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THE SOLVABILITIES OF EIGENVALUE OPTIMIZATION PROBLEMS ASSOCIATED WITH p-ORDER CONE AND CIRCULAR CONE

WEI-MING HSU, XIN-HE MIAO*, AND JEIN-SHAN CHEN†

ABSTRACT. In this paper, we consider two kinds of conic eigenvalue complementarity problems, which are eigenvalue complementarity problems and quadratic eigenvalue complementarity problem associated with circular cone or p-order cone, respectively. In the setting of the circular cone, we establish the relation between the solution of the circular cone eigenvalue complementarity problems (or the solution of the circular cone quadratic eigenvalue complementarity problems) and the solution of the corresponding circular cone complementarity problems. The similar results in the setting of p-order cone are achieved as well. These results build up a theoretical basis for making further study on their corresponding eigenvalue complementarity problems in the future.

1. INTRODUCTION

The traditional eigenvalue complementarity problem (EiCP) consists of finding a scalar $\lambda \in \mathbb{R}$ and a vector $0 \neq x \in \mathbb{R}^n$ such that

$$\begin{cases} y = \lambda Bx - Cx, \\ x \geq 0, \ y \geq 0, \ x^T y = 0, \end{cases} \tag{1.1}$$

where $B, C \in \mathbb{R}^{n \times n}$ and $y \in \mathbb{R}^n$. This problem is also called *the Pareto eigenvalue problem*, which has been studied in [23] and possesses a wide variety of applications in sciences and engineering, see [1, 21, 22]. In fact, the EiCP (1.1) is equivalent to seeking a scalar $\lambda \in \mathbb{R}$ and a vector $0 \neq x \in \mathbb{R}^n$ such that

$$\begin{cases} y = \lambda Bx - Cx, \\ x \geq 0, \ y \geq 0, \ x^T y = 0, \\ a^T x = 1, \end{cases}$$

with $a \succ 0$. Usually, we choose $a = (1, 1, \cdots, 1)^T \in \mathbb{R}^n$. Recently, an extension of the EiCP has been introduced in [24], which is named quadratic eigenvalue complementarity problem (QEiCP). It differs from the EiCP because there exists an additional quadratic term on λ attached to the eigenvalue problems.

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†Corresponding author. The author's work is supported by Ministry of Science and Technology, Taiwan.

More specifically, given matrices $A, B, C \in \mathbb{R}^{n\times n}$, the QEiCP consists of finding $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\begin{cases} y = \lambda^2 Ax + \lambda Bx + Cx, \\ x \geq 0, \quad y \geq 0, \quad x^T y = 0, \\ a^T x = 1. \end{cases}$$

In the literature, further extensions of the EiCP and the QEiCP are introduced in [1, 3, 11, 15], which include second-order cone eigenvalue complementarity problem (SOCEiCP) and general closed convex cone eigenvalue complementarity problem. Their general format are as below:

$$\begin{cases} y = \lambda Bx - Cx, \\ x \in K, \ y \in K^*, \ x^T y = 0, \\ a^T x = 1, \end{cases} \quad \text{or} \quad \begin{cases} y = \lambda^2 Ax + \lambda Bx + Cx, \\ x \in K, \ y \in K^*, \ x^T y = 0, \\ a^T x = 1, \end{cases} \tag{1.2}$$

where K is a closed convex cone, K^* denotes its dual cone, and $a \in \text{int}(K)$ is arbitrary fixed point. When K represents the second-order cone (denoted by $\mathcal{K}^n$), they become second-order cone eigenvalue complementarity problem (SOCEiCP) and second-order cone quadratic eigenvalue complementarity problem (SOCQEiCP) [17]. For the concept and related properties of second-order cone, please refer to [2, 6, 7, 8, 9, 12, 13, 14, 26].

Roughly, there are two main research directions regarding the eigenvalue complementarity problems. One is on the theoretical side in which their corresponding solution properties are investigated, see [3, 4, 5, 11, 21, 23, 24, 25]. The other one focuses on the numerical algorithm for solving the problems, which include the Lattice projection method, the semismooth Newton methods, the RLT-based branch and bound method (BBRLT) and so on, see [1, 3, 4, 5, 15, 21, 22, 23, 24] and references therein. As seen in the literature, almost all the attention is paid to the standard eigenvalue complementarity problems and second-order cone eigenvalue complementarity problems [17]. However, the study about other special conic eigenvalue complementarity problems is very limited. On the other hand, more and more nonsymmetric cones (for instance, circular cone, p-order cone) appear in plenty of real applications. Hence, in this manuscript, we are therefore motivated to extend the concepts and properties of the EiCP and the QEiCP to the setting of the circular cone $\mathcal{L}_\theta$ and p-order cone $\mathcal{K}_p$. In other words, the cone K in problem (1.2) is circular cone $\mathcal{L}_\theta$ and p-order cone $\mathcal{K}_p$. For the concepts of circular cone $\mathcal{L}_\theta$ and p-order cone $\mathcal{K}_p$, we will review them in details in the next section.

In this paper, several results about the solutions of the eigenvalue complementarity problem (EiCP) and the quadratic eigenvalue complementarity problems (QEiCP) are extended to the version of circular cones $\mathcal{L}_\theta$ and p-order cones $\mathcal{K}^n$, respectively. In the setting of circular cone, the relationship between the solution of the circular cone eigenvalue complementarity problems (CCEiCP) and the solution of the corresponding circular cone complementarity problems (or the solution of the circular cone quadratic eigenvalue complementarity problems (CCQEiCP) and the solution of the corresponding circular cone complementarity problems) are established. Besides, for the p-order cone eigenvalue complementarity problems (POCEiCP) and p-order cone eigenvalue complementarity problems (POCQEiCP), we also sum up the corresponding theoretical results. The solutions of the POCEiCP or the POCQEiCP and the solutions of the corresponding p-order cone complementarity problems correspond to each other. These results build up a theoretical basis for designing numerical solution algorithm or making further study about their corresponding eigenvalue complementarity problems in the future.

The remainder of this paper is organized as follows. In section 2, we recall some background materials regarding circular cone and p-order cone. In section 3, we study the properties of the solutions for the eigenvalue complementarity problems and quadratic eigenvalue complementarity problems in the setting of circular cone, respectively. In section 4, based on the cases of circular cone eigenvalue optimization problems, we study the corresponding properties of the solutions for p-order cone eigenvalue complementarity problems and p-order cone quadratic eigenvalue complementarity problems, respectively.

