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NEW SUFFICIENCY FOR GLOBAL OPTIMALITY AND DUALITY OF NONLINEAR MULTI-OBJECTIVE PROGRAMMING PROBLEMS VIA UNDERESTIMATORS

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ABSTRACT. Jeyakumar and Srisatkunarajah presented new sufficient conditions for a Karush-Kuhn-Tucker point to be a global minimizer of a mathematical programming problem by introducing the concept of underestimator. They also presented sufficient conditions for weak and strong duality results in terms of underestimators.

In this paper, we extend the above results to nonlinear multi-objective programming problems. We also present new type sufficient conditions for weak and strong duality results for nonlinear multi-objective programming problems.

1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be vector-valued maps. The multi-objective programming problem is formalized as follows:

$$\text{(MOP)} \begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n. \end{cases}$$

Many researchers have investigated a duality theory of (MOP), for instance, see [2, 3, 9].

The Karush-Kuhn-Tucker (KKT for short) condition gives a necessary condition for a feasible point to be a locally optimal solution of a mathematical programming problem under a certain constraint qualification. KKT condition is often used to developing criteria for identifying global solutions. However, much of the work in this area of KKT sufficiency requires that Lagrangian of the problem satisfies certain generalized convexity conditions, for instance, pseudo-convex functions and invex functions.

Jeyakumar and Srisatkunarajah [4] presented new sufficiency conditions for a Karush-Kuhn-Tucker point to be a global minimizer of a mathematical programming problem by introducing the concept of underestimator which has the property that every stationary point is a global minimizer. They [4] also presented sufficient conditions for weak and strong duality results in terms of underestimators. Their results rely on the fact that the biconjugate function of the Lagrangian is a convex underestimator at a point whenever it coincides with the Lagrangian at that point.

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In this paper, we aim to extend the above results in the framework of the nonlinear multi-objective programming problem. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ be vector-valued maps. We formulate the nonlinear multi-objective programming problem as follows (see also [9]):

$$\text{(P)} \begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \leq_{\mathbb{R}^l_+} 0 \quad (g(x) \in -\mathbb{R}^l_+) \end{cases}$$

where $\mathbb{R}^l_+ := \{x \in \mathbb{R}^l \mid x_1 \geq 0, x_2 \geq 0, ..., x_l \geq 0\}$ is a nonnegative orthant of $\mathbb{R}^l$.

The paper is organized as follows. First we introduce some mathematical preliminaries. Next, we introduce the concept of underestimators for vector-valued map and give Karush-Kuhn-Tucker sufficiency for nonlinear multi-objective programming problems by using the generalized underestimators. We also give new definitions of bi-conjugate of vector-valued maps and investigate its properties. Finally, we present Wolfe type weak and strong duality theorems for nonlinear multi-objective programming problems.

2. Mathematical preliminaries

2.1. **Preliminaries of vector optimization.** Let Y be a normed vector space and 0_Y the origin of Y. For a set $A \subset Y$, int$A$ and cl$A$ denote the topological interior and the topological closure of A, respectively. We denote the family of nonempty subsets of Y by $\mathcal{V}$. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

$$V_1 + V_2 := \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\}; \qquad \alpha V := \{\alpha v \mid v \in V\}.$$

We assume that $C \subset Y$ is a closed convex cone, that is, cl$C = C$, $C + C \subset C$ and $tC \subset C$ for all $t \in [0, \infty)$. A cone C is called pointed if $C \cap (-C) = \{0_Y\}$ and solid if int$C \neq \emptyset$.

Definition 2.1. For $a, b \in Y$ and a solid convex cone $C \subset Y$, we define

$$a \leq_C b \quad \text{by} \quad b - a \in C.$$

Proposition 2.2. *For $x \in Y$ and $y \in Y$, the following statements hold:*

(i) *$x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in Y$,*
(ii) *$x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$,*
(iii) *$\leq_C$ is reflexive and transitive. Moreover, if C is pointed, $\leq_C$ is antisymmetric and hence a partial order.*

We say that a point $a \in A \subset Y$ is a minimal [resp. weak minimal] point of A if there is no $\hat{a} \in A \setminus \{a\}$ such that $\hat{a} \leq_C a$ [resp. $\hat{a} \leq_{\text{int}C} a$]. The above definition is equivalent to

$$A \cap (a - C) = \{a\} \quad [\text{resp. } A \cap (a - \text{int}C) = \emptyset].$$

We denote the set of minimal [resp. weak minimal] points of A with respect to C [resp. int$C$] by Min$(A; C)$[resp. wMin$(A; \text{int}C)$], respectively

Similarly, we say that a point $a \in A \subset Y$ is a maximal [resp. weak maximal] point of A if there is no $\hat{a} \in A \setminus \{a\}$ such that $a \leq_C \hat{a}$ [resp. $a \leq_{\mathrm{int}C} \hat{a}$]. The above definition is equivalent to

$$A \cap (a + C) = \{a\} \quad [\text{resp. } A \cap (a + \mathrm{int}C) = \emptyset].$$

We denote the set of maximal [resp. weak maximal] points of A with respect to C [resp. int$C$] by $\mathrm{Max}(A; C)$[resp. $\mathrm{wMax}(A; \mathrm{int}C)$], respectively. We can easily see that

$$\mathrm{Min}(A; C) \subset \mathrm{wMin}(A; \mathrm{int}C) \subset A \qquad \text{and} \qquad \mathrm{Max}(A; C) \subset \mathrm{wMax}(A; \mathrm{int}C) \subset A.$$

Proposition 2.3. *Let $C \subset Y$ be a closed pointed convex cone. If $\mathrm{Max}(A; C) \neq \emptyset$ for all $A \in \mathcal{V}$ then the following statements hold:*

(i) $\mathrm{Max}(A + B; C) \subset \mathrm{Max}(A; C) + \mathrm{Max}(B; C)$ *for all $A, B \in \mathcal{V}$.*

(ii) $\mathrm{Max}(\alpha A; C) = \alpha \mathrm{Max}(A; C)$ *for all $A \in \mathcal{V}$ and $\alpha \geq 0$.*

Proof. (i) Let $x \in \mathrm{Max}(A + B; C)$. Then by the definition of maximal point, we have

$$(A + B) \cap (x + C) = \{x\},$$

that is,

(a) $x \in A + B$ and $x \in x + C$,

(b) for all $y \neq x$ satisfies $y \in A + B$ and $y \notin x + C$.

