17) imply that

$$\tau \leq 2^{-1}(1+\lambda)\tau.$$

Together with (2.15) this implies that

$$\tau = \lim_{i\to\infty} \rho(x_i, x_{i+1}) = 0.$$

Assume now that

$$\sum_{i=0}^{\infty} \Delta_i < \infty.$$

We show that the sequence $\{x_i\}_{i=0}^{\infty}$ converges to a fixed point of A.

Relation (2.17) implies that

$$\rho(x_{p_0+1}, x_{p_0+2}) \leq 2^{-1}(1+\lambda)\rho(x_{p_0}, x_{p_0+1}) + \Delta_{p_0+1}. \tag{2.18}$$

In view of (2.17) and (2.18),

$$\begin{aligned} \rho(x_{p_0+2}, x_{p_0+3}) &\leq 2^{-1}(1+\lambda)\rho(x_{p_0+1}, x_{p_0+2}) + \Delta_{p_0+2} \\ &\leq (2^{-1}(1+\lambda))^2\rho(x_{p_0}, x_{p_0+1}) \\ &\quad + 2^{-1}(1+\lambda)\Delta_{p_0+1} + \Delta_{p_0+2}. \end{aligned} \tag{2.19}$$

We show by induction that for each integer $n \geq 1$,

$$\begin{aligned} \rho(x_{p_0+n}, x_{p_0+n+1}) &\leq (2^{-1}(1+\lambda))^n\rho(x_{p_0}, x_{p_0+1}) \\ &\quad + \sum_{i=0}^{n-1}(2^{-1}(1+\lambda))^i\Delta_{p_0+n-i}. \end{aligned} \tag{2.20}$$

In view of (2.18) and (2.19), inequality (2.20) holds for $n = 1, 2$.

Assume that $k \geq 1$ is an integer and that (2.20) holds for $n = k$. Together with (2.17) this implies that

$$\begin{aligned}\rho(x_{p_0+k+1}, x_{p_0+k+2}) &\leq 2^{-1}(1+\lambda)\rho(x_{p_0+k}, x_{p_0+k+1}) + \Delta_{p_0+k+1} \\ &\leq (2^{-1}(1+\lambda))^{k+1}\rho(x_{p_0}, x_{p_0+1}) \\ &\quad + \sum_{i=0}^{k-1}(2^{-1}(1+\lambda))^{i+1}\Delta_{p_0+k-i} + \Delta_{p_0+k+1} \\ &= (2^{-1}(1+\lambda))^{k+1}\rho(x_{p_0}, x_{p_0+1}) \\ &\quad + \sum_{i=0}^{k}(2^{-1}(1+\lambda))^{i}\Delta_{p_0+k+1-i}.\end{aligned}$$

Thus (2.20) holds for $n = k+1$. Therefore we have shown that (2.20) holds for all integers $n \geq 1$.

Since $\sum_{i=0}^{\infty}\Delta_i < \infty$ equations (2.15) and (2.20) imply that

$$\begin{aligned}\sum_{n=1}^{\infty}\rho(x_{p_0+n}, x_{p_0+n+1}) &\leq \sum_{n=1}^{\infty}((2^{-1}(1+\lambda))^n\rho(x_{p_0}, x_{p_0+1}) \\ &\quad + \sum_{i=1}^{n}(2^{-1}(1+\lambda))^{n-i}\Delta_{p_0+i}) \\ &\leq \rho(x_{p_0}, x_{p_0+1})\sum_{n=1}^{\infty}(2^{-1}(1+\lambda))^n \\ &\quad + \sum_{i=1}^{\infty}\left(\sum_{j=0}^{\infty}(2^{-1}(1+\lambda))^j\right)\Delta_{p_0+i} \\ &\leq \left(\sum_{n=0}^{\infty}(2^{-1}(1+\lambda))^n\right)\left[\rho(x_{p_0}, x_{p_0+1}) + \sum_{i=1}^{\infty}\Delta_{p_0+i}\right] \\ &< \infty.\end{aligned}$$

