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THE COMPLETENESS AND SEPARABILITY OF THE LORENTZ SPACES DEFINED BY THE SUGENO AND SHILKRET INTEGRALS

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Abstract. As a continuation of our research for the Lorentz spaces defined by the Choquet integral, this paper is devoted to the study of the completeness and separability for the Lorentz spaces defined by the Sugeno and Shilkret integrals in the framework of nonadditive measure theory. It is also shown that the Lorentz spaces defined by the Sugeno integral coincide with the space of all Sugeno integrable functions and the Lorentz spaces defined by the Shilkret integral coincide with the Lorentz spaces of weak type.

1. Introduction

The Lorentz spaces, introduced by G. G. Lorentz [?, ?], are generalizations of the L^p spaces and real interpolation spaces between the spaces L^1 and L^∞. In [?] the Lorentz spaces were defined by the Choquet integral in the framework of nonadditive measure theory and their completeness and separability were discussed in quite generality. In nonadditive measure theory, in addition to the Choquet integral, there are two more integrals that are important and frequently used. They are the Sugeno integral and the Shilkret integral. Accordingly, it is natural to investigate the Lorentz spaces defined by those nonlinear integrals. The purpose of the paper is thus to study the completeness and separability of the Lorentz spaces defined by the Sugeno and Shilkret integrals.

The paper is organized as follows. Section ?? sets up notation and terminology. It also contains a discussion of an equivalence relation in the space of all measurable functions on a measurable space $(X, \mathcal{A})$. In Section ??, given a nonadditive measure μ on $(X, \mathcal{A})$, the Lorentz space $\mathfrak{Su}^{p,q}(\mu)$ is defined by the Sugeno integral for every $0 < p < \infty$ and $0 < q < \infty$. It is shown that $\mathfrak{Su}^{p,q}(\mu) = \mathfrak{Su}(\mu)$, where $\mathfrak{Su}(\mu)$ is the space of all Sugeno integrable functions on X. In Section ?? the completeness and separability of the Lorentz spaces $\mathfrak{Su}^{p,q}(\mu)$ are discussed by using the Cauchy criterion for measurable functions already established in our previous paper [?]. Section ?? is devoted to the study of the Lorentz spaces $\mathfrak{Sh}^{p,q}(\mu)$ defined by the Shilkret integral. It is shown that $\mathfrak{Sh}^{p,q}(\mu) = \mathcal{L}^{p,\infty}(\mu)$, where $\mathcal{L}^{p,\infty}(\mu)$ is the Lorentz space of weak type discussed in [?]. The completeness of $\mathfrak{Sh}^{p,q}(\mu)$ is also discussed. Section ?? provides a summary of our results.

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2. Preliminaries

Throughout the paper, $(X, \mathcal{A})$ is a measurable space, that is, X is a nonempty set and $\mathcal{A}$ is a σ-field of subsets of X. Let $\mathbb{R}$ denote the set of the real numbers and $\mathbb{N}$ the set of the natural numbers. Let $\overline{\mathbb{R}} := [-\infty, \infty]$ be the set of the extended real numbers with usual total order and algebraic structure. Assume that $(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0$ since this proves to be convenient in measure and integration theory.

For any $a, b \in \overline{\mathbb{R}}$, let $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ and for any $f, g \colon X \to \overline{\mathbb{R}}$, let $(f \vee g)(x) := f(x) \vee g(x)$ and $(f \wedge g)(x) := f(x) \wedge g(x)$ for every $x \in X$. Let $\mathcal{F}_0(X)$ denote the set of all $\mathcal{A}$-measurable real-valued functions on X. Then $\mathcal{F}_0(X)$ is a real linear space with usual pointwise addition and scalar multiplication. For any $f, g \in \mathcal{F}_0(X)$, the notation $f \leq g$ means that $f(x) \leq g(x)$ for every $x \in X$. Let $\mathcal{F}_0^+(X) := \{f \in \mathcal{F}_0(X) \colon f \geq 0\}$. A function taking only a finite number of real numbers is called a *simple function*. Let $\mathcal{S}(X)$ denote the set of all $\mathcal{A}$-measurable simple functions on X.