By property (a), there exist $a \in A$ and $b \in B$ such that $x = a + b$. Therefore, we have $a + b \in A + B$ and $a + b \in a + b + C$ and hence

$$a \in A \cap (a + C) \qquad \text{and} \qquad b \in B \cap (b + C).$$

We suppose contrary that

$$\hat{a} \in A \cap (a + C) \qquad \text{or} \qquad \hat{b} \in B \cap (b + C)$$

for some $\hat{a} \in \mathrm{Max}(A; C)(\hat{a} \neq a)$ and $\hat{b} \in \mathrm{Max}(B; C)(\hat{b} \neq b)$. Then we have

$$\hat{a} + \hat{b} \in \{A \cap (a + C)\} + \{B \cap (b + C)\} \subset (A + B) \cap (a + b + C),$$

that is, $\hat{a} + \hat{b} \in x + C$, which contradicts the maximality of x.

(ii) If $\alpha = 0$, then we have $\mathrm{Max}(\alpha A; C) = \alpha \mathrm{Max}(A; C) = \{0_Y\}$. Therefore, we consider the case $\alpha > 0$. Let $x \in \mathrm{Max}(\alpha A; C)$. Then by the definition of maximal point, we have

$$(\alpha A) \cap (x + C) = \{x\},$$

that is,

(a') $x \in \alpha A$ and $x \in x + C$,

(b') for all $y \neq x$ satisfies $y \in \alpha A$ and $y \notin x + C$.

We put $\hat{x} = \dfrac{x}{\alpha}$. Then by property (a'), we have $\hat{x} \in A$ and

$$\hat{x} \in \frac{1}{\alpha}(x + C) = \frac{x}{\alpha} + \frac{C}{\alpha} = \hat{x} + C.$$

Hence $\hat{x} \in A \cap (\hat{x} + C)$. We suppose contrary that

$$\hat{y} \in A \cap (\hat{x} + C)$$

for some $\hat{y} \in \mathrm{Max}(A; C)(\hat{y} \neq \hat{x})$. Then we have $\alpha\hat{y} \in \alpha A$ and $\alpha\hat{y} \in \alpha\hat{x} + C$. Hence

$$\alpha\hat{y} \in x + C,$$

which contradicts the maximality of x. The converse inclusion is similar as the above. □

Example 1. We see that the inverse inclusion $\mathrm{Max}(A + B; C) \supset \mathrm{Max}(A; C) + \mathrm{Max}(B; C)$ does not hold for all $A, B \in \mathcal{V}$. Let

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\},$$

$$A = B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, x_1 \geq 0, x_2 \geq 0\}.$$

We can confirm that

$$\mathrm{Max}(A; C) = \mathrm{Max}(B; C) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1, x_1 \geq 0, x_2 \geq 0\},$$

$$A + B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 4, x_1 \geq 0, x_2 \geq 0\} \qquad \text{and}$$

$$\mathrm{Max}(A+B; C) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2+x_2^2 = 4, x_1 \geq 0, x_2 \geq 0\} \subset \mathrm{Max}(A; C)+\mathrm{Max}(B; C).$$

2.2. **Preliminaries of set optimization.** We consider several types of binary relationships on $\mathcal{V}$ by using a solid convex cone $C \subset Y$.

Definition 2.4 ([1, 8])**.** For A, $B \in \mathcal{V}$ and a solid convex cone $C \subset Y$, we define

$$(\textbf{weak type}) \quad A \leq_C^w B \quad \text{by} \quad B - A \subset C,$$

$$(\textbf{lower type}) \quad A \leq_C^l B \quad \text{by} \quad B \subset A + C,$$

$$(\textbf{upper type}) \quad A \leq_C^u B \quad \text{by} \quad A \subset B - C,$$

$$(\textbf{strong type}) \quad A \leq_C^s B \quad \text{by} \quad 0_Y \in B - A - C.$$

Proposition 2.5 ([1])**.** *For A, $B \in \mathcal{V}$, the following statements hold:*

(i) *$A \leq_C^w B$ implies $A \leq_C^l B$ and $A \leq_C^l B$ implies $A \leq_C^s B$.*
(ii) *$A \leq_C^w B$ implies $A \leq_C^u B$ and $A \leq_C^u B$ implies $A \leq_C^s B$.*
(iii) *$A \leq_C^l B$ and $A \leq_C^u B$ are not comparable.*

Proposition 2.6 ([1])**.** *For A, B, $D \in \mathcal{V}$ and $\alpha \geq 0$, the following statements hold:*

(i) *$A \leq_C^l B$ implies $(A + D) \leq_C^l (B + D)$ and $A \leq_C^u B$ implies $(A + D) \leq_C^u (B + D)$.*
(ii) *$A \leq_C^l B$ implies $\alpha A \leq_C^l \alpha B$ and $A \leq_C^u B$ implies $\alpha A \leq_C^u \alpha B$.*
(iii) *$\leq_C^l$ and $\leq_C^u$ are reflexive and transitive.*

Introducing the equivalence relations

$$A \sim_l B \Longleftrightarrow A \leq_C^l B \quad \text{and} \quad B \leq_C^l A,$$

$$A \sim_u B \Longleftrightarrow A \leq_C^u B \quad \text{and} \quad B \leq_C^u A,$$

we can generate a partial ordering on the set of equivalence classes which are denoted by $[\cdot]^l$ and $[\cdot]^u$, respectively. We can easily see that

$$A \in [B]^l \Longleftrightarrow A + C = B + C,$$
$$A \in [B]^u \Longleftrightarrow A - C = B - C.$$

Definition 2.7. ($l[u]$-minimal and $l[u]$-maximal element [1]) Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u]$-minimal element if for any $A \in \mathcal{S}$,

$$A \leq_C^{l[u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_C^{l[u]} A.$$

Moreover, we say that $\bar{A} \in \mathcal{S}$ is a $l[u]$-maximal element if for any $A \in \mathcal{S}$,

$$\bar{A} \leq_C^{l[u]} A \quad \text{implies} \quad A \leq_C^{l[u]} \bar{A}.$$

We denote the family of $l[u]$-minimal elements of $\mathcal{S}$ by $l[u]$-Min$(\mathcal{S}, C)$ and the family of $l[u]$-maximal elements of $\mathcal{S}$ by $l[u]$-Max$(\mathcal{S}, C)$.

