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A RELAXED HYBRID STEEPEST DESCENT METHOD FOR SOLVING STRONGLY CONVEX MINIMIZATION PROBLEM VIA CONJUGATE GRADIENT DIRECTION

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Abstract. In this paper, we consider the strongly convex minimization problem in a Hilbert space. We propose a coupling iterative scheme of relaxed steepest descent method and conjugate gradient direction method for solving the considered problem. We prove that a sequence generated by the proposed algorithm converges strongly to the unique solution of the problem provided that some suitable assumptions are guaranteed.

1. Introduction

Let H be a real Hilbert space, $f : H \to \mathbb{R}$ be a real-valued strongly convex differentiable function and $C \subset H$ be a nonempty closed convex subset. The classical strongly convex minimization problem is to consider the following problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C. \end{array} \tag{1.1}$$

By the necessary and sufficient optimality conditions for convex constrained minimization, it is well known that the optimal solution to (1.1) is nothing else than the solution to the variational inequality problem of finding the point x^* in C such that

$$\langle \nabla f(x^*), v - x^* \rangle \geq 0, \tag{1.2}$$

for all $v \in C$.

In order to find the solution of (1.2), Goldstein [5] proposed the so-called project gradient method which started by given $x_1 \in C$ and constructed a sequence (x_n) by

$$x_{n+1} = P_C(x_n - \mu \nabla f(x_n)),$$

for all $n \geq 1$, where P_C is the metric projection operator onto the subset C and μ is a positive stepsize. However, in many practical situation, the metric projection may not have a closed form expression and the cost of computation can be very high. To avoid this limitation, the so-called hybrid steepest descent method has been

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proposed. Actually, in 2001 Yamada [16] proposed the method in the following form: given $x_1 \in H$ and define a sequence (x_n) by

$$x_{n+1} = T(x_n - \mu\alpha_n \nabla f(x_n))$$

for all $n \geq 1$, where $T : H \to H$ is a nonexpansive operator in which

$$C = \mathrm{Fix}(T) := \{x \in H : T(x) = x\},$$

the stepsize $\mu > 0$ and the stepsize (α_n) is nonincreasing sequence in $(0, 1]$. After that, many authors have improved both theoritical and numerical behaviour of the hybrid steepest descent method, see for instance [3, 4, 6, 8, 9, 11, 12, 13] and references therein.

A direction to improve numerical performance of the hybrid steepest descent method is due to the work of Iiduka and Yamada [9] which proposed the so-called hybrid conjugate gradient method by generating an iterative sequence (x_n) in the following:

$$x_{n+1} := T(x_n + \alpha_n d_n),$$

where $d_1 := -\nabla f(x_1)$ and for all $n \geq 2$,

$$d_n := -\nabla f(x_n) + \beta_n d_{n-1},$$

and (β_n) is a nonnegative step-size sequence. Under the boundedness of the generated sequence $(\nabla f(x_n))$, it has been proved that the sequence (x_n) converges to the unique solution to (1.2).

The aim of this paper is to present an iterative method by using the idea of the hybrid conjugate gradient descent method for minimizing the strongly convex minimization problem over the fixed-point constraint of the form:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \cap \mathrm{Fix}(T), \end{array} \tag{1.3}$$

where the following assumptions are assumed:

(A1) $f : H \to \mathbb{R}$ is an α-strongly convex Fréchet differentiable with L-Lipschitz continuous gradient.

(A2) $T : H \to H$ is a nonexpansive operator with $\mathrm{Fix}(T) \neq \emptyset$.

(A3) $X \subseteq H$ is a nonempty closed convex and bounded subset in which $\mathrm{Fix}(T) \cap X \neq \emptyset$.

Under some imposed control conditions on step-size sequences and a parameter, we prove the convergence of the generated sequence to the unique solution to the considered problem without assuming the boundedness of the generated sequence $(\nabla f(x_n))$.

2. Preliminaries

We summarize some useful notations, definitions and properties which we will utilize later. For further details, the reader can consult the well-known books, for instance, [1, 2, 14].

Let H be a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$.

Let $f : H \to \mathbb{R}$ be a real-valued function, and $x, v \in H$. The directional derivative of f at x in the direction v is given by

$$f'(x, v) := \lim_{h\to 0} \frac{f(x + hv) - f(x)}{h},$$

provided that the limit exists. We say that the function f is Gateaux differentiable at x if the directional derivative $f'(x, v)$ exists for all $v \in H$ and there is $g \in H$ such that

$$f'(x, v) = \langle g, v\rangle.$$

We call g by Gateaux derivative or Gateaux gradient of f at x and denote it by $\nabla f(x)$.

We say that the function f is Fréchet differentiable at $x \in H$ if there is $y \in H$ such that

$$\lim_{\|h\|\to 0} \frac{f(x + h) - f(x) - \langle y, h\rangle}{\|h\|} = 0,$$

we call y by Fréchet derivative or gradient of f at x and denote it by $D_f(x)$.

Note that if the function f is Fréchet differentiable at $x \in H$, then f is Gateaux differentiable at x, and $\nabla f(x) = D_f(x)$.