2. Preliminaries

In this section, we briefly review some basic concepts and background materials about the circular cone $\mathcal{L}_\theta$ and the p-order cone $\mathcal{K}_p$, which will be extensively used in the subsequent sections. More details can be found in [10, 16, 18, 20, 27, 28].

Let $\mathcal{L}_\theta$ denote the circular cone (see [10, 16, 18, 28]) in $\mathbb{R}^n$, which is defined by

$$\mathcal{L}_\theta := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1 \tan\theta\}$$

with $\|\cdot\|$ denoting the Euclidean norm and $\theta \in (0, \frac{\pi}{2})$. It is well known that the dual cone of $\mathcal{L}_\theta$ can be expressed as

$$(\mathcal{L}_\theta)^* = \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1 \cot\theta\} = \mathcal{L}_{\frac{\pi}{2}-\theta}.$$

Moreover, Zhou and Chen [28] gave the below spectral decomposition of $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with respect to $\mathcal{L}_\theta$:

$$x = \mu_1(x) v_x^{(1)} + \mu_2(x) v_x^{(2)},$$

where $\mu_1(x)$, $\mu_2(x)$, $v_x^{(1)}$, and $v_x^{(2)}$ are expressed as

$$\mu_1(x) = x_1 - \|x_2\| \cot\theta, \quad \mu_2(x) = x_1 + \|x_2\| \tan\theta,$$

and

$$v_x^{(1)} = \frac{1}{1+\cot^2\theta} \begin{bmatrix} 1 \\ -\cot\theta \cdot w \end{bmatrix}, \quad v_x^{(2)} = \frac{1}{1+\tan^2\theta} \begin{bmatrix} 1 \\ \tan\theta \cdot w \end{bmatrix}.$$

with $w = \frac{x_2}{\|x_2\|}$ if $x_2 \neq 0$, or any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $x_2 = 0$.

As for the p-order cone. Let $\mathcal{K}_p$ denote the p-order cone ($p > 1$) (see [19, 20, 27]) in $\mathbb{R}^n$, which is defined by

$$\mathcal{K}_p := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\|_p \leq x_1\}$$

with $\|\cdot\|_p$ denoting the p-norm. It is clear to see that when $p=2$, $\mathcal{K}_2$ is exactly the second-order cone, which confirms that the second-order cone is a special case of p-order cone. It is also well known that the dual cone of $\mathcal{K}_p$ can be expressed as

$$(\mathcal{K}_p)^* = \{x=(x_1,x_2)\in \mathbb{R}\times\mathbb{R}^{n-1} \,|\, \|x_2\|_q \le x_1\} = \mathcal{K}_q,$$

where $q>1$ and satisfies $\frac{1}{p}+\frac{1}{q}=1$. Miao, Qi and Chen [20] gave the following spectral decomposition of $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ with respect to $\mathcal{K}_p$:

$$x = \alpha_1(x)v_x^{(1)} + \alpha_2(x)v_x^{(2)},$$

where $\alpha_1(x)$, $\alpha_2(x)$, v_x^1, and v_x^2 are expressed as

$$\alpha_1(x) = \frac{(x_1+\|x_2\|_p)}{2}, \quad \alpha_2(x) = \frac{(x_1-\|x_2\|_p)}{2},$$

and

$$v_x^{(1)} = \begin{bmatrix} 1 \\ w \end{bmatrix}, \quad v_x^{(2)} = \begin{bmatrix} 1 \\ -w \end{bmatrix}.$$

with $w=\frac{x_2}{\|x_2\|_p}$ if $x_2\neq 0$, or any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\|_p=1$ if $x_2=0$.

3. Circular cone eigenvalue complementarity problems

Now, we consider the eigenvalue complementarity problems and circular cone quadratic eigenvalue complementarity problems in the setting of the circular cone $\mathcal{L}_\theta$. We will sum up the relationship between the solution of circular cone eigenvalue complementarity problems (or the solution of circular cone quadratic eigenvalue complementarity problems) and the solution of the corresponding circular cone complementarity problems, respectively.

3.1. **Circular cone eigenvalue complementarity problems.**

Consider the circular cone eigenvalue complementarity problems (CCEiCP for short): find $(x,y,\lambda)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ such that

$$\text{(3.1)} \qquad \mathrm{CCEiCP}(B,C): \begin{cases} y = \lambda Bx - Cx, \\ x\in\mathcal{L}_\theta,\ y\in(\mathcal{L}_\theta)^*,\ x^Ty=0, \\ a^Tx=1, \end{cases}$$

where $B,C\in\mathbb{R}^{n\times n}$ and a is an arbitrary fixed point with $a\in\mathrm{int}((\mathcal{L}_\theta)^*)$. We first study the solvability of the CCEiCP that will be used in subsequent analysis.

Proposition 1. *Suppose that $x:=(x_1,x_2)\in\mathcal{L}_\theta$, $y:=(y_1,y_2)\in(\mathcal{L}_\theta)^*$. Then, the following hold.*

(a) $x^Ty\ge 0$.

(b) *If $y\in\mathrm{int}(\mathcal{L}_\theta)$, then $x^Ty>0$ if and only if $x\neq 0$.*

(c) *If $x\neq 0$ and $y\neq 0$, then*

$$x^Ty=0 \iff x_1\tan\theta = \|x_2\|\, y_1\cot\theta = \|y_2\| \ \text{ and } \ y=\alpha(x_1\tan^2\theta, -x_2),$$

where α is a positive constant, or $x=\beta(y_1\cot^2\theta,-y_2)$ with a positive constant β.

Proof. (a) It is obvious by the definition of dual cone.