2.3. **Preliminaries of multi-objective optimization.** We denote n-dimensional Euclidean space by $\mathbb{R}^n$. We clearly see that $\mathbb{R}^m_+$ is closed pointed convex cone. We denote the set of $m \times l$ matrix by $\mathbb{R}^{m\times l}$, that is,

$$\mathbb{R}^{m\times l} := \left\{ A \;\middle|\; A = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{ml} \end{pmatrix}, a_{ij} \in \mathbb{R}, i \in \{1, ..., m\}, j \in \{1, ..., l\} \right\},$$

$$\mathbb{R}^{m\times l}_+ := \left\{ A \in \mathbb{R}^{m\times l} \;\middle|\; A = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{ml} \end{pmatrix}, a_{ij} \geq 0, i \in \{1, ..., m\}, j \in \{1, ..., l\} \right\}.$$

For a matrix $A \in \mathbb{R}^{m\times l}$, we denote a transpose of A by A^T. We denote the inner product of the two vectors $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)^T$ by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i,$$

and the norm of the vector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^T$ by $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. We denote the ε-neighborhood of $\bar{x}$ by

$$\mathcal{N}(\bar{x}, \varepsilon) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| < \varepsilon\}.$$

Definition 2.8 (global minimizer, global maximizer)**.** We say that $\bar{x} \in \mathbb{R}^n$ is a global minimizer of (P) if there does not exist $x \in \mathbb{R}^n$ such that $f(x) \leq_{\mathbb{R}^m_+ \setminus \{0\}} f(\bar{x})$. Similarly, we say that $\bar{x} \in \mathbb{R}^n$ is a global maximizer of (P) if there does not exist $x \in \mathbb{R}^n$ such that $f(\bar{x}) \leq_{\mathbb{R}^m_+ \setminus \{0\}} f(x)$.

Definition 2.9 (local minimizer, local maximizer)**.** We say that $\bar{x}$ is a local minimizer of (P) if for some $\mathcal{N}(\bar{x}, \varepsilon)$ there does not exist $x \in \mathcal{N}(\bar{x}, \varepsilon)$ such that $f(x) \leq_{\mathbb{R}^m_+ \setminus \{0\}} f(\bar{x})$. Similarly, we say that $\bar{x} \in \mathbb{R}^n$ is a local maximizer of (P) if for some $\mathcal{N}(\bar{x}, \varepsilon)$ there does not exist $x \in \mathcal{N}(\bar{x}, \varepsilon)$ such that $f(\bar{x}) \leq_{\mathbb{R}^m_+ \setminus \{0\}} f(x)$.

Definition 2.10 (differentiability of vector-valued function)**.** We say that a vector-valued function $f = (f_1, f_2, \cdots, f_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable if for each i $(1 \leq i \leq m)$ $f_i : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Then we have the following equality:

$$\nabla f(x) = \begin{pmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_m(x)^T \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix},$$

where $\nabla f_i(x)^T = \left(\frac{\partial f_i}{\partial x_1}(x), \cdots, \frac{\partial f_i}{\partial x_n}(x) \right)$ $(1 \leq i \leq m)$.

Definition 2.11 ($\mathbb{R}^m_+$-convex function)**.** We say that a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex function if for each i $(1 \leq i \leq m)$ $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex function, that is,

$$f(\lambda x + (1-\lambda)y) \leq_{\mathbb{R}^m_+} \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

Proposition 2.12. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. If ϕ is a $\mathbb{R}^m_+$-convex function, then every stationary point of ϕ is a global minimizer of ϕ.*

Proof. The proof is straightforward from the definition of $\mathbb{R}^m_+$-convex function and global minimizer. ☐

3. Karush-Kuhn-Tucker sufficiency and underestimators for vector-valued map

Let $f_j : \mathbb{R}^n \to \mathbb{R}$ $(j = 0, 1, 2, \cdots, m)$ be continuously differentiable functions on an open subset of $\mathbb{R}^n$. The mathematical programming problem is formalized as follows:

$$(\text{MP}) \begin{cases} \text{Minimize} \quad f_0(x) \\ \text{subject to} \quad f_j(x) \leq 0 \quad (j = 1, 2, \cdots, m). \end{cases}$$

Definition 3.1 (Underestimator [4])**.** A function $h : \mathbb{R}^n \to \mathbb{R}$ is called an underestimator of a function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$, if h satisfies the following conditions.

(a) For each $x \in \mathbb{R}^n$, $h(x) \leq f(x)$ and $f(\bar{x}) = h(\bar{x})$.
(b) A point $\bar{x} \in \mathbb{R}^n$ is a stationary point of the function $f : \mathbb{R}^n \to \mathbb{R}$ if f is differentiable at $\bar{x}$ and $\nabla f(\bar{x}) = 0$.

The concept of underestimator, which is introduced by Jeyakumar-Srisatkunarajah [4], is very powerful tool in mathematical programming problem because various generalized convex functions such as pseudo-convex functions, and the invex functions satisfy the property of underestimator, that is, every stationary point is a global minimizer.

Theorem 3.2 ([4])**.** *Let $\bar{x}$ be a feasible point of* (MP) *at which Karush-Kuhn-Tucker conditions hold with the multiplier λ. We assume that the Lagrangian $L(\cdot, \lambda)$ admits an underestimator at $\bar{x}$, at which it is differentiable. If every stationary point of the underestimator is a global minimizer, then $\bar{x}$ is a global minimizer of* (MP).

The aim of this section is to generalize the above results to vector-valued function.

3.1. **Underestimators for vector-valued map.** Throughout the paper, we assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable function on an open subset of $\mathbb{R}^n$. The Lagrangian of (P), denoted by $L(x, A)$, is given by

$$L(x, A) = f(x) + Ag(x),$$

where A is an $m \times l$ matrix. If $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^T$ is a local minimizer of (P) and if a certain constraint qualification (see [2, 3, 7] for detail) holds then the following Karush-Kuhn-Tucker conditions hold:

(KKT) there exists $A \in \mathbb{R}_+^{m\times l}$ such that $Ag(\bar{x}) = 0$ and $\nabla L(\bar{x}, A) = 0$.