A real-valued function $f : H \to \mathbb{R}$ is said to be convex if

$$f((1 - \lambda)x + \lambda y) \le (1 - \lambda)f(x)) + \lambda f(y),$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. We say that f is α-strongly convex with a parameter $\alpha > 0$ if

$$f(1 - \lambda)x + \lambda y) \le (1 - \lambda)f(x) + \lambda f(y) - \frac{1}{2}\alpha\lambda(1 - \lambda)\|x - y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It can be noted that the gradient of a α-strongly convex function f is α-strongly monotone, that is

$$\langle\nabla f(x) - \nabla f(y), x - y\rangle \ge \alpha\|x - y\|^2,$$

for all $x, y \in H$.

An operator $T : H \to H$ is said to be L-Lipschitz continuous with $L > 0$ if

$$||Tx - Ty|| \le L||x - y||,$$

for all $x, y \in H$. We call an operator T nonexpansive if it is 1-Lipschitz continuous, that is

$$||Tx - Ty|| \le ||x - y||,$$

for all $x, y \in H$. It is worth noting that the set of all fixed points of a nonexpansive operator is a closed and convex set.

The following theorem states an important property of a nonexpansive operator with a nonempty fixed-point set.

Proposition 2.1 ([14])**.** *Let $T : H \to H$ be a nonexpansive operator with* $\mathrm{Fix}(T) \neq \emptyset$*. If (x_n) is a sequence in H and $x \in H$ such that $x_n \rightharpoonup x$ and $\|T(x_n) - x_n\| \to 0$, then we have $x \in \mathrm{Fix}(T)$.*

Let X be a nonempty subset of H and $x \in H$. If there exists a point $y \in X$ such that

$$||x - y|| \leq ||x - z||,$$

for all $z \in X$, then the point y is called the metric projection of x onto X, and it is denoted by $P_X x$. Note that if the set X is a nonempty closed and convex set, then for every $x \in H$, there is the unique metric projection $P_X x$.

The following proposition will play an important role in the convergence analysis.

Proposition 2.2 ([16, Lemma 3.1(b)])**.** *Let $T : H \to H$ be a nonexpansive operator and $f : H \to \mathbb{R}$ be a α-strongly convex Fréchet differentiable with its gradient $\nabla f : H \to H$ which is L-Lipschitz continuous. Define the operator $T^\lambda : H \to H$ by $T^\lambda(x) := T(x - \mu\lambda\nabla f(x))$ for all $x \in H$. If the parameters $\mu \in (0, \frac{2\alpha}{L^2})$ and $\lambda \in [0, 1]$, then for every $x, y \in H$, we have*

$$||T^\lambda(x) - T^\lambda(y)|| \leq (1 - \lambda\tau)||x - y||,$$

where $\tau := 1 - \sqrt{1 - \mu(2\alpha - \mu L^2)} \in (0, 1]$.

In order to prove our main theorem, we need the following lemma which can be found in [10, Lemma 3.1].

Proposition 2.3. *Let (a_n) be a sequence of nonnegative real numbers such that there exists a subsequence (a_{n_j}) of (a_n) with $a_{n_j} < a_{n_{j+1}}$ for all $j \in \mathbb{N}$, and define the set of indexes $(\nu(n))_{n \geq n_0}$ by*

$$\nu(n) = \max\left\{k \in [n_0, n] : a_k < a_{k+1}\right\}.$$

Then the following properties hold:

(i) *$(\nu(n))_{n \geq n_0}$ is nondecreasing.*
(ii) $\lim_{n\to\infty} \nu(n) = \infty$.
(iii) *$a_{\nu(n)} \leq a_{\nu(n)+1}$ and $a_n \leq a_{\nu(n)+1}$ for all $n \geq n_0$.*

Another important tool for proving our main result is stated in the next proposition which can be found in [15, Lemma 2.5].

Proposition 2.4. *Let (a_n) be a positive real sequence, (t_n) be a real sequence and (α_n) be a real sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n t_n,$$

for all $n \geq 1$. If $\limsup_{n\to\infty} t_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Proposition 2.5 ([14])**.** *If $x, y \in H$ and $\alpha \in \mathbb{R}$, then it holds that*

$$||\alpha x + (1 - \alpha)y||^2 + \alpha(1 - \alpha)||x - y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2.$$

3. The proposed method and its convergence

In this section we will present an iterative method for solving the considered minimization problem and subsequently prove its convergence result.

Algorithm 1: Relaxed hybrid steepest descent method

Initialization: Select a starting point $x_1 \in H$, the parameters $\mu \in (0, 2\alpha/L^2)$, $(\lambda_n) \subset (0,1)$, $(\alpha_n) \subset (0,1]$ and $(\beta_n) \subset [0, +\infty)$. Set $d_1 := -\nabla f(x_1)$.

Step 1: For current iterates $x_n \in H$ and $d_n \in H$, compute $x_{n+1} \in H$ by

$$x_{n+1} := P_X((1-\lambda_n)x_n + \lambda_n T(x_n + \mu\alpha_n d_n)). \tag{3.1}$$

Step 2: Define the search direction $d_{n+1} \in H$ as

$$d_{n+1} := -\nabla f(x_{n+1}) + \beta_{n+1} d_n. \tag{3.2}$$

Update $n := n+1$ and go to **Step 1**.