(b) The result that $x^T y > 0$ implies $x \neq 0$ is trivial. We only prove the other direction as follows. Since $x \neq 0$, we know that $x_1 > 0$. Otherwise, we have that $\|x_2\| \leq x_1 = 0$ implies $x = 0$. Then, it follows that

$$x^T y = x_1 y_1 + x_2^T y_2 \geq x_1 y_1 - \|x_2\| \|y_2\| > x_1 y_1 - x_1 \tan\theta \, y_1 \cot\theta = 0.$$

(c) The proof of this direction "$\Leftarrow$" is trivial. We only need to prove the direction "$\Rightarrow$". By the assumption $x \neq 0$, we know that $x_1 > 0$. Similarly, we obtain that $y_1 > 0$. Using the condition $x^T y = 0$, the Schwarz's inequality and the definition of circular cones , we have

$$x_1 y_1 = |x_2^T y_2| \leq \|x_2\| \|y_2\| \leq (x_1 \tan\theta)(y_1 \cot\theta) = x_1 y_1. \tag{3.2}$$

Thus, it follows that

$$(x_1 \tan\theta)(y_1 \cot\theta) = x_1 y_1 = \|x_2\| \|y_2\|. \tag{3.3}$$

By combining $\|x_2\| \leq x_1 \tan\theta$, $\|y_2\| \leq y_1 \cot\theta$ and (3.3), it yields

$$x_1 \tan\theta = \|x_2\| \quad \text{and} \quad y_1 \cot\theta = \|y_2\|. \tag{3.4}$$

Besides, by (3.2), (3.3) and (3.4) again, we have

$$x_2^T y_2 = -x_1 y_1 = -\|x_2\| \|y_2\|.$$

This implies the equality holds in the Schwarz's inequality. Hence, there exists a constant k such that

$$y_2 = k x_2. \tag{3.5}$$

Since $k\|y_2\|^2 = x_2^T y_2 = -x_1 y_1 < 0$, we have $k < 0$. Choosing $\alpha = -k$, it leads to $\alpha > 0$ and $y_2 = -\alpha x_2$. Then, it follows from (3.4) and (3.5) that

$$y_1 = \|y_2\| \tan\theta = \| - \alpha x_2\| \tan\theta = | - \alpha| \|x_2\| \tan\theta = \alpha \|x_2\| \tan\theta = \alpha x_1 \tan^2\theta.$$

By this, we have $y = \alpha(x_1 \tan^2\theta, -x_2)$. Thus, the proof is complete. □

In the next theorem, we establish the relation between CCEiCP and the below special circular cone complementarity problem (CCCP):

$$\text{CCCP}(F): \; x \in \mathcal{L}_\theta, \; F(x) \in (\mathcal{L}_\theta)^*, \; x^T F(x) = 0, \tag{3.6}$$

where

$$F(x) = \begin{cases} \frac{x^T C x}{x^T B x} Bx - Cx & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{3.7}$$

Theorem 3.1. *Consider the CCEiCP(B, C) given as in (3.1) where B is positive definite. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (3.7). Then, the following hold.*

(a) *If (x^*, λ^*) solves the CCEiCP(B, C), then x^* solves the CCCP(F) (3.6).*

(b) *If $\bar{x}$ is a nonzero solution of the CCCP(F) (3.6), then (x^*, λ^*) solves the CCEiCP(B, C) with $\lambda^* = \frac{\bar{x}^T C \bar{x}}{\bar{x}^T B \bar{x}}$ and $x^* = \frac{\bar{x}}{a^T \bar{x}}$.*

Proof. Part(a) is trivial and we only need to prove part(b). Suppose that $\bar{x}$ is a nonzero solution to the CCCP(F) (3.6). Then, we have

$$\bar{x} \in \mathcal{L}_\theta, \quad \frac{\bar{x}^T C\bar{x}}{\bar{x}^T B\bar{x}} \cdot B\bar{x} - C\bar{x} \in (\mathcal{L}_\theta)^*, \quad \text{and} \quad \bar{x}^T \left(\frac{\bar{x}^T C\bar{x}}{\bar{x}^T B\bar{x}} B\bar{x} - C\bar{x} \right) = 0.$$

Using $a \in \mathrm{int}((\mathcal{L}_\theta)^*)$ and $\bar{x} \in \mathcal{L}_\theta$, we have $\frac{1}{a^T\bar{x}} > 0$. In view of all the above, we conclude that

$$\begin{array}{rcl} x^* & := & \frac{1}{a^T\bar{x}}\bar{x} \in \mathcal{L}_\theta, \\ w^* & := & \lambda^* B x^* - C x^* = \frac{1}{a^T\bar{x}} \left[\left(\frac{\bar{x}^T C\bar{x}}{\bar{x}^T B\bar{x}} \right) B\bar{x} - C\bar{x} \right] \in (\mathcal{L}_\theta)^*, \\ (x^*)^T w^* & = & \left(\frac{1}{a^T\bar{x}} \right)^2 \left[\bar{x}^T \left(\frac{\bar{x}^T C\bar{x}}{\bar{x}^T B\bar{x}} B\bar{x} - C\bar{x} \right) \right] = 0, \\ a^T x^* & = & \frac{a^T\bar{x}}{a^T\bar{x}} = 1. \end{array}$$

Thus, (x^*, λ^*) is a solution to the CCEiCP(B, C). □

Remark 3.2. For the CCCP(F) (3.6), based on Proposition 1, we note that if x is a solution of the CCCP(F) and x has the spectral decomposition: $x = \mu_1(x)v_x^{(1)} + \mu_2(x)v_x^{(2)}$, $F(x)$ can be written by $F(x) = \mu_1(F(x))v_x^{(2)} + \mu_2(F(x))v_x^{(1)}$. Therefore, we only need to check both $x \in \mathrm{bd}(\mathcal{L}_\theta)$ and $F(x) \in \mathrm{bd}((\mathcal{L}_\theta)^*)$. In addition, we define

$$H_1(x) = \begin{bmatrix} \mu_1(x) \\ \vdots \\ \mu_1(x) \\ \mu_1(F(x)). \end{bmatrix} \in \mathbb{R}^n \quad \textit{and} \quad g_1(z) := \frac{1}{2}\|H_1(z)\|^2 = \frac{1}{2}H_1(z)^T H_1(z).$$

In fact, g is a merit function for the CCCP(F). Then, we can obtain a solution of the CCEiCP (3.1) by using Newton method for solving the equations $H_1(x) = 0$.