Definition 3.3. A vector-valued function $h : \mathbb{R}^n \to \mathbb{R}^m$ is called an underestimator of a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ at $\bar{x} \in \mathbb{R}^n$, if h satisfies the following conditions.

(a) For each $x \in \mathbb{R}^n$, $h(x) \leq_{\mathbb{R}_+^m} f(x)$ and $f(\bar{x}) = h(\bar{x})$.
(b) A point $\bar{x} \in \mathbb{R}^n$ is a stationary point of the function $f : \mathbb{R}^n \to \mathbb{R}^m$ if f is differentiable at $\bar{x}$ and $\nabla f(\bar{x}) = 0$.

We generalize Theorem 3.2 to the case of a multi-objective optimization problem (P).

Theorem 3.4. *Let $\bar{x}$ be a feasible point of* (P) *at which* **(KKT)** *hold with the multiplier $A \in \mathbb{R}_+^{m\times l}$. We assume that the Lagrangian $L(\cdot, A)$ admits an underestimator at $\bar{x}$, at which it is differentiable. If every stationary point of the underestimator is a global minimizer, then $\bar{x}$ is a global minimizer of* (P).

Proof. We set $\boldsymbol{b_1}^T := (a_{11}, a_{12}, \cdots, a_{1l}), \cdots, \boldsymbol{b_m}^T := (a_{m1}, a_{m2}, \cdots, a_{ml})$, that is,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{ml} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b_1}^T \\ \vdots \\ \boldsymbol{b_m}^T \end{pmatrix}.$$

By the assumption $a_{ij} \geq 0$ $(1 \leq i \leq m, 1 \leq j \leq l)$, we have

$$Ag(x) = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{ml} \end{pmatrix} \begin{pmatrix} g_1(x) \\ \vdots \\ g_l(x) \end{pmatrix} = \begin{pmatrix} \boldsymbol{b_1}^T g(x) \\ \vdots \\ \boldsymbol{b_m}^T g(x) \end{pmatrix} \leq_{\mathbb{R}^m} 0.$$

Of course, we see that the first component of $Ag(x)$ is $\boldsymbol{b_1}^T g(x) = \langle \boldsymbol{b_1}, g(x)\rangle$. Hence, we have $L_1(x, \boldsymbol{b_1}) = f_1(x) + \langle \boldsymbol{b_1}, g(x)\rangle$.

Let $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ be the underestimator of $L(\cdot, A)$ at $\bar{x}$, and let $d \in \mathbb{R}^n$ be arbitrary. Then for each $\alpha > 0$ we have

$$\frac{h_1(\bar{x} + \alpha d) - h_1(\bar{x})}{\alpha} \leq \frac{L_1(\bar{x} + \alpha d, \boldsymbol{b_1}) - L_1(\bar{x}, \boldsymbol{b_1})}{\alpha}.$$

Letting $\alpha \to 0$, we obtain $\nabla h_1(\bar{x})^T d \leq \nabla L_1(\bar{x}, \boldsymbol{b_1})^T d = 0$. Hence, $\nabla h_1(\bar{x})^T d \leq 0$ for each $d \in \mathbb{R}^n$. Therefore, we obtain $\nabla h_1(\bar{x}) = 0$ and $\bar{x}$ is a stationary point of h_1. By the assumption, we have that $h_1(\bar{x}) \leq h_1(x)$ for each $x \in \mathbb{R}^n$.

Finally, we show that $\bar{x}$ is a global minimizer of (P). Let x be a feasible point of (P). Since $\langle \boldsymbol{b_1}, g(x)\rangle \leq 0$ and $\langle \boldsymbol{b_1}, g(\bar{x})\rangle = 0$, we have

$$f_1(x) + \langle \boldsymbol{b_1}, g(x)\rangle - f_1(\bar{x}) - \langle \boldsymbol{b_1}, g(\bar{x})\rangle \leq f_1(x) - f_1(\bar{x}),$$

that is,

$$L_1(x, \boldsymbol{b_1}) - L_1(\bar{x}, \boldsymbol{b_1}) \leq f_1(x) - f_1(\bar{x}).$$

Since $h_1(x) \leq L_1(x, \boldsymbol{b_1})$ and $h_1(\bar{x}) = L_1(\bar{x}, \boldsymbol{b_1})$, we have

$$h_1(x) - h_1(\bar{x}) \leq f_1(x) - f_1(\bar{x}).$$

Furthermore, since $h_1(\bar{x}) \leq h_1(x)$, we obtain $f_1(\bar{x}) \leq f_1(x)$. Similarly, with the same argument as the above, we obtain

$$f_2(\bar{x}) \leq f_2(x), \quad f_3(\bar{x}) \leq f_3(x), \quad \cdots, \quad f_m(\bar{x}) \leq f_m(x)$$

and hence

$(*)$ $f(\bar{x}) \leq_{\mathbb{R}^m_+} f(x)$ for all $x \in \mathbb{R}^n$.

Then there does not exist $x \in \mathbb{R}^n$ such that $f(x) \leq_{\mathbb{R}^m_+\setminus\{0\}} f(\bar{x})$, that is, $\bar{x}$ is a global minimizer of (P). Suppose contrary that there exists $x \in \mathbb{R}^n$ such that $f(x) \leq_{\mathbb{R}^m_+\setminus\{0\}} f(\bar{x})$. By the definition of $\leq_{\mathbb{R}^m_+}$, we have

$(**)$ $f(x) \leq_{\mathbb{R}^m_+\setminus\{0\}} f(\bar{x}) \iff f(\bar{x}) - f(x) \in \mathbb{R}^m_+\setminus\{0\} \iff f(x) - f(\bar{x}) \in -\mathbb{R}^m_+\setminus\{0\}$.