Remark 3.1. (i) Since the constrained set X is bounded, the iterate $P_X((1-\lambda_n)x_n + \lambda_n T(x_n + \mu\alpha_n d_n))$ is bounded for all $n \in \mathbb{N}$. This yields that the sequence (x_n) is also bounded.

(ii) It is worth noting that the proposed method differs from the hybrid conjugated gradient [9]. Actually, if we set the relaxation parameter $\lambda_n = 1$ for all $n \in \mathbb{N}$, the iterate x_{n+1} is in the form $x_{n+1} := P_X(T(x_n + \mu\alpha_n d_n))$ which cannot reduce to the hybrid conjugate gradient method since the constrained set X cannot be the whole Hilbert space H.

Now, we are in a position to present our main theorem.

Theorem 3.2. *Let the sequence (x_n) be given by Algorithm 1. Suppose that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$. Then the sequence (x_n) converges strongly to the unique solution of the problem (1.3).*

Proof. Let $n \geq 1$ and $x^* \in X \cap \mathrm{Fix}(T)$ be the unique solution of the problem (1.3). We first note that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||P_X((1-\lambda_n)x_n + \lambda_n T(x_n + \mu\alpha_n d_n)) - x^*||^2 \\ &= ||P_X((1-\lambda_n)x_n + \lambda_n T(x_n + \mu\alpha_n d_n)) - P_X(x^*)||^2 \\ &\leq ||(1-\lambda_n)x_n + \lambda_n T(x_n + \mu\alpha_n d_n) - x^*||^2 \\ &= ||(1-\lambda_n)(x_n - x^*) + \lambda_n(T(x_n + \mu\alpha_n d_n) - x^*)||^2, \end{aligned}$$

Proposition 3.17 yields that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||(1-\lambda_n)(x_n - x^*) + \lambda_n(T(x_n + \mu\alpha_n d_n) - x^*)||^2 \\ &= (1-\lambda_n)||x_n - x^*||^2 + \lambda_n||T(x_n + \mu\alpha_n d_n) - x^*||^2 \\ &\quad -\lambda_n(1-\lambda_n)||(T(x_n + \mu\alpha_n d_n) - x^*) - (x_n - x^*)||^2 \\ &= (1-\lambda_n)||x_n - x^*||^2 + \lambda_n||T(x_n + \mu\alpha_n d_n) - x^*||^2 \\ &\quad -\lambda_n(1-\lambda_n)||T(x_n + \mu\alpha_n d_n) - x_n||^2. \end{aligned} \tag{3.3}$$

Next, we will show that the sequences $(\nabla f(x_n))$ and (d_n) are bounded. Since (x_n) is bounded, there is an $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$. Let us consider

$$\begin{aligned} ||\nabla f(x_n)|| &\leq ||\nabla f(x_n) - \nabla f(x^*)|| + ||\nabla f(x^*)|| \\ &\leq L||x_n - x^*|| + ||\nabla f(x^*)|| \\ &\leq L(||x_n|| + ||x^*||) + ||\nabla f(x^*)|| \\ &\leq LM + L||x^*|| + ||\nabla f(x^*)||, \end{aligned}$$

which means that $(\nabla f(x_n))$ is a bounded sequence. Furthermore, since we know that the stepsize $(\beta_n) \subset [0, +\infty)$ satisfies $\lim_{n\to\infty} \beta_n = 0$, there exists $m_1 \in \mathbb{N}$ such that $\beta_n \leq \frac{1}{2}$ for all $n \geq m_1$. Denote $K_1 := \sup_{n\geq 1} ||\nabla f(x_n)|| < \infty$ and $K_2 := \max\{K_1, ||d_{m_1}||\}$, we see that $2K_2 \geq ||d_{m_1}||$. Moreover, by the definition of (d_n), we note that

$$\begin{aligned} ||d_{n+1}|| &\leq ||-\nabla f(x_{n+1})|| + ||\beta_{n+1} d_n|| \\ &\leq ||\nabla f(x_{n+1})|| + \beta_{n+1}||d_n|| \\ &\leq ||\nabla f(x_{n+1})|| + \frac{1}{2}||d_n|| \\ &\leq K_1 + \frac{1}{2}||d_n|| \leq K_2 + \frac{1}{2}||d_n||, \end{aligned} \tag{3.4}$$

We claim that $||d_n|| \leq 2K_2$ for all $n \geq m_1$. Indeed, if $n = m_1$, then we have $||d_n|| = ||d_{m_1}|| \leq 2K_2$. For $n > m_1$, we assume that $||d_n|| \leq 2K_2$. By using the inequality (3.4), we note that

$$||d_{n+1}|| \leq K_2 + \frac{1}{2}||d_n|| \leq K_2 + \frac{1}{2}(2K_2) \leq 2K_2,$$

which implies that $||d_n|| \leq 2K_2$ for all $n \geq m_1$. Now, denote $K^* := \max\{||d_1||, ||d_2||, \ldots, ||d_{m_1-1}||, 2K_2\}$, we have $||d_n|| \leq K^*$ for all $n \in \mathbb{N}$, which means that the sequence (d_n) is bounded.