Thus $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{2.21}$$

We show that $x_* \in A(x_*)$. Let $\epsilon > 0$. In view of (2.21), there exists an integer $n_0 \geq 1$ such that for each integer $n \geq n_0$,

$$\rho(x_n, x_*) \leq \epsilon/4. \tag{2.22}$$

Let $n \geq n_0$ be an integer. By (2.2) and (2.22),

$$H(A(x_n), A(x_*)) \leq \rho(x_n, x_*) \leq \epsilon/4. \tag{2.23}$$

Relation (2.4) implies that

$$x_{n+1} \in A(x_n). \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$\rho(x_{n+1}, A(x_*)) \le \epsilon/4.$$

Thus there exists

$$y \in A(x_*) \tag{2.25}$$

such that

$$\rho(x_{n+1}, y) < \epsilon/4 + \epsilon/8. \tag{2.26}$$

By (2.22), (2.25) and (2.26),

$$\rho(x_*, A(x_*)) \le \rho(x_*, y) \le \rho(x_*, x_{n+1}) + \rho(x_{n+1}, y) \le \epsilon/4 + \epsilon/4 + \epsilon/8.$$

Since ϵ is an arbitrary positive number, we conclude that

$$x_* \in A(x_*).$$

Theorem 2.1 is proved. □

The next theorem is our second main result. It shows that our fixed point problem is well-posed.

Theorem 2.2. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $x \in X$ satisfying $\rho(x, A(x)) < \delta$ there exists $\bar{x} \in X$ such that $\bar{x} \in A(\bar{x})$ and $\rho(x, \bar{x}) < \epsilon$.*

Proof. By (2.1), there exist $\delta_0 > 0$ and $\lambda_0 \in (0, 1)$ such that

$$\phi(t) < \lambda_0 \text{ for all } t \in (0, \delta_0]. \tag{2.27}$$

Fix

$$\lambda_1 \in (\lambda_0, 1). \tag{2.28}$$

Choose a positive number δ such that

$$\delta < \delta_0 \text{ and } \delta(1 - \lambda_1)^{-1} < \epsilon. \tag{2.29}$$

Let $x \in X$ satisfy

$$\rho(x, A(x)) < \delta. \tag{2.30}$$

Set

$$x_0 = x. \tag{2.31}$$

By (2.30) and (2.31), there exists $x_1 \in X$ such that

$$x_1 \in A(x_0), \ \rho(x_0, x_1) < \delta, \tag{2.32}$$

$$\text{if } x_0 \in A(x_0), \text{ then } x_1 = x_0. \tag{2.33}$$

By induction we define a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that for each integer $n \ge 1$ the following properties hold:

(P2) if $x_n \in A(x_n)$, then $x_{n+1} = x_n$;

(P3) if $x_n \notin A(x_n)$, then $x_{n+1} \in A(x_n)$ and

$$\rho(x_n, x_{n+1}) < \rho(x_n, A(x_n))\lambda_1\lambda_0^{-1}; \tag{2.34}$$

(P4) if

$$\rho(x_n, A(x_n)) < \rho(x_{n-1}, x_n),$$

then

$$\rho(x_n, x_{n+1}) < \rho(x_{n-1}, x_n).$$

If $x_1 \in A(x_1)$, then in view of (2.29), (2.31) and (2.32), the assertion of the theorem holds. Thus we may assume without loss of generality that

$$x_1 \notin A(x_1). \tag{2.35}$$

By (P3),

$$x_2 \neq x_1. \tag{2.36}$$

It follows from (2.32), (2.36) and (P2) that

(2.37). $$x_1 \neq x_0.$$

Assume that an integer $k \geq 2$ and

$$x_k \neq x_{k-1}. \tag{2.38}$$

(Note that in view of (2.36), (2.38) is true for $k = 2$.) By (2.33), (2.35)-(2.38) and properties (P2) and (P3),

$$x_i \neq x_{i-1},\ x_{i-1} \notin A(x_{i-1}) \tag{2.39}$$

for all integers i satisfying $1 \leq i \leq k$.