By $(*)$ and $(**)$, we have $f(x) - f(\bar{x}) \in \mathbb{R}^m_+ \cap \{-\mathbb{R}^m_+\setminus\{0\}\} = \emptyset$, which is a contradiction. □

Example 2. We consider the multi-objective nonconvex programming

$$(\text{P1}) \begin{cases} \text{Minimize} & (\frac{1}{4}x^4 + x^3 - 2x^2 + 1, x^4 + 4x^3 - 8x^2 + 4) \\ \text{subject to} & (x^2 - 25, x^2 + 4x - 12) \leq_{\mathbb{R}^2_+} (0, 0). \end{cases}$$

Then $\bar{x} = -4$ and $x = 1$ are local minimizers of (P1). **(KKT)** are satisfied at both points with $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The function $h : \mathbb{R} \to \mathbb{R}^2$ defined by

$$h(x) = \begin{cases} (\frac{1}{4}x^4 + x^3 - 2x^2 + 1, x^4 + 4x^3 - 8x^2 + 4) & \text{if } x \leq -4 \\ (-31, -124) & \text{otherwise} \end{cases}$$

is a nonconvex underestimator of the Lagrangian $L(\cdot, O)$ at $\bar{x} = -4$, where $L(x, 0) = (\frac{1}{4}x^4 + x^3 - 2x^2 + 1, x^4 + 4x^3 - 8x^2 + 4)$. We have also that

$$\nabla h(x) = \begin{cases} (x^3 + 3x^2 - 4x, 4x^3 + 12x^2 - 16x) & \text{if } x \leq -4 \\ (0, 0) & \text{otherwise.} \end{cases}$$

We easily see that all the stationary points of h are indeed its global minimizers and the point $\bar{x} = -4$ is the unique global minimizer of (P1).

Corollary 3.5. *Let $\bar{x}$ be a feasible point of* (P) *at which* **(KKT)** *holds with the multiplier $A \in \mathbb{R}_+^{m\times l}$. If the Lagrangian $L(\cdot, A)$ admits a $\mathbb{R}_+^m$-convex underestimator at $\bar{x}$ then $\bar{x}$ is a global minimizer of* (P)*.*

Proof. Let $\phi = (\phi_1, \phi_2, \cdots, \phi_m)$ be the convex underestimator of $L(\cdot, A)$ at $\bar{x}$, and let $d \in \mathbb{R}^n$ be arbitrary. Then for each $\alpha > 0$ we have

$$\frac{\phi_1(\bar{x}+\alpha d) - \phi_1(\bar{x})}{\alpha} \leq \frac{L_1(\bar{x}+\alpha d, \boldsymbol{b_1}) - L_1(\bar{x}, \boldsymbol{b_1})}{\alpha}.$$

Letting $\alpha \to 0$, we obtain $(\phi_1)'(\bar{x}, d) \leq \nabla L_1(\bar{x}, \boldsymbol{b_1})^T d = 0$. Since ϕ is $\mathbb{R}_+^m$-convex and $(\phi_i)'(\bar{x}, d) \leq 0$ for each $d \in \mathbb{R}^n$ and $1 \leq i \leq m$, ϕ is differentiable at $\bar{x}$ and $\nabla\phi(\bar{x}) = 0$. Moreover, every stationary point of ϕ is a global minimizer of ϕ. Therefore, the conclusion follows from Theorem 3.4. ☐

3.2. **Conjugate maps and its applications.** In this subsection, we consider the conjugate maps for vector-valued function f and its applications to (P).

In this subsection, we set $\mathcal{V} := 2^{\mathbb{R}^m} \setminus \{\emptyset\}$.

Definition 3.6 (Tanino-Sawaragi [10, 11, 12])**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. Then the set-valued map $f^* : \mathbb{R}^{m\times n} \to \mathcal{V}$ defined by

$$f^*(A) := \text{Max}\left(\bigcup_{x\in\mathbb{R}^n} \{Ax - f(x)\}; \mathbb{R}_+^m\right)$$

is called the conjugate map of f.

Definition 3.7 (Tanino-Sawaragi [10, 11, 12])**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function and let $f^* : \mathbb{R}^{m\times n} \to \mathcal{V}$ be the conjugate map of f. By reiterating the operation $f \to f^*$ on f^*, we define the bi-conjugate of f, $f^{**} : \mathbb{R}^n \to \mathcal{V}$, by the following form:

$$f^{**}(x) := \text{Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}} \{Ax - f^*(A)\}; \mathbb{R}_+^m\right).$$

However, generally speaking, $f^*(A)$ is a set-valued mapping. To overcome the difficulty, Kawasaki [5, 6] introduced set relation on $\mathcal{V}$. He also defined conjugate relation and introduced the concept of Γ^n-regularization to derive a duality theorem in multi-objective programming. Based on his results, we give new definitions of the bi-conjugate of f.

Definition 3.8. For $f^*(A) \neq \emptyset$, we define $f_l^{**}, f_u^{**} : \mathbb{R}^n \to \mathcal{V}$ by

$$f_l^{**}(x) := l\text{-Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}} [Ax - f^*(A)], \mathbb{R}_+^m\right),$$

$$f_u^{**}(x) := u\text{-Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}} [Ax - f^*(A)], \mathbb{R}_+^m\right).$$

Proposition 3.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. Then the bi-conjugate of f has the following properties.*

(a) *If* $f^*(A)$ *is a singleton for all* $A \in \mathbb{R}^{m\times n}$, *then* $f^{**}(x) \leq^w_{\mathbb{R}^m_+} f(x)$ *for all* $x \in \mathbb{R}^n$.
(b) *If* $f^*(A)$ *satisfies the condition* $f^*(A) \subset Ax - f(x) + \mathbb{R}^m_+$ *for all* $x \in \mathbb{R}^n$ *and* $A \in \mathbb{R}^{m\times n}$, *then* $f_l^{**}(x) \leq^l_{\mathbb{R}^m_+} f(x)$ *for all* $x \in \mathbb{R}^n$.
(c) *If* $f^*(A)$ *satisfies the condition* $f^*(A) - f^*(A) \subset -\mathbb{R}^m_+$ *for all* $A \in \mathbb{R}^{m\times n}$, *then* $f_u^{**}(x) \leq^u_{\mathbb{R}^m_+} f(x)$ *for all* $x \in \mathbb{R}^n$.

Proof. (a) By the definition of f^*, we have

$$Ax - f(x) \leq_{\mathbb{R}^m_+} f^*(A) \qquad \forall x \in \mathbb{R}^n, \quad \forall A \in \mathbb{R}^{m\times n}$$

and hence

$$Ax - f^*(A) \leq_{\mathbb{R}^m_+} f(x) \qquad \forall x \in \mathbb{R}^n, \quad \forall A \in \mathbb{R}^{m\times n}.$$

By the definition of f^{**}, we obtain the conclusion.