For a sequence $\{a_n\}_{n\in\mathbb{N}} \subset \overline{\mathbb{R}}$ and $a \in \overline{\mathbb{R}}$, the notation $a_n \uparrow a$ means that $\{a_n\}_{n\in\mathbb{N}}$ is nondecreasing and $a_n \to a$, and $a_n \downarrow a$ means that $\{a_n\}_{n\in\mathbb{N}}$ is nonincreasing and $a_n \to a$. For a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$, the notation $A_n \uparrow A$ means that $\{A_n\}_{n\in\mathbb{N}}$ is nondecreasing and $A = \bigcup_{n=1}^\infty A_n$, and $A_n \downarrow A$ means that $\{A_n\}_{n\in\mathbb{N}}$ is nonincreasing and $A = \bigcap_{n=1}^\infty A_n$. The characteristic function of a set A, denoted by χ_A, is the function on X such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Given two sets A and B, let $A \triangle B := (A \setminus B) \cup (B \setminus A)$ and $A^c := X \setminus A$. Let 2^X denote the collection of all subsets of X.

2.1. **Nonadditive measures.** A *nonadditive measure* is a set function $\mu \colon \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$. This type of set function is also called a monotone measure [**?**], a capacity [**?**], or a fuzzy measure [**?**, **?**] in the literature.

Let $\mathcal{M}(X)$ denote the set of all nonadditive measures $\mu \colon \mathcal{A} \to [0, \infty]$. We say that μ is *order continuous* [**?**] if $\mu(A_n) \to 0$ whenever $A_n \downarrow \emptyset$, *conditionally continuous from above* if $\mu(A_n) \to \mu(A)$ whenever $A_n \downarrow A$ and $\mu(A_1) < \infty$, *continuous from above* if $\mu(A_n) \to \mu(A)$ whenever $A_n \downarrow A$, *continuous from below* if $\mu(A_n) \to \mu(A)$ whenever $A_n \uparrow A$, *continuous* if it is continuous from above and from below, and *null-continuous* [**?**] if $\mu(\bigcup_{n=1}^\infty N_n) = 0$ whenever $\{N_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ is nondecreasing and $\mu(N_n) = 0$ for every $n \in \mathbb{N}$. If μ is continuous from above, then it is order continuous. The conditional continuity from above follows from the continuity from above, but the converse does not hold even for the Lebesgue measure on the real line. If μ is continuous from below, then it is null-continuous.

Following the terminology used in [**?**], μ is called *weakly null-additive* if $\mu(A \cup B) = 0$ whenever $A, B \in \mathcal{A}$ and $\mu(A) = \mu(B) = 0$ and *null-additive* if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{A}$ and $\mu(B) = 0$, Furthermore, we say that μ satisfies the *pseudometric generating property* ((p.g.p.) for short) [**?**] if $\mu(A_n \cup B_n) \to 0$ whenever $A_n, B_n \in \mathcal{A}$ and $\mu(A_n) \vee \mu(B_n) \to 0$, and is *monotone autocontinuous from below* [**?**] if $\mu(A \setminus B_n) \to \mu(A)$ whenever $A, B_n \in \mathcal{A}$, $\mu(B_n) \to 0$, and $\{B_n\}_{n\in\mathbb{N}}$ is nonincreasing. It is easy to see that μ satisfies the (p.g.p.) if and only if for any $\varepsilon > 0$ there is $\delta > 0$

such that $\mu(A \cup B) < \varepsilon$ whenever $A, B \in \mathcal{A}$ and $\mu(A) \vee \mu(B) < \delta$. If μ satisfies the (p.g.p.), then it is weakly null-additive. If μ is monotone autocontinuous from below, then it is null-additive, hence weakly null-additive.

A nonadditive measure μ is called *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for every disjoint $A, B \in \mathcal{A}$, *relaxed subadditive* if there is a constant $K \geq 1$ such that $\mu(A \cup B) \leq K\{\mu(A) + \mu(B)\}$ for every disjoint $A, B \in \mathcal{A}$ (in this case μ is called *K-relaxed subadditive*). Every subadditive nonadditive measure is relaxed subadditive. If μ is relaxed subadditive, then it satisfies the (p.g.p.). See [**?**, **?**, **?**] for further information on nonadditive measures.

Remark 2.1. The relaxed subadditivity is also called the quasi-subadditivity according to the terminology used in metric space theory.

2.2. **The Sugeno and Shilkret integrals.** The Choquet integral [**?**, **?**] is one of important integrals that is widely used in nonadditive measure theory and its applications. In addition to the Choquet integral, the following nonlinear integrals are also important.