Next, by using the nonexpansiveness of T, we note that

$$\|T(x_n + \mu\alpha_n d_n) - T(x^*)\| \leq \|x_n + \mu\alpha_n d_n - x^*\| \leq \|x_n - x^*\| + \mu\alpha_n\|d_n\|,$$

which is

$$\begin{aligned} \|T(x_n + \mu\alpha_n d_n) - T(x^*)\|^2 &\leq (\|x_n - x^*\| + \mu\alpha_n\|d_n\|)^2 \\ &= \|x_n - x^*\|^2 + 2\mu\alpha_n\|d_n\|\|x_n - x^*\| \\ &\quad + \mu^2\alpha_n^2\|d_n\|^2. \end{aligned} \tag{3.5}$$

By substituting the inequality (3.5) in the inequality (3.3), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\lambda_n)||x_n - x^*||^2 + \lambda_n||x_n - x^*||^2 \\ &\quad + 2\mu\lambda_n\alpha_n||d_n||||x_n - x^*|| + \lambda_n\mu^2\alpha_n^2||d_n||^2 \\ &\quad - \lambda_n(1-\lambda_n)||T(x_n + \mu\alpha_n d_n) - x_n||^2 \\ &= ||x_n - x^*||^2 + 2\mu\lambda_n\alpha_n||d_n||||x_n - x^*||^2 + \mu^2\lambda_n\alpha_n^2||d_n||^2 \\ &\quad - \lambda_n(1-\lambda_n)||T(x_n + \mu\alpha_n d_n) - x_n||^2. \end{aligned} \tag{3.6}$$

By setting

$$\varphi_n := 2\mu\lambda_n\alpha_n||d_n||||x_n - x^*||^2 + \mu^2\lambda_n{\alpha_n}^2||d_n||^2,$$

for all $n \in \mathbb{N}$, we have

$$\|x_{n+1} - x^*\|^2 \leq ||x_n - x^*||^2 + \varphi_n - \lambda_n(1-\lambda_n)||T(x_n + \mu\alpha_n d_n) - x_n||^2. \tag{3.7}$$

Note that

$$\begin{aligned}
\limsup_{n\to\infty}[2\mu\lambda_n\alpha_n||d_n||||x_n - x^*||^2 \quad &+ \quad \mu^2\lambda_n\alpha_n^2||d_n||^2] \\
&= \limsup_{n\to\infty} 2\mu\lambda_n\alpha_n||d_n||||x_n - x^*||^2 \\
&\quad + \limsup_{n\to\infty} \mu^2\lambda_n\alpha_n^2||d_n||^2 \\
&= 2\mu \limsup_{n\to\infty} \lambda_n\alpha_n||d_n||||x_n - x^*||^2 \\
&\quad + \mu^2 \limsup_{n\to\infty} \lambda_n\alpha_n^2||d_n||^2.
\end{aligned}$$

Since $\limsup_{n\to\infty}\lambda_n < 1$, the boundednesses of the sequences (d_n), (x_n), we have

$$\lim_{n\to\infty} \varphi_n = 0. \tag{3.8}$$

By setting $z_n := x_n + \mu\alpha_n d_n$, the fact that $\|x + y\| \leq \|x\|^2 + 2\langle y, x + y\rangle$ for all $x, y \in H$, and using Proposition 2.2, we note that

$$\begin{aligned}
\|T(x_n \quad &+ \quad \mu\alpha_n d_n) - T(x^*)\|^2 \\
&\leq \|x_n + \mu\alpha_n(-\nabla f(x_n) + \beta_n d_{n-1}) - x^*\|^2 \\
&= \|x_n - \mu\alpha_n\nabla f(x_n) + \mu\alpha_n\beta_n d_{n-1} - x^* + \mu\alpha_n\nabla f(x^*) - \mu\alpha_n\nabla f(x^*)\|^2 \\
&= \|(x_n - \mu\alpha_n\nabla f(x_n)) - (x^* - \mu\alpha_n\nabla f(x^*)) + \mu\alpha_n\beta_n d_{n-1} - \mu\alpha_n\nabla f(x^*)\|^2 \\
&\leq \|(x_n - \mu\alpha_n\nabla f(x_n)) - (x^* - \mu\alpha_n\nabla f(x^*))\|^2 \\
&\quad +2\langle x_n - \mu\alpha_n\nabla f(x_n) - x^* + \mu\alpha_n\beta_n d_{n-1}, \mu\alpha_n\beta_n d_{n-1} - \mu\alpha_n\nabla f(x_*)\rangle \\
&\leq (1-\tau\alpha_n)^2\|x_n - x^*\|^2 \\
&\quad +2\alpha_n\langle x_n + \mu\alpha_n(-\nabla f(x_n) + \beta_n d_{n-1}) - x^*, \mu\beta_n d_{n-1} - \mu\nabla f(x^*)\rangle \\
&\leq (1-\tau\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle x_n + \mu\alpha_n d_n - x^*, \mu\beta_n d_{n-1} - \mu\nabla f(x^*)\rangle \\
&= (1-\tau\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\mu\beta_n\langle x_n + \mu\alpha_n d_n - x^*, d_{n-1}\rangle \\
&\quad +2\alpha_n\mu\langle x_n + \mu\alpha_n d_n - x^*, -\nabla f(x^*)\rangle \\
&= (1-\tau\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\mu\beta_n\langle z_n - x^*, d_{n-1}\rangle \\
&\quad +2\alpha_n\mu\langle x_n - x^*, -\nabla f(x^*)\rangle + 2\alpha_n\mu\langle\mu\alpha_n d_n, -\nabla f(x^*)\rangle \\
&= (1-\tau\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\mu\beta_n\langle z_n - x^*, d_{n-1}\rangle \\
&\quad +2\alpha_n\mu\langle x^* - x_n, \nabla f(x^*)\rangle + 2\alpha_n^2\mu^2\langle d_n, -\nabla f(x^*)\rangle \\
&= (1-\tau\alpha_n)\|x_n - x^*\|^2 + 2\tau\alpha_n\xi_n,
\end{aligned}$$