(b) By the definition of f^* and assumption (b), we have

$$\{Ax - f(x)\} \leq^l_{\mathbb{R}^m_+} f^*(A) \qquad \forall x \in \mathbb{R}^n, \quad \forall A \in \mathbb{R}^{m\times n},$$

and hence

$$\{Ax\} \leq^l_{\mathbb{R}^m_+} f(x) + f^*(A). \qquad (f(x) + f^*(A) \subset Ax + \mathbb{R}^m_+).$$

Then we have

$$f(x) \in f(x) + f^*(A) - f^*(A) \subset Ax - f^*(A) + \mathbb{R}^m_+,$$

that is, $Ax - f^*(A) \leq^l_{\mathbb{R}^m_+} f(x)$. By the definition of f_l^{**}, we obtain the conclusion.

(c) By the definition of f^*, we have

$$\{Ax - f(x)\} \leq^u_{\mathbb{R}^m_+} f^*(A) \qquad \forall x \in \mathbb{R}^n, \quad \forall A \in \mathbb{R}^{m\times n},$$

and hence

$$\{Ax\} \leq^u_{\mathbb{R}^m_+} f(x) + f^*(A). \qquad (Ax \in f(x) + f^*(A) - \mathbb{R}^m_+).$$

Then by assumption (c), we have

$$Ax - f^*(A) \subset f(x) + f^*(A) - f^*(A) - \mathbb{R}^m_+ \subset f(x) - \mathbb{R}^m_+ - \mathbb{R}^m_+ = f(x) - \mathbb{R}^m_+,$$

that is, $Ax - f^*(A) \leq^u_{\mathbb{R}^m_+} f(x)$. By the definition of f_u^{**}, we obtain the conclusion. □

Proposition 3.10. *If* $f^*(A)$ *is a singleton, then* f^{**} *is* u-$\mathbb{R}^m_+$-*convex function, that is,*

$$f^{**}(\lambda x_1 + (1-\lambda)x_2) \leq^u_{\mathbb{R}^m_+} \lambda f^{**}(x_1) + (1-\lambda) f^{**}(x_2)$$

for any $x_1, x_2 \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Then we have

$$f^{**}(\lambda x_1 + (1-\lambda)x_2) = \mathrm{Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}} \{A(\lambda x_1 + (1-\lambda)x_2) - f^*(A)\}; \mathbb{R}^m_+\right)$$

$$= \mathrm{Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}} \{\lambda A x_1 + (1-\lambda)Ax_2 - \lambda f^*(A) - (1-\lambda) f^*(A)\}; \mathbb{R}^m_+\right)$$

$$= \mathrm{Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}}\{\lambda(Ax_1 - f^*(A)) + (1-\lambda)(Ax_2 - f^*(A))\}; \mathbb{R}^m_+\right)$$

$$= \mathrm{Max}\left(\bigcup_{A\in\mathbb{R}^{m\times n}}\{\lambda(Ax_1 - f^*(A))\} + \bigcup_{A\in\mathbb{R}^{m\times n}}\{(1-\lambda)(Ax_2 - f^*(A))\}; \mathbb{R}^m_+\right) = (\star).$$

By using (i) of Proposition 2.3, we have

$$(\star) \subset \mathrm{Max}\left(\bigcup_A \lambda\{Ax_1 - f^*(A)\}; \mathbb{R}^m_+\right) + \mathrm{Max}\left(\bigcup_A (1-\lambda)\{Ax_2 - f^*(A)\}; \mathbb{R}^m_+\right) = (\star\star).$$

Moreover, by using (ii) of Proposition 2.3, we have

$$(\star\star) = \lambda\cdot\mathrm{Max}\left(\bigcup_A\{Ax_1 - f^*(A)\}; \mathbb{R}^m_+\right) + (1-\lambda)\cdot\mathrm{Max}\left(\bigcup_A\{Ax_2 - f^*(A)\}; \mathbb{R}^m_+\right)$$

$$= \lambda f^{**}(x_1) + (1-\lambda) f^{**}(x_2) \subset \lambda f^{**}(x_1) + (1-\lambda) f^{**}(x_2) - \mathbb{R}^m_+.$$

□

Remark 1. By (i) of Proposition 2.3, it is difficult to obtain l-type convexity of vector-valued biconjugate function.

Corollary 3.11. *Let $\bar{x}$ be a feasible point of* (P) *at which* **(KKT)** *holds with the multiplier $A \in \mathbb{R}^{m\times l}_+$. If $L^{**}(\bar{x}, A)$ is a singleton and $L^{**}(\bar{x}, A) = L(\bar{x}, A)$ then $\bar{x}$ is a global minimizer of* (P).

Proof. By Proposition 3.10, L^{**} is $\mathbb{R}^m_+$-convex function. Moreover, by Proposition 3.9, we have $L^{**}(\bar{x}, A) \leq_{\mathbb{R}^m_+} L(\bar{x}, A)$. Thus $L^{**}(\cdot, A)$ is a $\mathbb{R}^m_+$-convex underestimator of $L(\cdot, A)$ at $\bar{x}$. Therefore, conclusion follows from Corollary 3.5. □

Example 3. We consider the multi-objective nonconvex programming

$$(\mathrm{P2})\begin{cases}\text{Minimize} & (\frac{1}{4}x^4 - x^3 + x^2, \frac{1}{4}x^4 - x^3 + x^2 + 3)\\ \text{subject to} & (x-5, x-7) \leq_{\mathbb{R}^2_+} (0,0).\end{cases}$$

(KKT) conditions are satisfied at $\bar{x} = 0$ and $\bar{x} = 2$ with $A = \begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$. The Lagrangian $L(x, O) := (\frac{1}{4}x^4 - x^3 + x^2, \frac{1}{4}x^4 - x^3 + x^2 + 3)$ and

$$L^{**}(x, O) = \begin{cases}(0,3) & \text{if} \quad 0 \leq x \leq 2\\ (\frac{1}{4}x^4 - x^3 + x^2, \frac{1}{4}x^4 - x^3 + x^2 + 3) & \text{otherwise.}\end{cases}$$

We easily see that $L^{**}(\bar{x}, O) = L(\bar{x}, O) = (0,3)$ at both points $\bar{x} = 0$ and $\bar{x} = 2$ which are global minimizers of (P2).