Next, we consider the three special kinds of the circular cone complementarity problems, which have the following forms:

$$\text{CCLCP}(-C, 0): \; x \in \mathcal{L}_\theta, \quad -Cx \in (\mathcal{L}_\theta)^*, \quad x^T(-Cx) = 0; \tag{3.8}$$

$$\text{CCCP}(G_1): \left\{ \begin{array}{l} x \in \mathcal{L}_\theta, \quad \lambda \geq 0, \\ \lambda Bx - Cx \in (\mathcal{L}_\theta)^*, \quad a^T x - 1 \geq 0, \\ x^T(\lambda Bx - Cx) + \lambda(a^T x - 1) = 0; \end{array} \right. \tag{3.9}$$

and

$$\text{CCCP}(G_2): \left\{ \begin{array}{l} x \in \mathcal{L}_\theta, \quad \lambda \geq 0, \\ -\lambda Bx - Cx \in (\mathcal{L}_\theta)^*, \quad a^T x - 1 \geq 0, \\ x^T(-\lambda Bx - Cx) + \lambda(a^T x - 1) = 0. \end{array} \right. \tag{3.10}$$

In fact, the solutions of the CCEiCP (3.1) have a close correspondence with the solutions of the above three kinds of complementarity problems. We shall show the relationship between the CCEiCP and the CCCP in the following theorems.

Theorem 3.3. *Let (x^*, λ^*) solves the CCEiCP(B,C) (3.1). Then, the following hold.*

(a) *If $\lambda^* > 0$, then (x^*, λ^*) solves the CCCP(G_1) (3.9).*
(b) *If $\lambda^* < 0$, then $(x^*, -\lambda^*)$ solvesthe CCCP(G_2) (3.10).*
(c) *If $\lambda^* = 0$, then x^* solves the CCLCP($-C$,0) (3.8).*

Proof. From the assumption that (x^*, λ^*) solves the CCEiCP(B,C) (3.1), we have

$$x^* \in \mathcal{L}_\theta,\ \lambda^* Bx^* - Cx^* \in (\mathcal{L}_\theta)^*,\ (x^*)^T(\lambda^* Bx^* - Cx^*) = 0 \ \text{ and } \ a^T x^* = 1.$$

To proceed, we discuss three cases of λ^*.

(a) If $\lambda^* > 0$, then we have

$$(x^*)^T(\lambda^* Bx^* - Cx^*) + \lambda^*(a^T x^* - 1) = (x^*)^T(\lambda^* Bx^* - Cx^*) = 0,$$

which implies that (x^*, λ^*) solves the CCCP (G_1) (3.9).

(b) If $\lambda^* < 0$, then we get that $-\lambda^* > 0$ and

$$(x^*)^T(\lambda^* Bx^* - Cx^*) - \lambda^*(a^T x^* - 1) = (x^*)^T(\lambda^* Bx^* - Cx^*) = 0.$$

This indicates that $(x^*, -\lambda^*)$ solves the CCCP(G_2) (3.10).

(c) If $\lambda^* = 0$, then it follows that

$$\lambda^* Bx^* - Cx^* = -Cx^* \in (\mathcal{L}_\theta)^* \text{ and } (x^*)^T(-Cx^*) = (x^*)^T(\lambda^* Bx^* - Cx^*) = 0.$$

This implies that $(x^*, 0)$ solves the CCLCP($-C$,0) (3.8).

Based on the above arguments, the proof is complete. □

Theorem 3.4. (a) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_1) (3.9), then (x^*, λ^*) solves the CCEiCP(B,C) (3.1).*
(b) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_2) (3.10), then $(x^*, -\lambda^*)$ solves the CCEiCP(B,C) (3.1).*
(c) *If x^* solves the CCLCP($-C, 0$) (3.8) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ solves the CCEiCP(B, C) (3.1).*

Proof. The proof is done by the below three cases..

Case (a): if $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_1) (3.9), then we have

$$\begin{cases} x^* \in \mathcal{L}_\theta, \quad \lambda^* > 0, \\ \lambda^* Bx^* - Cx^* \in (\mathcal{L}_\theta)^*, \quad a^T x^* - 1 \geq 0, \\ (x^*)^T(\lambda^* Bx^* - Cx^*) + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

Combining with the definition of the dual cone again, it gives

$$(x^*)^T(\lambda^* Bx^* - Cx^*) = 0 \ \text{ and } \ \lambda^*(a^T x^* - 1) = 0.$$

It follows from $\lambda^* > 0$ that $a^T x^* - 1 = 0$. Hence, (x^*, λ^*) solves the CCEiCP(B, C) (3.1).

Case (b): if $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_2) (3.10), we have

$$\begin{cases} x^* \in \mathcal{L}_\theta, \quad \lambda^* > 0, \\ -\lambda Bx^* - Cx^* \in (\mathcal{L}_\theta)^*, \quad a^T x^* - 1 \geq 0, \\ (x^*)^T(\lambda^* Bx^* - Cx^*) + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

Then, it follows that $(x^*)^T(-\lambda^* Bx^* - Cx^*) = 0$ and $a^T x^* - 1 = 0$. Hence, $(x^*, -\lambda^*)$ solves the CCEiCP(B, C) (3.1).

Case (c): if x^* is a solution of CCLCP$(-C, 0)$ (3.8) and $x^* \neq 0$, then it is easy to see that $(\frac{x^*}{a^T x^*}, 0)$ solves the CCEiCP(B, C) (3.1).

Based on the above arguments, the proof is complete. □

Remark 3.5. If let $\phi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a circular cone complementarity function [16], based on Theorem 3.3 and Theorem 3.4, then the CCEiCP(B, C) in (3.1) can be reformulated as the following nonsmooth system of equations

$$\Phi_1(z) = \Phi_1(x, y, \lambda) := \begin{bmatrix} \phi_1(x, y) \\ \lambda Bx - Cx - y \\ a^T x - 1 \end{bmatrix} = 0. \tag{3.11}$$

By this, we can solve the CCEiCP (3.1) via using semismooth Newton method to solve the nonsmooth system of equations (3.11).

Example 1. In the CCEiCP(B, C) (3.1) or the CCCP(G_1) (3.9) and the CCCP(G_2) (3.10), let $\theta = \frac{\pi}{3}$. Suppose that the matrix B is the identity matrix, i.e., $B := I$, and the matrix C takes the following matrices, respectively:

$$C_1 := \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad C_2 := \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad C_3 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For the CCEiCP(B, C_1), we find that $w_1^* := (x_1^* = (\alpha, 0)^T, \lambda_1^* = 1)$ for any $\alpha > 0$ are the solutions of this problems. At this moment, $\lambda_1^* = 1 > 0$, by Theorem 3.3, it follows that $w_1^* = (x_1^* = (\alpha, 0)^T, \lambda_1^* = 1)$ are also the solutions of CCCP(G_1) (3.9).