Let $\mu \in \mathcal{M}(X)$. The *Sugeno integral* [**?**, **?**] is defined by

$$\mathrm{Su}(\mu, f) := \sup_{t \in [0,\infty)} t \wedge \mu(\{f > t\})$$

for every $f \in \mathcal{F}_0^+(X)$ and the *Shilkret integral* [**?**, **?**] is defined by

$$\mathrm{Sh}(\mu, f) := \sup_{t \in [0,\infty)} t\mu(\{f > t\})$$

for every $f \in \mathcal{F}_0^+(X)$. In the above definitions the nonincreasing distribution function $\mu(\{f > t\})$ may be replaced with $\mu(\{f \geq t\})$ and the interval of the range in which the variable t moves may be replaced with $[0, \infty]$ or $(0, \infty)$ without changing the integral value.

The following elementary properties of the Sugeno and Shilkret integrals are easy to prove; see also [**?**] and [**?**]. Note that these integrals are not additive in general.

- Monotonicity: For any $f, g \in \mathcal{F}_0^+(X)$, if $f \leq g$, then $\mathrm{Su}(\mu, f) \leq \mathrm{Su}(\mu, g)$ and $\mathrm{Sh}(\mu, f) \leq \mathrm{Sh}(\mu, g)$.
- Generativity: For any $A \in \mathcal{A}$ and $c \geq 0$, it follows that $\mathrm{Su}(\mu, c\chi_A) = c \wedge \mu(A)$ and $\mathrm{Sh}(\mu, c\chi_A) = c\mu(A)$.
- Truncated subhomogeneousness: For any $f \in \mathcal{F}_0^+(X)$ and $c \geq 0$, it follows that $\mathrm{Su}(\mu, cf) \leq \max\{1, c\}\mathrm{Su}(\mu, f)$.
- Positive homogeneousness: For any $f \in \mathcal{F}_0^+(X)$ and $c \geq 0$, it follows that $\mathrm{Sh}(\mu, cf) = c\,\mathrm{Sh}(\mu, f)$.
- Elementariness: If $h \in \mathcal{S}(X)$ is represented by

$$h = \sum_{k=1}^{n} (c_k - c_{k-1})\chi_{A_k} = \bigvee_{k=1}^{n} c_k \chi_{A_k},$$

where $n \in \mathbb{N}$, $c_0 = 0 < c_1 < c_2 < \cdots < c_n < \infty$, and $A_1 \supset A_2 \supset \cdots \supset A_n$, then it follows that

$$\mathrm{Su}(\mu, h) = \bigvee_{k=1}^{n} c_k \wedge \mu(A_k) \quad \text{and} \quad \mathrm{Sh}(\mu, h) = \bigvee_{k=1}^{n} c_k \mu(A_k).$$

- Upper marginal continuity: For any $f \in \mathcal{F}_0^+(X)$, it follows that
$$\mathrm{Su}(\mu, f) = \sup_{r>0} \mathrm{Su}(\mu, f \wedge r) \quad \text{and} \quad \mathrm{Sh}(\mu, f) = \sup_{r>0} \mathrm{Sh}(\mu, f \wedge r).$$
- Measure-truncation: For any $f \in \mathcal{F}_0^+(X)$, it follows that
$$\mathrm{Su}(\mu, f) = \sup_{s>0} \mathrm{Su}(\mu \wedge s, f) \quad \text{and} \quad \mathrm{Sh}(\mu, f) = \sup_{s>0} \mathrm{Sh}(\mu \wedge s, f),$$
where $(\mu \wedge s)(A) := \mu(A) \wedge s$ for every $A \in \mathcal{A}$ and $s > 0$.
- Exponentiation: Let $0 < p < \infty$. For any $f \in \mathcal{F}_0^+(X)$ it follows that
$$\mathrm{Su}(\mu, f^p) = \mathrm{Su}(\mu^{1/p}, f)^p \quad \text{and} \quad \mathrm{Sh}(\mu, f^p) = \mathrm{Sh}(\mu^{1/p}, f)^p.$$
- Integrability: For any $f \in \mathcal{F}_0^+(X)$ and $\alpha \in [0, \infty]$, it follows that
$$\mathrm{Su}(\mu, f) \leq \alpha \vee \mu(\{f > \alpha\}).$$
Consequently, it follows that $\mathrm{Su}(\mu, f) < \infty$ if and only if there is $\alpha_0 \in [0, \infty)$ such that $\mu(\{f > \alpha_0\}) < \infty$.

The relaxed subadditivity of the Sugeno and Shilkret integrals can be characterized in the following way.