4. Wolfe type duality

In this section, we present weak and strong duality results in the framework of not necessarily convex programming problems. For (P), we consider Wolfe type dual problem.

$$\text{(D)}\begin{cases}\text{Maximize} & L(y,A) \quad (y,A)\in\mathbb{R}^n\times\mathbb{R}^{m\times l}\\ \text{subject to} & \nabla L(y,A)=0, \quad A\in\mathbb{R}_+^{m\times l}\end{cases}$$

4.1. **Weak duality.**

Theorem 4.1 (Weak Duality)**.** *For the problem* (P) *and* (D)*, we suppose that at each feasible solution* (y,A) *of* (D)*, the Lagrangian admits an underestimator* $\tilde{L}(\cdot,A)$ *which satisfies the following conditions.*

(1) $\tilde{L}(\cdot,A)$ *is differentiable at that point.*

(2) $\tilde{L}(\cdot,A)$ *has the property that every stationary point is a global minimizer.*

Then $\mathrm{Max(D)} \leq^w_{\mathbb{R}^m_+} \mathrm{Min(P)}$*.*

Proof. Let $x\in\mathbb{R}^n$ be feasible for (P), $(y,A)\in\mathbb{R}^n\times\mathbb{R}_+^{m\times l}$ be feasible for (D) and

$$L(y,A):=\big(L_1(y,A),L_2(y,A),\cdots,L_m(y,A)\big),$$
$$\tilde{L}(y,A):=\big(\tilde{L}_1(y,A),\tilde{L}_2(y,A),\cdots,\tilde{L}_m(y,A)\big).$$

Then for each $d\in\mathbb{R}^n$ and for each $\alpha>0$ we have

$$\frac{\tilde{L}_1(y+\alpha d,A)-\tilde{L}_1(y,A)}{\alpha}\leq\frac{L_1(y+\alpha d,A)-L_1(y,A)}{\alpha}.$$

Letting $\alpha\to 0$, we obtain $\nabla\tilde{L}_1(y,A)^Td\leq\nabla L_1(y,A)^Td=0$. Therefore, we have $\nabla\tilde{L}_1(y,A)=0$. By the definition of L_1, we have

$$f_1(x)-L_1(y,A)\geq L_1(x,A)-L_1(y,A). \qquad (\dagger)$$

Since $\tilde{L}_1$ is an underestimator of L_1, we have

$$L_1(x,A)-L_1(y,A)\geq\tilde{L}_1(x,A)-L_1(y,A). \qquad (\dagger\dagger)$$

Moreover, by the definition of the underestimator and assumption (2), we obtain

$$\tilde{L}_1(x,A)-L_1(y,A)=\tilde{L}_1(x,A)-\tilde{L}_1(y,A)\geq 0. \qquad (\dagger\dagger\dagger)$$

Combining (†), (††) and († † †), we obtain $L_1(y,A)\leq f_1(x)$. Similarly, with the same argument as the above, we obtain

$$L_2(y,A)\leq f_2(x),\quad L_3(y,A)\leq f_3(x),\quad\cdots,\quad L_m(y,A)\leq f_m(x)$$

and hence $L(y,A)\leq_{\mathbb{R}^m_+} f(x)$.

Therefore, we have that $\mathrm{Max(D)}\leq^w_{\mathbb{R}^m_+}\mathrm{Min(P)}$ holds, that is, weak duality holds. □

Corollary 4.2. *For the problem* (P) *and* (D)*, we suppose that at each feasible solution* (y,A) *of* (D)*, the Lagrangian admits a* $\mathbb{R}^m_+$*-convex underestimator* $\tilde{L}(\cdot,A)$*. Then we have* $\mathrm{Max(D)}\leq^w_{\mathbb{R}^m_+}\mathrm{Min(P)}$*.*

Proof. Let $x \in \mathbb{R}^n$ be feasible for (P), $(y, A) \in \mathbb{R}^n \times \mathbb{R}_+^{m\times l}$ be feasible for (D) and

$$L(y,A) := \big(L_1(y,A), L_2(y,A), \cdots, L_m(y,A)\big)$$
$$\tilde{L}(y,A) := \big(\tilde{L}_1(y,A), \tilde{L}_2(y,A), \cdots, \tilde{L}_m(y,A)\big).$$

Then for each $d \in \mathbb{R}^n$ and for each $\alpha > 0$ we have

$$\frac{\tilde{L}_1(y+\alpha d, A) - \tilde{L}_1(y,A)}{\alpha} \leq \frac{L_1(y+\alpha d, A) - L_1(y,A)}{\alpha}$$

Letting $\alpha \to 0$, we obtain $(\tilde{L}_1)'(y,A)^T d \leq \nabla L_1(y,A)^T d = 0$. Therefore, we have $\nabla \tilde{L}_1(y,A) = 0$. Similarly, we obtain

$$(\tilde{L}_2)'(\bar{x}, d) = 0, \cdots, (\tilde{L}_m)'(\bar{x}, d) = 0.$$

Therefore, we have that $\tilde{L}$ is differentiable at $\bar{x}$ and $\nabla \tilde{L}(\bar{x}) = 0$. Since $\tilde{L}$ is $\mathbb{R}_+^m$-convex, every stationary point of $\tilde{L}$ is a global minimizer of $\tilde{L}$. Therefore, the conclusion follows from Theorem 4.1.