where

$$\xi_n := \left\{\frac{\mu\beta_n}{\tau}\langle z_n - x^*, d_{n-1}\rangle + \frac{\mu}{\tau}\langle x^* - x_n, \nabla f(x^*)\rangle + \frac{\mu^2\alpha_n}{\tau}\langle d_n, -\nabla f(x^*)\rangle\right\}.$$

Substituting the above inequality in (3.3), we get

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq (1-\lambda_n)||x_n - x^*||^2 + \lambda_n(1-\tau\alpha_n)\|x_n - x^*\|^2 \\ &\quad +2\tau\alpha_n\lambda_n\xi_n - \lambda_n(1-\lambda_n)||T(x_n+\mu\alpha_n d_n) - x_n||^2. \end{aligned} \tag{3.9}$$

Since $\lambda_n \in (0,1)$, we have $0 < \lambda_n(1-\lambda_n) < \lambda_n$, which together with the inequalities (3.9) yield that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq (1-\lambda_n)||x_n - x^*||^2 + \lambda_n(1-\tau\alpha_n)\|x_n - x^*\|^2 \\ &\quad +2\tau\lambda_n\alpha_n\xi_n \\ &= (1-\lambda_n+\lambda_n-\tau\alpha_n\lambda_n)||x_n - x^*||^2 + 2\tau\lambda_n\xi_n\alpha_n \\ &= (1-\tau\alpha_n\lambda_n)||x_n - x^*||^2 + 2\xi_n(\tau\alpha_n\lambda_n). \end{aligned} \tag{3.10}$$

We claim that $\tau\alpha_n\lambda_n \in (0,1)$ and $\sum_{n=1}^{\infty}\tau\alpha_n\lambda_n = +\infty$. Now, since $\tau \in (0,1]$, $(\alpha_n) \subset (0,1)$ and $(\lambda_n) \subset (0,1)$, we have $\tau\alpha_n\lambda_n \in (0,1)$ for all $n \geq 1$. Furthermore, since $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n = \lambda$, where $\lambda \in (0,1)$. Thus, for all $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\lambda_n \geq \lambda + \epsilon > \epsilon$ for all $n \geq n_0$.

Note that

$$\sum_{n=n_0}^{\infty}\tau\alpha_n\lambda_n = \tau\sum_{n=n_0}^{\infty}\alpha_n\lambda_n > \tau\sum_{n=n_0}^{\infty}\alpha_n\epsilon = \tau\epsilon\sum_{n=n_0}^{\infty}\alpha_n.$$

Since $\sum_{n=n_0}^{\infty}\alpha_n = +\infty$, we have $\tau\epsilon\sum_{n=n_0}^{\infty}\alpha_n = +\infty$, which implies that $\sum_{n=n_0}^{\infty}\tau\alpha_n\lambda_n = +\infty$.

Denote $a_n := \|x_n - x^*\|^2$ for all $n \geq 1$, we will show that $\lim_{n\to\infty}||x_n - x^*||^2 = 0$ by dividing the proof into two cases.

Case I: suppose that there is $n_0 \geq 1$ such that $a_{n+1} \leq a_n$ for all $n \geq n_0$. In this situation, the sequence (a_n) converges and let $\lim_{n\to\infty} a_n = r$. We will show that $\lim_{n\to\infty}||T(x_n+\mu\alpha_n d_n) - x_n|| = 0$. Now, by using the inequalities (3.6) and (3.8), we note that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\lambda_n(1-\lambda_n)||T(x_n+\mu\alpha_n d_n) - x_n||^2 \\ &\leq \limsup_{n\to\infty}(a_n - a_{n+1} + \varphi_n) \\ &= \lim_{n\to\infty} a_n - \lim_{n\to\infty} a_{n+1} + \lim_{n\to\infty}\varphi_n = r - r + 0 = 0, \end{aligned}$$

that is

$$\limsup_{n\to\infty}\lambda_n(1-\lambda_n)||T(x_n+\mu\alpha_n d_n) - x_n||^2 = 0,$$

which yields that

$$\lim_{n\to\infty}||T(x_n+\mu\alpha_n d_n) - x_n|| = 0. \tag{3.11}$$

Since (x_n) is bounded, the sequence $(\langle x^* - x_n, \nabla f(x^*)\rangle)$ is also bounded. Thus, there exists a subsequence (x_{n_i}) of (x_n) such that

$$\limsup_{n\to\infty}\langle x^* - x_n, \nabla f(x^*)\rangle = \lim_{i\to\infty}\langle x^* - x_{n_i}, \nabla f(x^*)\rangle.$$