Proposition 2.2. *Let $\mu \in \mathcal{M}(X)$.*

(1) *The following assertions are equivalent.*
  (i) *μ is K-relaxed subadditive for some $K \geq 1$.*
  (ii) *For any $f, g \in \mathcal{F}_0^+(X)$, it follows that*
$$\mathrm{Su}(\mu, f + g) \leq K\{\mathrm{Su}(\mu, f) + \mathrm{Su}(\mu, g)\}.$$
In particular, the Sugeno integral is subadditive if and only if μ is subadditive.

(2) *If μ is K-relaxed subadditive, then for any $f, g \in \mathcal{F}_0^+(X)$ it follows that*
$$\mathrm{Sh}(\mu, f + g) \leq 2K\{\mathrm{Sh}(\mu, f) + \mathrm{Sh}(\mu, g)\}.$$

Proof. (1) (i)⇒(ii) Let $f, g \in \mathcal{F}_0^+(X)$. Let $a := \mathrm{Su}(\mu, f)$ and $b := \mathrm{Su}(\mu, g)$. We may assume that both a and b are finite. Let $\varepsilon > 0$. If it were true that $\mu(\{f > a+\varepsilon\}) > a$ then we would have

$$a = \mathrm{Su}(\mu, f) \geq (a + \varepsilon) \wedge \mu(\{f > a + \varepsilon\}) > a,$$

which is impossible. It thus follows that $\mu(\{f > a + \varepsilon\}) \leq a$. Similarly, we have $\mu(\{f > b + \varepsilon\}) \leq b$. Then the K-relaxed subadditivity of μ yields

$$\mu(\{f + g > a + b + 2\varepsilon\}) \leq \mu(\{f > a + \varepsilon\} \cup \{g > b + \varepsilon\}) \leq K(a + b),$$

so that

$$\mathrm{Su}(\mu, f + g) \leq (a + b + 2\varepsilon) \vee \mu(\{f + g > a + b + 2\varepsilon\})$$

$$\leq (a+b+2\varepsilon) \vee \big\{K(a+b)\big\}.$$

Letting $\varepsilon \downarrow 0$ implies the desired inequality.

(ii)⇒(i) Let $A, B \in \mathcal{A}$ and assume that $A \cap B = \emptyset$. For each $r > 0$, let $f := r\chi_A$ and $g := r\chi_B$. Then, assertion (ii) implies that

$$r \wedge \mu(A \cup B) \leq K\big\{r \wedge \mu(A) + r \wedge \mu(B)\big\}. \tag{1}$$

First consider the case where $\mu(A \cup B) = \infty$. Suppose, contrary to our claim, that both $\mu(A)$ and $\mu(B)$ are finite. Then, letting $r := K\big\{\mu(A) + \mu(B)\big\} + 1 > 0$ in (**??**) yields

$$K\big\{\mu(A) + \mu(B)\big\} + 1 \leq K\big\{\mu(A) + \mu(B)\big\},$$

which is impossible. Therefore, at least one of $\mu(A)$ and $\mu(B)$ is infinite, so that

$$\mu(A \cup B) = \infty = K\big\{\mu(A) + \mu(B)\big\}.$$

If $\mu(A \cup B) < \infty$ then letting $r := \mu(A \cup B) + 1 > 0$ in (**??**) yields the desired inequality. Therefore, μ is K-relaxed subadditive.

(2) It can be easily proved by the fact that

$$\{f + g > t\} \subset \{f > t/2\} \cup \{g > t/2\}$$

for every $f, g \in \mathcal{F}_0^+(X)$ and $t \geq 0$. □

Remark 2.3. (1) The subadditivity of the Sugeno integral was proved in [**?**, Proposition 5] for nonadditive measures on the discrete space $(\mathbb{N}, 2^{\mathbb{N}})$.

(2) The Shilkret integral is not subadditive even if μ is σ-additive. To show this, let $X := (0,1)$ and let $\mathcal{A}$ be the σ-field of all Borel subsets of X. Let λ be the Lebesgue measure on $\mathbb{R}$. Define the functions $f, g\colon X \to [0,\infty)$ by $f(x) := 1/x$ and $g(x) := 1/(1-x)$ for every $x \in X$. Then it follows that $\mathrm{Sh}(\lambda, f) = \mathrm{Sh}(\lambda, g) = 1$ and $\mathrm{Sh}(\lambda, f+g) = 4$. In fact, the Shilkret integral is subadditive if and only if μ is maxitive, that is, $\mu(A \cup B) = \mu(A) \vee \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$; see [**?**, Theorem 4].