□

Corollary 4.3. *For the problem* (P) *and* (D)*, we suppose that at each feasible solution* (y, A) *of* (D)*,* $L^{**}(y,A)$ *is a singleton and* $L^{**}y, A) = L(y,A)$*. Then we have* Max(D) $\leq^w_{\mathbb{R}_+^m}$ Min(P)*.*

Proof. Since $L^{**}(\cdot, A)$ is a $\mathbb{R}_+^m$-convex underestimator of $L(\cdot, A)$ at each feasible (y, A) of (D), the conclusion follows from Corollary 4.2. □

4.2. **Strong duality.**

Theorem 4.4 (Strong Duality)**.** *For the problem* (P) *and* (D)*, we suppose that at each feasible solution* (y, A) *of* (D)*, the Lagrangian admits an underestimator* $\tilde{L}(\cdot, A)$ *which satisfies the following conditions.*

(1) $\tilde{L}(\cdot, A)$ *is differentiable at that point.*
(2) $\tilde{L}(\cdot, A)$ *has the property that every stationary point is a global minimizer.*

If **(KKT)** *holds at a minimizer of* (P) *then the following statements hold.*

(a) *If we assume the condition* $L(\cdot, A) \leq^l_{\mathbb{R}_+^m}$ Max(D)*, then we have* Max(D)$\in$ $[\text{Min(P)}]^l$*.*
(b) *If we assume the condition* Min(P) $\subset$ Max(D)$-\mathbb{R}_+^m$*, then we have* Max(D)$\in$ $[\text{Min(P)}]^u$*.*
(c) *If we assume the condition* $L(\cdot, A) \leq^w_{\mathbb{R}_+^m}$ Max(D)*, then we have* Max(D) $=$ Min(P)*.*

Proof. (a) Let $\bar{x} \in \mathbb{R}^n$ be a minimizer of (P). Then, by the assumption, there exists $\bar{A} \in \mathbb{R}_+^{m\times l}$ such that $\bar{A}g(\bar{x}) = 0$ and $\nabla L(\bar{x}, \bar{A}) = 0$, that is, $(\bar{x}, \bar{A})$ is feasible for (D). Therefore, by assumption (a) and Theorem 4.1, we have

$$L(\bar{x}, \bar{A}) \leq^l_{\mathbb{R}_+^m} \text{Max(D)} \leq^l_{\mathbb{R}_+^m} \text{Min(P)}.$$

Since $f(\bar{x}) = L(\bar{x}, \bar{A}) \in$ Min(P), we have

$$\text{Max(D)} \subset L(\bar{x}, \bar{A}) + \mathbb{R}_+^m \subset \text{Min(P)} + \mathbb{R}_+^m,$$

that is, Min(P)$\leq^l_{\mathbb{R}^m_+}$ Max(D). By the definition of $[\cdot]^l$, we obtain the conclusion.

(b) Let $\bar{x} \in \mathbb{R}^n$ be a minimizer of (P). Then, by the assumption, $(\bar{x}, \bar{A})$ is feasible for (D). Therefore, by Theorem 4.1, we have

$$L(\bar{x}, \bar{A}) \leq^u_{\mathbb{R}^m_+} \text{Max(D)} \leq^u_{\mathbb{R}^m_+} \text{Min(P)}.$$

(We remark that by the definition of maximal point, $L(\bar{x}, \bar{A}) \leq^u_{\mathbb{R}^m_+}$ Max(D) is always true.) By assumption (b), we have

$$\text{Min(P)} \leq^u_{\mathbb{R}^m_+} \text{Max(D)} \leq^u_{\mathbb{R}^m_+} \text{Min(P)}.$$

By the definition of $[\cdot]^u$, we obtain the conclusion.

(c) With the same argument as the proof of (a) and taking account of the fact that $\mathbb{R}^m_+$ is a pointed convex cone, we obtain the conclusion. □

Remark 2. If we consider some scalarizing functions which satisfies monotonicity condition, that is, for $A, B \in \mathcal{V}$ and scalarizing function $s : \mathcal{V} \to \mathbb{R}$ we have that $A \leq^l_{\mathbb{R}^m_+} B$ implies $s(A) \leq s(B)$, then

$$A \in [B]^l \quad \Longrightarrow \quad s(A) = s(B).$$

Corollary 4.5. *For the problem* (P) *and* (D)*, we suppose that at each feasible solution* (y, A) *of* (D)*, the Lagrangian admits a* $\mathbb{R}^m_+$*-convex underestimator* $\tilde{L}(\cdot, A)$*. If* **(KKT)** *holds at a minimizer of* (P) *then the following statements hold.*

(a) *If we assume the condition* $L(\cdot, A) \leq^l_{\mathbb{R}^m_+}$ Max(D)*, then we have* Max(D)$\in$ $[\text{Min(P)}]^l$.
(b) *If we assume the condition* Min(P) $\subset$ Max(D)$-\mathbb{R}^m_+$*, then we have* Max(D)$\in$ $[\text{Min(P)}]^u$.
(c) *If we assume the condition* $L(\cdot, A) \leq^w_{\mathbb{R}^m_+}$ Max(D)*, then we have* Max(D) $=$ Min(P).

Corollary 4.6. *For the problem* (P) *and* (D)*,* $L^{**}(y, A)$ *is a singleton and* $L^{**}(y, A) = L(y, A)$ *for each feasible* (y, A) *of* (D)*. If* **(KKT)** *holds at a minimizer of* (P) *then the following statements hold.*

(a) *If we assume the condition* $L(\cdot, A) \leq^l_{\mathbb{R}^m_+}$ Max(D)*, then we have* Max(D)$\in$ $[\text{Min(P)}]^l$.
(b) *If we assume the condition* Min(P) $\subset$ Max(D)$-\mathbb{R}^m_+$*, then we have* Max(D)$\in$ $[\text{Min(P)}]^u$.
(c) *If we assume the condition* $L(\cdot, A) \leq^w_{\mathbb{R}^m_+}$ Max(D)*, then we have* Max(D) $=$ Min(P).

5. Conclusions

In this paper, we have given Karush-Kuhn-Tucker sufficiency criteria for a feasible point to be a global minimizer of nonlinear multi-objective programming problems by introducing the concept of underestimator for Lagrangian of the nonlinear multi-objective optimization problem. We also presented duality results for nonlinear multi-objective programming problems in terms of underestimator in the framework of set optimization problem. Moreover, we have given new definitions of bi-conjugate of vector-valued function and investigated convexity property of bi-conjugate map in the framework of set optimization problem.

Since for a given vector-valued function, its conjugate map is set-valued map (see also [5, 6]), we think that set optimization problem plays a significant role to derive duality theory for multi-objective programming problems. Especially, in Corollary 3.11/4.3/4.6, the assumption that "L^{**} is a singleton" is very strong. To relax the above condition, we have to introduce the concept of underestimator for set-valued map and it will be the subject of forthcoming research.

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