For the CCEiCP(B, C_2), we know that $w_2^* := (x_2^* = (\alpha, 0)^T, \lambda_2^* = -1)$ for any $\alpha > 0$ are the solutions of this problems. Here, $\lambda_2^* = -1 < 0$, by Theorem 3.3, it follows that $(x_2^* = (\alpha, 0)^T, -\lambda_2^* = 1)$ are the solutions of the CCCP(G_2) (3.10).

For the CCEiCP(B, C_3), it is easy to see that $w_3^* := (x_3^* = (1, -\sqrt{3})^T, \lambda_3^* = 0)$ is a solution of this problems. Now, $\lambda_3^* = 0$, by Theorem 3.3 again, we have $x_3^* = (1, -\sqrt{3})^T$ is a solution of the CCCP$(-C_3, 0)$ (3.8).

Conversely, we know that w_1^*, w_2^* and x_3^* solve the CCCP(G_1) (3.9), the CCCP(G_2) (3.10) and the CCCP$(-C_3, 0)$ (3.8), respectively. By Theorem 3.4, based on the value of λ_i^* for $i = 1, 2, 3$, it follows that w_1^*, $(x_2^*, -\lambda_2^*)$ and $(\frac{x_3^*}{a^T x_3^*}, 0)$ are the solutions of CCEiCP(B, C_i) for $i = 1, 2, 3$.

3.2. **Circular cone quadratic eigenvalue complementarity problems.** In this subsection, we consider the following circular cone quadratic eigenvalue complementarity problems (CCQEiCP for short). Given matrices $A, B, C \in \mathbb{R}^{n\times n}$, the CCQEiCP seeks to find $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\text{CCQEiCP}(A,B,C): \begin{cases} y = \lambda^2 Ax + \lambda Bx + Cx, \\ x \in \mathcal{L}_\theta,\ y \in (\mathcal{L}_\theta)^*,\ x^T y = 0, \\ a^T x = 1, \end{cases} \tag{3.12}$$

where a is an arbitrary fixed point with $a \in \text{int}((\mathcal{L}_\theta)^*)$.

Before discussing the properties of the CCQEiCP (3.12), we first introduce the definition of $\mathcal{L}_\theta$-hyperbolic that will be used later.

Definition 1. A triple (A, B, C), with $A, B, C \in \mathbb{R}^{n\times n}$ is called $\mathcal{L}_\theta$-*hyperbolic* if

$$(x^T Bx)^2 \geq 4(x^T Ax)(x^T Cx),$$

for all nonzero $x \in \mathcal{L}_\theta$.

Again, similar to Theorem 3.1, we build up the relation between the CCQEiCP (3.12) and the CCCP (3.6).

Theorem 3.6. *Consider the CCQEiCP(A,B,C) given as in (3.12) where A is positive definite and a triple (A,B,C) is $\mathcal{L}_\theta$-hyperbolic. Let $F_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows:*

$$F_i(x) = \begin{cases} \lambda_i^2(x)Ax + \lambda_i(x)Bx + Cx & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{3.13}$$

where $\lambda_i(x) = \frac{-(x^TBx)+(-1)^{i+1}\sqrt{(x^TBx)^2-4(x^TAx)(x^TCx)}}{2(x^TAx)}$ *for $i = 1, 2$. Then the following hold.*

(a) *If (x^*, λ^*) is a solution to the CCQEiCP(A,B,C), then x^* solves either the CCCP(F_1) or the CCCP(F_2).*

(b) *If $\bar{x}$ is a nonzero solution to the CCCP(F_1) or the CCCP(F_2), then (x^*, λ^*) is a solution of the CCQEiCP(A,B,C) with $x^* = \frac{\bar{x}}{a^T\bar{x}}$ and $\lambda^* = \lambda_1(\bar{x})$ or $\lambda_2(\bar{x})$.*

Proof. The result of part(a) is trivial. We only need to prove part(b). Suppose that $\bar{x}$ is a nonzero solution to the CCCP(F_i) with F_i as in (3.13) for $i = 1, 2$. Then, for each i, we know that

$$\bar{x} \in \mathcal{L}_\theta,\ \ \lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x} \in \mathcal{L}_\theta \ \text{ and } \ \bar{x}^T(\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x}) = 0.$$

Since $a \in \text{int}((\mathcal{L}_\theta)^*)$ and $\bar{x} \in \mathcal{L}_\theta$, we have $\frac{1}{a^T\bar{x}} > 0$. From all the above, we conclude that

$$\begin{aligned} x^* &:= \tfrac{1}{a^T\bar{x}}\bar{x} \in \mathcal{L}_\theta, \\ w^* &:= (\lambda^*)^2Ax^* + \lambda^*Bx^* + Cx^* = \tfrac{1}{a^T\bar{x}}\left(\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x}\right) \in (\mathcal{L}_\theta)^*, \\ (x^*)^Tw^* &= \left(\tfrac{1}{a^T\bar{x}}\right)^2\left[\bar{x}^T(\lambda_i^2(\bar{x})A\bar{x} + \lambda_i(\bar{x})B\bar{x} + C\bar{x})\right] = 0, \\ a^Tx^* &= \tfrac{a^T\bar{x}}{a^T\bar{x}} = 1. \end{aligned}$$

Thus, (λ^*, x^*) solves the CCQEiCP(A, B, C) and the proof is complete. $\square$

Remark 3.7. Similar to Remark 3.2, for the CCCP(F_i), where F_i is defined as in (3.13) $(i = 1, 2)$, we use the same skills to define

$$H_{i1}(x) = \begin{bmatrix} \mu_1(x) \\ \vdots \\ \mu_1(x) \\ \mu_1(F_i(x)). \end{bmatrix} \in \mathbb{R}^n$$

with $i = 1, 2$. Then, we obtain a solution of the CCQEiCP (3.12) via using Newton method for solving the equation $H_{i1}(x) = 0$ $(i = 1, 2)$.