2.3. **Various modes of convergence of measurable functions.** Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ and $f \in \mathcal{F}_0(X)$. There are several ways to define the convergence of sequences of measurable functions. We say that $\{f_n\}_{n\in\mathbb{N}}$ converges *μ-almost everywhere* to f, denoted by $f_n \to f$ μ-a.e., if there is $N \in \mathcal{A}$ such that $\mu(N) = 0$ and $f_n(x) \to f(x)$ for every $x \notin N$. We also say that $\{f_n\}_{n\in\mathbb{N}}$ converges *μ-almost uniformly* to f, denoted by $f_n \to f$ μ-a.u., if for any $\varepsilon > 0$ there is $E_\varepsilon \in \mathcal{A}$ such that $\mu(E_\varepsilon) < \varepsilon$ and f_n converges to f uniformly on $X \setminus E_\varepsilon$. Another concept of convergence is not quite intuitive, but it has some advantages in analysis. We say that $\{f_n\}_{n\in\mathbb{N}}$ converges *in μ-measure* to f, denoted by $f_n \xrightarrow{\mu} f$, if $\mu(\{|f_n - f| > \varepsilon\}) \to 0$ for every $\varepsilon > 0$. Every sequence of measurable functions converging μ-almost uniformly converges μ-almost everywhere and in μ-measure to the same limit function.

The three modes of convergence introduced above require that the differences between the elements f_n of the sequence and the limit function f should become small in some sense as n increases. The following definition involves only the elements of the sequence. We say that $\{f_n\}_{n\in\mathbb{N}}$ is *Cauchy in μ-measure* if for any $\varepsilon > 0$ and

$\delta > 0$ there is $n_0 \in \mathbb{N}$ such that $\mu(\{|f_m - f_n| > \varepsilon\}) < \delta$ whenever $m, n \in \mathbb{N}$ and $m, n \geq n_0$.

See a survey paper [**?**] for further information on various modes of convergence of measurable functions in nonadditive measure theory.

2.4. **Equivalence relation and quotient space.** The quotient space of $\mathcal{F}_0(X)$ is constructed by an equivalence relation determined by a nonadditive measure μ. The proof of the following statements is routine and left it to the reader.

- Assume that μ is weakly null-additive. Given $f, g \in \mathcal{F}_0(X)$, define the binary relation $f \sim g$ on $\mathcal{F}_0(X)$ by $\mu(\{|f - g| > c\}) = 0$ for every $c > 0$ so as to become an equivalence relation on $\mathcal{F}_0(X)$. For every $f \in \mathcal{F}_0(X)$ the equivalence class of f is the set of the form $\{g \in \mathcal{F}_0(X) \colon f \sim g\}$ and denoted by $[f]$. Then the quotient space of $\mathcal{F}_0(X)$ is defined by $F_0(X) := \{[f] \colon f \in \mathcal{F}_0(X)\}$.
- Assume that μ is weakly null-additive. Given equivalence classes $[f], [g] \in F_0(X)$ and $c \in \mathbb{R}$, define addition and scalar multiplication on $F_0(X)$ by $[f] + [g] := [f + g]$ and $c[f] := [cf]$. They are well-defined, that is, they are independent of which member of an equivalence class we choose to define them. Then $F_0(X)$ is a real linear space.

The binary relation on $\mathcal{F}_0(X)$ defined above may not be transitive unless μ is weakly null-additive; see [**?**, Example 5.1]. In what follows, let $S(X) := \{[h] \colon h \in \mathcal{S}(X)\}$.

2.5. **Prenorms.** Let V be a real linear space. A *prenorm* on V is a nonnegative real-valued function $\|\cdot\|$ defined on V such that $\|0\| = 0$ and $\|-x\| = \|x\|$ for every $x \in V$. Then the pair $(V, \|\cdot\|)$ is called a *prenormed space*. A prenorm $\|\cdot\|$ is called *homogeneous* if it follows that $\|cx\| = |c|\|x\|$ for every $x \in V$ and $c \in \mathbb{R}$ and *truncated subhomogeneous* if it follows that $\|cx\| \leq \max(1, |c|)\|x\|$ for every $x \in V$ and $c \in \mathbb{R}$. A *seminorm* is a prenorm that is homogeneous and satisfies the triangle inequality, that is, $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in V$. Then a *norm* is a seminorm that separates points of V, that is, for any $x \in V$, if $\|x\| = 0$ then $x = 0$. Following [**?**], a prenorm $\|\cdot\|$ is called relaxed if it satisfies a *relaxed triangle inequality*, that is, there is a constant $K \geq 1$ such that $\|x + y\| \leq K\{\|x\| + \|y\|\}$ for every $x, y \in V$ (in this case, we say that $\|\cdot\|$ satisfies the K-relaxed triangle inequality). A *quasi-seminorm* on V is a prenorm that is homogeneous and satisfies a relaxed triangle inequality. Then a *quasi-norm* is a quasi-seminorm that separates points of V.