Since (x_{n_i}) is bounded, there are $\widehat{x} \in H$ and a subsequence $(x_{n_{i_j}})$ such that $x_{n_{i_j}} \rightharpoonup \widehat{x}$. We will show that $\widehat{x} \in \mathrm{Fix}(T) \cap X$. Let us note from (3) that

$$\begin{aligned}
\lim_{j\to\infty} \|T(x_{n_{i_j}}) - x_{n_{i_j}}\| &= \lim_{n\to\infty} \|T(x_{n_{i_j}}) - T(x_{n_{i_j}} + \mu\alpha_{n_{i_j}} d_{n_{i_j}}) \\
&\quad + T(x_{n_{i_j}} + \mu\alpha_{n_{i_j}} d_{n_{i_j}}) - x_{n_{i_j}}\| \\
&= \lim_{j\to\infty} \|T(x_{n_{i_j}}) - T(x_{n_{i_j}} + \mu\alpha_{n_{i_j}} d_{n_{i_j}})\| \\
&\quad + \lim_{j\to\infty} \|T(x_{n_{i_j}} + \mu\alpha_{n_{i_j}} d_{n_{i_j}}) - x_{n_{i_j}}\| \\
&= \lim_{j\to\infty} \|T(x_{n_{i_j}}) - T(x_{n_{i_j}} + \mu\alpha_{n_{i_j}} d_{n_{i_j}})\| \\
&\leq \lim_{j\to\infty} \|x_{n_{i_j}} - (x_{n_{i_j}} + \mu\alpha_{n_{i_j}} d_{n_{i_j}})\| \\
&= \lim_{j\to\infty} \|\mu\alpha_{n_{i_j}} d_{n_{i_j}}\| \\
&= \mu \lim_{j\to\infty} \alpha_{n_{i_j}} \|d_{n_{i_j}}\|.
\end{aligned}$$

By using the boundedness of (d_n), we obtain that

$$\lim_{j\to\infty} \|T(x_{n_{i_j}}) - x_{n_{i_j}}\| = 0. \tag{3.12}$$

Now the fact that $x_{n_{i_j}} \rightharpoonup \widehat{x}$, the inequality (3.12) and Proposition 2.1, we get that $\widehat{x} \in \mathrm{Fix}(T) \cap X$.

Since x^* is the unique solution of the considered problem, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle x^* - x_n, \nabla f(x^*)\rangle &= \lim_{i\to\infty} \langle x^* - x_{n_i}, \nabla f(x^*)\rangle \\
&= \lim_{j\to\infty} \langle x^* - x_{n_{i_j}}, \nabla f(x^*)\rangle \\
&= \langle x^* - \widehat{x}, \nabla f(x^*)\rangle \\
&\leq 0. \qquad (3.13)
\end{aligned}$$

By using the assumptions $\lim_{n\to\infty} \alpha_n = 0, \lim_{n\to\infty} \beta_n = 0$, and the facts that (d_n) and (z_n) are bounded, we have

$$\limsup_{n\to\infty} \frac{\mu\beta_n}{\tau} \langle z^n - x^*, d^{n-1}\rangle \leq \limsup_{n\to\infty} \frac{\mu\beta_n}{\tau} ||z_n - x^*|| ||d_{n-1}|| = 0, \tag{3.14}$$

and then

$$\limsup_{n\to\infty} \frac{\mu^2\alpha_n}{\tau} \langle d^n, -\nabla f(x^*)\rangle \leq \limsup_{n\to\infty} \frac{\mu^2\alpha_n}{\tau} ||d_n|| ||\nabla f(x^*)|| = 0. \tag{3.15}$$

Invoking the inequalities (3.13), (3.14) and (3.15), we have

$$\limsup_{n\to\infty} \left(\frac{\mu\beta_n}{\tau} \langle z^n - x^*, d^{n-1}\rangle + \frac{\mu}{\tau} \langle x^* - x^n, \nabla f(x^*)\rangle + \frac{\mu^2\alpha_n}{\tau} \langle d^n, -\nabla f(x^*)\rangle \right) \leq 0.$$

Thus, we have $\limsup_{n\to\infty} 2\xi_n = 2 \limsup_{n\to\infty} \xi_n \leq 0$, Hence, Proposition 2.4 and the inequality (3.10), we conclude that $\lim_{n\to\infty} ||x_n - x^*||^2 = 0$.