Let an integer i satisfy $1 \leq i \leq k$. By (2.2), (2.39), (P2) and (P3),

$$\rho(x_i, A(x_i)) \leq H(A(x_{i-1}), A(x_i)) \leq \phi(\rho(x_{i-1}, x_i))\rho(x_{i-1}, x_i). \tag{2.40}$$

In view of (2.39) and (2.40),

$$\rho(x_i, A(x_i)) < \rho(x_{i-1}, x_i). \tag{2.41}$$

It follows from (5.41) and (P4) that

$$\rho(x_i, x_{i+1}) < \rho(x_{i-1}, x_i).$$

Since the inequality above holds for all integers i satisfying $1 \leq i \leq k$ we conclude using (2.30) that

$$\rho(x_k, x_{k+1}) \leq \rho(x_{k-1}, x_k) \leq \cdots \leq \rho(x_1, x_2) \leq \rho(x_0, x_1) < \delta.$$

By the relation above, (2.27), (2.29), (2.34), (2.39), (2.40) and property (P3), for each integer i satisfying $1 \leq i < k$,

$$\begin{aligned}\rho(x_i, x_{i+1}) &< \rho(x_i, A(x_i))\lambda_1\lambda_0^{-1} \\ &\leq \lambda_1\lambda_0^{-1}\phi(\rho(x_{i-1}, x_i))\rho(x_{i-1}, x_i) \\ &\leq \lambda_1\lambda_0^{-1}\lambda_0\rho(x_{i-1}, x_i) = \lambda_1\rho(x_{i-1}, x_i).\end{aligned}$$

Together with (2.32) this implies that for each integer i satisfying $1 \leq i < k$,

$$\rho(x_i, x_{i+1}) \leq \lambda_1^i\rho(x_0, x_1) \leq \lambda_1^i\delta. \tag{2.42}$$

Thus we have shown that the following property holds:

(P5) if an integer $k \geq 2$ satisfies $x_k \neq x_{k-1}$, then for each integer i satisfying $1 \leq i < k$ the inequality

$$\rho(x_i, x_{i+1}) \leq \lambda_1^i \delta$$

holds.

There are two cases:

(a) there is an integer $k \geq 2$ such that

$$x_k \neq x_{k-1},\ x_{k+1} = x_k;$$

(b) $x_k \neq x_{k-1}$ for all integers $k \geq 2$.

Assume that the case (a) holds and let an integer $k \geq 2$ be as guaranteed in the case (a). Then by property (P5),

$$\rho(x_i, x_{i+1}) \leq \lambda_1^i \delta,\ i = 1, \dots, k-1. \tag{2.43}$$

In view of (P2) and (P3),

$$x_i = x_k \text{ for all integers } i \geq k,\ x_k \in A(x_k). \tag{2.44}$$

It follows from (2.29), (2.32) and (2.43) that

$$\rho(x_0, x_k) \leq \sum_{i=0}^{k-1} \rho(x_i, x_{i+1}) \leq \sum_{i=0}^{\infty} \lambda_1^i \delta = \delta(1-\lambda_1)^{-1} < \epsilon.$$

Thus

$$x_k \in A(x_k),\ \rho(x, x_k) < \epsilon$$

and in the case (a) Theorem 2.2 is proved.

Assume that the case (b) holds. By property (P5), for all integers $i \geq 1$,

$$\rho(x_i, x_{i+1}) \leq \lambda_1^i \delta. \tag{2.45}$$

In view of (2.45),

$$\sum_{i=0}^{\infty} \rho(x_i, x_{i+1}) < \infty.$$

Thus $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence. Let

$$\bar{x} = \lim_{i \to \infty} x_i. \tag{2.46}$$

Arguing as in the proof of Theorem 2.1 we can show that

$$\bar{x} \in A(\bar{x}).$$

By (2.29), (2.31), (2.45) and (2.46),

$$\begin{aligned} \rho(x, \bar{x}) = \rho(x_0, \bar{x}) &= \lim_{n \to \infty} \rho(x_0, x_n) \\ &\leq \sum_{i=0}^{\infty} \rho(x_i, x_{i+1}) \leq \sum_{i=0}^{\infty} \lambda_1^i \delta \\ &= \delta(1-\lambda_1)^{-1} < \epsilon. \end{aligned}$$

This completes the proof of Theorem 2.2. □

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