Next, we consider two special classes of the circular cone complementarity problems, which have the following forms:

$$\text{CCCP}(G_4) : \begin{cases} x \in \mathcal{L}_\theta, \quad \lambda \geq 0, \\ \lambda^2 Ax + \lambda Bx + Cx \in (\mathcal{L}_\theta)^*, \quad a^T x - 1 \geq 0, \\ x^T(\lambda^2 Ax + \lambda Bx + Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{3.14}$$

and

$$\text{CCCP}(G_5) : \begin{cases} x \in \mathcal{L}_\theta, \quad \lambda \geq 0, \\ \lambda^2 Ax - \lambda Bx + Cx \in (\mathcal{L}_\theta)^*, \quad a^T x - 1 \geq 0, \\ x^T(\lambda^2 Ax - \lambda Bx + Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{3.15}$$

Similar to the cases of the CCEiCP (3.1), We will show the relationship between the CCQEiCP(A, B, C), the CCLCP$(C, 0)$ and the CCCP(G_i) for $i = 4, 5$.

Theorem 3.8. *Let (x^*, λ^*) solves the CCQEiCP(A,B,C) (3.12). Then, the following hold.*

(a) *If $\lambda^* > 0$, then (x^*, λ^*) solves the CCCP(G_4) (3.14).*
(b) *If $\lambda^* < 0$, then $(x^*, -\lambda^*)$ solves the CCCP(G_5) (3.15).*
(c) *If $\lambda^* = 0$, then x^* solves the CCLCP(C,0) (3.8).*

Proof. From the assumption that (x^*, λ^*) solves the CCQEiCP(A,B,C) (3.12), we have

$$x^* \in \mathcal{L}_\theta, \ (\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^* \in (\mathcal{L}_\theta)^*, \ (x^*)^T\left[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*\right] = 0$$

and $a^T x^* = 1$. To proceed, we discuss three cases.

Case (a): if $\lambda^* > 0$, then we have

$$(x^*)^T[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*] + \lambda^*(a^T x^* - 1) = 0.$$

This implies that (x^*, λ^*) solves the CCCP(G_4) (3.14).

Case (b): if $\lambda^* < 0$, then we have $-\lambda^* > 0$ and

$$(x^*)^T\left[(\lambda^*)^2 Ax^* - \lambda^* Bx^* + Cx^*\right] - \lambda^*(a^T x^* - 1) = 0.$$

This indicates that $(x^*, -\lambda^*)$ solves the CCCP(G_5) (3.15).

Case (c): if $\lambda^* = 0$, then we have

$$(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^* = Cx^* \in \mathcal{L}_\theta \quad \text{and} \quad (x^*)^T Cx^* = 0.$$

This says that $(x^*, 0)$ solves the CCLCP(C,0) (3.8).

Based on the above arguments, we prove the desired result. $\square$

Theorem 3.9. (a) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_4) (3.14), then (x^*, λ^*) solves the CCQEiCP(A,B,C) (3.12).*

(b) *If (x^*, λ^*) solves the CCCP(G_5) (3.15) and $\lambda^* \neq 0$, then $(x^*, -\lambda^*)$ solves the CCQEiCP(A,B,C) (3.12).*

(c) *If x^* solves CCLCP($-C$, 0) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ is a solution to the CCQEiCP(A,B,C) (3.12).*

Proof. Again, the proof is done by discussing three cases.

Case (a): if $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_4) (3.14), then we have

$$\begin{cases} x^* \in \mathcal{L}_\theta, \quad \lambda^* > 0, \\ (\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^* \in (\mathcal{L}_\theta)^*, \quad a^T x^* - 1 \geq 0, \\ (x^*)^T \left[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*\right] + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

By the definition of the dual cone, this implies $(x^*)^T[(\lambda^*)^2 Ax^* + \lambda^* Bx^* + Cx^*] = 0$ and $\lambda^*(a^T x^* - 1) = 0$. In addition, it follows from $\lambda^* > 0$ that $a^T x^* - 1 = 0$. Hence, we obtain that (x^*, λ^*) solves the CCQEiCP(A,B,C) (3.12).

Case (b): if $\lambda^* \neq 0$ and (x^*, λ^*) solves the CCCP(G_5) (3.15), then we have

$$\begin{cases} x^* \in \mathcal{L}_\theta, \quad \lambda^* > 0, \\ (-\lambda^*)^2 Ax^* + (-\lambda^*) Bx^* + Cx^* \in (\mathcal{L}_\theta)^*, \quad a^T x^* - 1 \geq 0, \\ (x^*)^T \left[(-\lambda^*)^2 Ax^* + (-\lambda^*) Bx^* + Cx^*\right] + \lambda^*(a^T x^* - 1) = 0. \end{cases}$$

This says that $(x^*)^T[(-\lambda^*)^2 Ax^* + (-\lambda^*) Bx^* + Cx^*] = 0$ and $a^T x^* - 1 = 0$. Hence, we obtain that $(x^*, -\lambda^*)$ solves the CCQEiCP(A,B,C) (3.12).

Case (c): if x^* solves the CCLCP($-C$, 0) (3.8) and $x^* \neq 0$, then it is easy to see that $(\frac{x^*}{a^T x^*}, 0)$ solves the CCQEiCP(A,B,C) (3.12).

Based on the above arguments, the proof is complete. $\square$

Remark 3.10. Similar to Remark 3.5, the CCQEiCP(A, B, C) in (3.12) can be reformulated as the following nonsmooth system of equations

$$\Psi_1(z) = \Psi_1(x, y, \lambda) := \begin{bmatrix} \phi_1(x, y) \\ \lambda^2 Ax + \lambda Bx + Cx - y \\ a^T x - 1 \end{bmatrix} = 0, \tag{3.16}$$

From this, we can solve the CCQEiCP(A, B, C) (3.12) via using semismooth Newton method to solve the nonsmooth system of equations (3.16).

4. p-ORDER CONE EIGENVALUE COMPLEMENTARITY PROBLEMS

In this section, we consider p-order cone eigenvalue complementarity problems and p-order cone quadratic eigenvalue complementarity problems. Similar to the cases of the circular cone, we will show the relationship between the solution of p-order cone eigenvalue complementarity problems (or the solution of p-order cone quadratic eigenvalue complementarity problems) and the solution of the corresponding p-order cone complementarity problems, respectively.