Case II: suppose that there exists a subsequence (a_{n_k}) of (a_n) such that $a_{n_k} < a_{n_k+1}$ for all $k \geq 1$ and let $\{\nu(n)\}_{n=1}^{\infty}$ be defined as in Proposition 2.3. Then, for all $n \geq n_0$, we have

$$a_{\nu(n)} \leq a_{\nu(n)+1}, \tag{3.16}$$

and

$$a_n \leq a_{\nu(n)+1}. \tag{3.17}$$

By utilizing the inequality (3.7), we have

$$a_{\nu(n)+1} - a_{\nu(n)} \leq -\lambda_{\nu(n)}(1-\lambda_{\nu(n)})||T(x_{\nu(n)} + \mu\alpha_{\nu(n)}d_{\nu(n)}) - x_{\nu(n)}||^2 + \varphi_{\nu(n)}. \tag{3.18}$$

It follows from the inequalities (3.16) and (3.18) that

$$0 \leq -\lambda_{\nu(n)}(1-\lambda_{\nu(n)})||T(x_{\nu(n)} + \mu\alpha_{\nu(n)}d_{\nu(n)}) - x_{\nu(n)}||^2 + \varphi_{\nu(n)}.$$

Thus, we obtain that

$$\lambda_{\nu(n)}(1-\lambda_{\nu(n)})||T(x_{\nu(n)} + \mu\alpha_{\nu(n)}d_{\nu(n)}) - x_{\nu(n)}||^2 \leq \varphi_{\nu(n)}.$$

Since $\lim_{n\to\infty} \varphi_{\nu(n)} = \lim_{n\to\infty} \varphi_n = 0$, we have

$$\lim_{n\to\infty} \lambda_{\nu(n)}(1-\lambda_{\nu(n)})||T(x_{\nu(n)} + \mu\alpha_{\nu(n)}d_{\nu(n)}) - x_{\nu(n)}||^2 \leq 0,$$

and then

$$\lim_{n\to\infty} ||T(x_{\nu(n)} + \mu\alpha_{\nu(n)}d_{\nu(n)}) - x_{\nu(n)}|| = 0.$$

Now, let $(x_{\nu(n_i)})$ be a subsequence of $(x_{\nu(n)})$ such that

$$\limsup_{n\to\infty} \langle x^* - x_{\nu(n)}, \nabla f(x^*)\rangle = \lim_{i\to\infty} \langle x^* - x_{\nu(n_i)}, \nabla f(x^*)\rangle.$$

Since $(x_{\nu(n_i)})$ is a bounded sequence, there are $\widehat{x} \in H$ and a subsequence $(x_{\nu(n_{i_j})})$ in which $x_{\nu(n_{i_j})} \rightharpoonup \widehat{x}$. In a similar fashion to **Case I**, we also obtain that $\widehat{x} \in \mathrm{Fix}(T) \cap X$. Now, since x^* is the unique solution to the considered problem, we have

$$\begin{aligned} \limsup_{n\to\infty} \langle x^* - x_{\nu(n)}, \nabla f(x^*)\rangle &= \lim_{i\to\infty} \langle x^* - x_{\nu(n_i)}, \nabla f(x^*)\rangle \\ &= \lim_{j\to\infty} \langle x^* - x_{\nu(n_{i_j})}, \nabla f(x^*)\rangle \\ &= \langle x^* - \widehat{x}, \nabla f(x^*)\rangle \\ &\leq 0, \end{aligned} \tag{3.19}$$

and by using the same ideas of the inequalities (3.14) and (3.15), we obtain that

$$\limsup_{n\to\infty} 2\xi_{\nu(n)} \leq 0.$$

We note from the inequality (3.10) that

$$\begin{aligned} a_{\nu(n)+1} &\leq (1-\tau\alpha_{\nu(n)}\lambda_{\nu(n)})a_{\nu(n)} + 2\xi_{\nu(n)}(\tau\alpha_{\nu(n)}\lambda_{\nu(n)}) \\ &= a_{\nu(n)} - \tau\alpha_{\nu(n)}\lambda_n(a_{\nu(n)}) + 2\xi_{\nu(n)}(\tau\alpha_{\nu(n)}\lambda_{\nu(n)}) \\ &= a_{\nu(n)} + \tau\alpha_{\nu(n)}\lambda_{\nu(n)}(2\xi_{\nu(n)} - a_{\nu(n)}), \end{aligned}$$

and then

$$0 \leq a_{\nu(n)+1} - a_{\nu(n)} \leq \tau \alpha_{\nu(n)} \lambda_{\nu(n)} (2\xi_{\nu(n)} - a_{\nu(n)}).$$

Since $\tau \in (0,1], (a_{\nu(n)}) \subset (0,1)$ and $(\lambda_{\nu(n)}) \subset (0,1)$, we have $\tau a_{\nu(n)} \lambda_{\nu(n)} \in (0,1)$, thus $\tau a_{\nu(n)} \lambda_{\nu(n)} > 0$.

Now, we have

$$0 \leq \tau \alpha_{\nu(n)} \lambda_{\nu(n)} (2\xi_{\nu(n)} - a_{\nu(n)}),$$

which is

$$0 \leq 2\xi_{\nu(n)} - a_{\nu(n)},$$

which implies that $0 \leq a_{\nu(n)} \leq 2\xi_{\nu(n)}$. Since $\limsup_{n\to\infty} 2\xi_{\nu(n)} \leq 0$, we have $0 \leq \limsup_{n\to\infty} a_{\nu(n)} \leq \limsup_{n\to\infty} 2\xi_{\nu(n)} \leq 0$, and then $\limsup_{n\to\infty} a_{\nu(n)} = 0$. Thus, we obtain that $\lim_{n\to\infty} a_{\nu(n)} = 0$ and $\lim_{n\to\infty} a_{\nu(n)+1} - a_{\nu(n)} = 0$. By using the relation in (3.17), we have

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} a_n &\leq \limsup_{n\to\infty} a_{\nu(n)+1} \\ &= \limsup_{n\to\infty} [a_{\nu(n)+1} - a_{\nu(n)}] + \limsup_{n\to\infty} a_{\nu(n)} = 0 + 0 = 0, \end{aligned}$$

that is $\limsup_{n\to\infty} a_n = 0$, which implies that $\lim_{n\to\infty} a_n = 0$, which means that $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$. From these two cases, we conclude that the sequence (x_n) converges to the unique solution to the considered problem. □

4. Some extensions

In this section, we will consider the solving of the minimization problem over the finite intersection of fixed-point sets of nonexpansive operators and a simple closed convex and bounded constraint by applying the results obtained in the previous section.

Let us consider the problem of the form: of the form:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X \cap \bigcap_{i=1}^m \operatorname{Fix}(T_i), \end{array} \tag{4.1}$$

where the following assumptions are assumed:

- (A1) $f : H \to \mathbb{R}$ is an α-strongly convex Fréchet differentiable with L-Lipschitz continuous gradient.
- (A2) $T_i : H \to H, i = 1, \ldots, m$, are nonexpansive operators with $\operatorname{Fix}(T_i) \neq \emptyset$.
- (A3) $X \subseteq H$ is a nonempty closed convex and bounded subset in which $\bigcap_{i=1}^m \operatorname{Fix}(T_i) \cap X \neq \emptyset$.

The following algorithm allows us to compute the operators $T_i, i = 1, \ldots, m$ cyclically.

Algorithm 2: Relaxed cyclic hybrid steepest descent method

Initialization: Select a starting point $x_1 \in H$, the parameters $\mu \in (0, 2\alpha/L^2)$, $(\lambda_n) \subset (0,1)$, $(\alpha_n) \subset (0,1]$ and $(\beta_n) \subset [0, +\infty)$. Set $d_1 := -\nabla f(x_1)$.

Step 1: For current iterates $x_n \in H$ and $d_n \in H$, compute $x_{n+1} \in H$ by

$$x_{n+1} := P_X((1-\lambda_n)x_n + \lambda_n T_m T_{m-1} \cdots T_2 T_1(x_n + \mu\alpha_n d_n)). \tag{4.2}$$

Step 2: Define the search direction $d_{n+1} \in H$ as

$$d_{n+1} := -\nabla f(x_{n+1}) + \beta_{n+1} d_n.$$

Update $n := n+1$ and go to **Step 1**.

By applying Theorem 3.2 together with the fact that the composition of a finite numbers of nonexpansive operators is also nonexpansive, we immediately obtain the the following corollary.

Corollary 4.1. *Let the sequence (x_n) be given by Algorithm 2. Suppose that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$. Then the sequence (x_n) converges strongly to the unique solution of the problem (4.1).*

Remark 4.2. Algorithm 2 and the corresponding convergence result in Corollary 4.1 are different from the iterative method presented in [12]. In fact, in order to update the next iterate x_{n+1} in (4.2), we compute the convex combination of x_n and $T_m T_{m-1} \cdots T_2 T_1(x_n + \mu\alpha_n d_n)$, whereas in [12], the authors utilized the sum of $x_n + \mu\alpha_n d_n$ and $T_m T_{m-1} \cdots T_2 T_1(x_n + \mu\alpha_n d_n)$.

The next algorithm allows us to compute the operators $T_i, i = 1, \ldots, m$ simultaneously.

Algorithm 3: Relaxed simultaneous hybrid steepest descent method

Initialization: Select a starting point $x_1 \in H$, the parameters $\mu \in (0, 2\alpha/L^2)$, $(\lambda_n) \subset (0,1)$, $(\alpha_n) \subset (0,1]$, $(\beta_n) \subset [0, +\infty)$, and $(\omega_i)_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m \omega_i = 1$. Set $d_1 := -\nabla f(x_1)$.

Step 1: For current iterates $x_n \in H$ and $d_n \in H$, compute $x_{n+1} \in H$ by

$$x_{n+1} := P_X((1-\lambda_n)x_n + \lambda_n \sum_{i=1}^{m} \omega_i T_i(x_n + \mu\alpha_n d_n)). \tag{4.3}$$

Step 2: Define the search direction $d_{n+1} \in H$ as

$$d_{n+1} := -\nabla f(x_{n+1}) + \beta_{n+1} d_n.$$

Update $n := n+1$ and go to **Step 1**.

By applying Theorem 3.2 together with the fact that the convex combination of a finite numbers of nonexpansive operators is also nonexpansive, we immediately obtain the the following corollary.

Corollary 4.3. *Let the sequence (x_n) be given by Algorithm 3. Suppose that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$ and $0 < \liminf_{n\to\infty} \lambda_n \leq$*

$\limsup_{n\to\infty} \lambda_n < 1$. *Then the sequence* (x_n) *converges strongly to the unique solution of the problem (4.1).*

5. Conclusion

The object of this work was the solving of a strongly convex minimization problem over the intersection of fixed-point set of a nonexpansive operator and a nonempty closed convex bounded set. We associated to it the so-called relaxed hybrid steepest descent method. We proved strong convergence of the generated sequence of iterates to the unique solution to the considered problem.

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