4.1. **p-order cone eigenvalue complementarity problems.** Consider the p-order cone eigenvalue complementarity problems (POCEiCP for short): find $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$\text{POCEiCP}(B, C): \begin{cases} y = \lambda Bx - Cx, \\ x \in \mathcal{K}_p, \quad y \in (\mathcal{K}_p)^*, \quad x^T y = 0, \\ a^T x = 1, \end{cases} \tag{4.1}$$

where $B, C \in \mathbb{R}^{n \times n}$ and a is an arbitrary fixed point with $a \in \text{int}((\mathcal{K}_p)^*)$. First, we introduce some notations for the sake of convenience. If $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $x_2 = (x_2^{(1)}, ..., x_2^{(n-1)})^T$, then we denote $\left(|x_2^{(1)}|^p, \cdots, |x_2^{(n-1)}|^p\right)^T$ by $|x_2|^p$. Similar to the CCEiCP (Proposition 1), by Hölder's inequality, we achieve the version of p-order cone as the following proposition.

Proposition 2. *Suppose that $x := (x_1, x_2) \in \mathcal{K}_p$, $y := (y_1, y_2) \in (\mathcal{K}_p)^*$. Then, the following hold.*

(a) $x^T y \geq 0$.

(b) *If $y \in \text{int}(\mathcal{K}_p)$, then $x^T y > 0$ if and only if $x \neq 0$.*

(c) *If $x \neq 0$ and $y \neq 0$, then*

$$x^T y = 0 \quad \Rightarrow \quad x_1 = \|x_2\|_p \ \text{ and } \ |y_2|^q = \alpha |x_2|^p,$$

where α is a positive constant. Similarly, if $x \neq 0$ and $y \neq 0$, then

$$x^T y = 0 \quad \Rightarrow \quad y_1 = \|y_2\|_q \ \text{ and } \ |x_2|^p = \beta |y_2|^q,$$

where β is a positive constant.

Proof. By the definition of dual cones, part(a) is obvious. For part(b), it is also trivial that $x^T y > 0$ implies $x \neq 0$. We only prove the other direction as follows. Since $x \neq 0$, we know that $x_1 > 0$. Otherwise, we have that $\|x_2\|_p \leq x_1 = 0$ implies $x = 0$. Thus, it yields

$$x^T y = x_1 y_1 + x_2^T y_2 \geq x_1 y_1 - \|x_2\|_p \|y_2\|_q > x_1 y_1 - x_1 y_1 = 0.$$

Hence, we prove the result of part(b).

(c) "$\Leftarrow$" The proof of this direction is trivial. "$\Rightarrow$" By the assumption $x \neq 0$, we know that $x_1 > 0$. Similarly, we know that $y_1 > 0$. Using the condition of $x^T y = 0$, the Hölder's inequality and the definition of circular cones, we have

$$x_1 y_1 = |x_2^T y_2| \leq \|x_2\|_p \|y_2\|_q \leq x_1 y_1 \tag{4.2}$$

which leads to

$$x_1 y_1 = \|x_2\|_p \|y_2\|_q. \tag{4.3}$$

From $x_1 > 0, y_1 > 0, x_1 \geq \|x_2\|_p, y_1 \geq \|y_2\|_q$ and the equality (4.3), it follows that

$$x_1 = \|x_2\|_p \ \text{ and } \ y_1 = \|y_2\|_q. \tag{4.4}$$

Moreover, by (4.2), (4.3) and (4.4) again, we have

$$x_2^T y_2 = -x_1 y_1 = -\|x_2\| \|y_2\|.$$

This implies the equality holds in the Hölder's inequality. Thus, there exists a constant α such that

$$|y_2|^q = \alpha |x_2|^p.$$

Then, the proof is complete. □

Based on the cases of circular cone, we build up the relation between the POCEiCP and the below p-order cone complementarity problem (POCCP):

$$\text{POCCP}(F): \quad x \in \mathcal{K}_p, \quad F(x) \in (\mathcal{K}_p)^*, \quad x^T F(x) = 0, \tag{4.5}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (3.7), i.e.,

$$F(x) = \begin{cases} \frac{x^T C x}{x^T B x} Bx - Cx & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Similar to Theorem 3.1, we use the same technique to establish the relation between the POCEiCP and the POCCP.

Theorem 4.1. *Consider the POCEiCP(B, C) given as in (4.1) where B is positive definite. Then, the following hold.*

(a) *If (x^*, λ^*) solves the POCEiCP(B, C), then x^* solves the POCCP(F) (4.5).*

(b) *If $\bar{x}$ is a nonzero solution to the POCCP(F) (4.5), then (x^*, λ^*) solves the POCEiCP(B, C) with $\lambda^* = \frac{\bar{x}^T C \bar{x}}{\bar{x}^T B \bar{x}}$ and $x^* = \frac{\bar{x}}{a^T \bar{x}}$.*

Proof. The proof is similar to Theorem 3.1. Hence, we omit it. □

Likewise, we consider the following three special classes of the p-order cone complementarity problems:

$$\text{POCLCP}(-C, 0): \ x \in \mathcal{K}_p, \quad -Cx \in (\mathcal{K}_p)^*, \quad x^T(-Cx) = 0, \tag{4.6}$$

$$\text{POCCP}(G_1): \begin{cases} x \in \mathcal{K}_p, \quad \lambda \geq 0, \\ \lambda Bx - Cx \in (\mathcal{K}_p)^*, \quad a^T x - 1 \geq 0, \\ x^T(\lambda Bx - Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{4.7}$$

and

$$
\text{POCCP}(G_2): \begin{cases} x \in \mathcal{K}_p, \quad \lambda \geq 0, \\ -\lambda Bx - Cx \in (\mathcal{K}_p)^*, \quad a^T x - 1 \geq 0, \\ x^T(-\lambda Bx - Cx) + \lambda(a^T x - 1) = 0, \end{cases} \tag{4.8}
$$

where the matrixes B, C are given as in the POCEiCP (4.1).

Analogous to Theorem 3.3 and Theorem 3.4, we have the relation between the POCEiCP and the POCCP.

Theorem 4.2. *Let (x^*, λ^*) solves thePOCEiCP(B,C) (4.1). Then, the following hold.*

(a) *If $\lambda^* > 0$, then (x^*, λ^*) solves the POCCP(G_1) (4.7).*
(b) *If $\lambda^* < 0$, then $(x^*, -\lambda^*)$ solves the POCCP(G_2) (4.8).*
(c) *If $\lambda^* = 0$, then x^* solves the POCLCP($-C$,0)(4.6).*

Proof. The proof is similar to Theorem 3.3. Hence, we omit it. □

Theorem 4.3. (a) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the POCCP(G_1) (4.7), then (x^*, λ^*) solves the POCEiCP(B,C) (4.1).*

(b) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the POCCP(G_2) (4.8), then $(x^*, -\lambda^*)$ solves the POCEiCP(B,C) (4.1).*
(c) *If x^* solves the POCLCP($-C, 0$)(4.6) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ solves the POCEiCP(B, C) (4.1).*

Proof. The proof is similar to Theorem 3.4. Hence, we omit it. □

Remark 4.4. If let $\phi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a p-order cone complementarity function, based on Theorem 4.2 and Theorem 4.3, then the POCEiCP(B, C) in (4.1) can be reformulated as the following nonsmooth system of equations:

$$
\Phi_2(z) = \Phi_2(x, y, \lambda) := \begin{bmatrix} \phi_2(x, y) \\ \lambda Bx - Cx - y \\ a^T x - 1 \end{bmatrix} = 0. \tag{4.9}
$$

Accordingly, we can solve the POCEiCP (4.1) via using semismooth Newton method to solve the nonsmooth system of equations (4.9).

4.2. **p-order Cone quadratic eigenvalue complementarity problems.** Now, we consider the p-order cone quadratic eigenvalue complementarity problems (POCQEiCP for short) as follows: given matrices $A, B, C \in \mathbb{R}^{n\times n}$, the POCQEiCP seeks to find $(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that

$$
\text{POCQEiCP}(A, B, C): \begin{cases} y = \lambda^2 Ax + \lambda Bx + Cx, \\ x \in \mathcal{K}_p, \quad y \in (\mathcal{K}_p)^*, \quad x^T y = 0, \\ a^T x = 1, \end{cases} \tag{4.10}
$$

where a is an arbitrary fixed point with $a \in \text{int}((\mathcal{K}_p)^*)$. We first introduce the definition of $\mathcal{K}_p$-hyperbolic, which will be used later.

Definition 2. A triple (A, B, C), with $A, B, C \in \mathbb{R}^{n\times n}$ is called *$\mathcal{K}_p$-hyperbolic* if

$$(x^T Bx)^2 \geq 4(x^T Ax)(x^T Cx),$$

for all nonzero $x \in \mathcal{K}_p$.

Again, analogous to Theorem 3.6, we build up the relation between POCQEiCP and the POCCP.

Theorem 4.5. *Consider the POCQEiCP(A, B, C) given as in (4.10) where A is positive definite and the triple (A, B, C) is $\mathcal{K}_p$-hyperbolic. Let $F_i : \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (3.13). Then, the following hold.*

(a) *If (x^*, λ^*) solves the POCQEiCP(A, B, C), then x^* solves either the POCCP(F_1) or the POCCP(F_2).*
(b) *If $\bar{x}$ is a nonzero solution to the POCCP(F_1) or the POCCP(F_2), then (x^*, λ^*) solves the POCQEiCP(A, B, C) with $x^* = \frac{\bar{x}}{a^T\bar{x}}$ and $\lambda^* = \lambda_1(\bar{x})$ or $\lambda_2(\bar{x})$, where*

$$\lambda_i(x) = \frac{-(x^T Bx) + (-1)^{i+1}\sqrt{(x^T Bx)^2 - 4(x^T Ax)(x^T Cx)}}{2(x^T Ax)}$$

for $i = 1, 2$.

Proof. The proof is similar to Theorem 3.6. Hence, we omit it. □

Lastly, we consider the following two special kinds of p-order cone complementarity problems:

$$\text{(4.11)} \qquad \text{POCCP}(G_4) : \begin{cases} x \in \mathcal{K}_p, \quad \lambda \geq 0, \\ \lambda^2 Ax + \lambda Bx + Cx \in (\mathcal{K}_p)^*, \quad a^T x - 1 \geq 0, \\ x^T(\lambda^2 Ax + \lambda Bx + Cx) + \lambda(a^T x - 1) = 0, \end{cases}$$

and

$$\text{(4.12)} \qquad \text{POCCP}(G_5) : \begin{cases} x \in \mathcal{K}_p, \quad \lambda \geq 0, \\ \lambda^2 Ax - \lambda Bx + Cx \in (\mathcal{K}_p)^*, \quad a^T x - 1 \geq 0, \\ x^T(\lambda^2 Ax - \lambda Bx + Cx) + \lambda(a^T x - 1) = 0. \end{cases}$$

Accordingly, we have the relation between the POCEiCP and the POCCP.

Theorem 4.6. *Let (x^*, λ^*) solves the POCQEiCP(A,B,C) (4.10). Then, the following hold.*

(a) *If $\lambda^* > 0$, then (x^*, λ^*) solves the POCCP (G_4) (4.11).*
(b) *If $\lambda^* < 0$, then $(x^*, -\lambda^*)$ solves the POCCP(G_5) (4.12).*
(c) *If $\lambda^* = 0$, then x^* solves the POCLCP(C,0).*

Proof. The proof is similar to Theorem 3.8. Hence, we omit it. □

Theorem 4.7. (a) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the POCCP(G_4) (4.11), then (x^*, λ^*) solves the POCQEiCP(A,B,C) (4.10).*

(b) *If $\lambda^* \neq 0$ and (x^*, λ^*) solves the POCCP(G_5) (4.12), then $(x^*, -\lambda^*)$ solves POCQEiCP(A,B,C) (4.10).*

(c) *If x^* solves POCLCP($-C, 0$) and $x^* \neq 0$, then $(\frac{x^*}{a^T x^*}, 0)$ is a solution of the POCLCP($-C$,0).*

Proof. The proof is similar to Theorem 3.9. Hence, we omit it. □

Remark 4.8. Similar to Remark 3.10, the POCQEiCP(A, B, C) in (4.10) can be reformulated as the following nonsmooth system of equations:

$$\Psi_2(z) = \Psi_2(x, y, \lambda) := \begin{bmatrix} \phi_2(x, y) \\ \lambda^2 Ax + \lambda Bx + Cx - y \\ a^T x - 1 \end{bmatrix} = 0. \tag{4.13}$$

From this, we can solve the POCQEiCP(A, B, C) (4.10) vis using semismooth Newton method to solve the nonsmooth system of equations (4.13).

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Wei-Ming Hsu
Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan
E-mail address: `a32410@hotmail.com`

Xin-He Miao
School of Mathematics, Tianjin University, China, Tianjin 300072, China
E-mail address: `xinhemiao@tju.edu.cn`

Jein-Shan Chen
Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan
E-mail address: `jschen@math.ntnu.edu.tw`