To associate with similar characteristics of nonadditive measures, a prenorm $\|\cdot\|$ is called *weakly null-additive* if $\|x + y\| = 0$ whenever $x, y \in V$ and $\|x\| = \|y\| = 0$ and *null-additive* if $\|x + y\| = \|x\|$ whenever $x, y \in V$ and $\|y\| = 0$.

Let $(V, \|\cdot\|)$ be a prenormed space. Let $\{x_n\}_{n\in\mathbb{N}} \subset V$ and $x \in V$. We say that $\{x_n\}_{n\in\mathbb{N}}$ *converges* to x, denoted by $x_n \to x$, if $\|x_n - x\| \to 0$. We may simply say that $\{x_n\}_{n\in\mathbb{N}}$ converges if the limit x is not needed to specify. The notion of a Cauchy sequence involves only the elements of the sequence and we say that $\{x_n\}_{n\in\mathbb{N}}$ is *Cauchy* if for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ whenever

$m, n \in \mathbb{N}$ and $m, n \geq n_0$. Not every converging sequence is Cauchy since prenorms satisfy neither the triangle inequality nor its relaxed ones in general. A subset B of V is called *bounded* if $\sup_{x\in B} \|x\| < \infty$.

A prenormed space $(V, \|\cdot\|)$ is called *complete* if every Cauchy sequence in V converges to an element in V. It is called *quasi-complete* if every bounded Cauchy sequence in V converges to an element in V. The denseness and the separability can be defined in the same way as in ordinary normed spaces. We say that V is *separable* if there is a countable subset D of V such that D is *dense* in V, that is, for any $x \in V$ and $\varepsilon > 0$ there is $y \in D$ such that $\|x - y\| < \varepsilon$.

If the prenorm $\|\cdot\|$ is needed to emphasize in the above terms, then the phrase "with respect to $\|\cdot\|$" is added to each term.

3. The Lorentz spaces defined by the Sugeno integral

Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. When μ is σ-additive, the ordinary Lorentz space is defined by

$$\mathcal{L}^{p,q}(\mu) := \{f \in \mathcal{F}_0(X) \colon \|f\|_{p,q} < \infty\},$$

where $\|\cdot\|_{p,q}$ is the Lorentz quasi-seminorm on $\mathcal{L}^{p,q}(\mu)$ defined by the Lebesgue integral as

$$\|f\|_{p,q} := p^{1/q} \left(\int_0^\infty \left[t\mu(\{|f| > t\})^{1/p} \right]^q \frac{dt}{t} \right)^{1/q} \tag{2}$$

for every $f \in \mathcal{F}_0(X)$ [?, Theorem 6.6]. In this section, a type of the Lorentz spaces is defined by using the Sugeno integral as an analog of the ordinary Lorentz spaces.

The right side of (??) can be expressed as

$$\left(\frac{p}{q}\right)^{1/q} \mathrm{Ch}(\mu^{q/p}, |f|^q)^{1/q}$$

in terms of the Choquet integral defined by

$$\mathrm{Ch}(\mu, f) := \int_0^\infty \mu(\{f > t\})dt$$

for every $f \in \mathcal{F}_0^+(X)$ and $\mu \in \mathcal{M}(X)$, which is due to the fact that

$$\mathrm{Ch}(\mu^{q/p}, |f|^q) = \int_0^\infty q t^{q-1} \mu(\{|f| > t\})^{q/p} dt.$$

This observation leads to the study of the Lorentz spaces defined by the Choquet integral [?] and also to the following definition.

Definition 3.1. Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Define the function $(\,\cdot\,)_{p,q} \colon \mathcal{F}_0(X) \to [0, \infty]$ by

$$(f)_{p,q} := \left(\frac{p}{q}\right)^{1/q} \mathrm{Su}(\mu^{q/p}, |f|^q)^{1/q}$$

for every $f \in \mathcal{F}_0(X)$ and let

$$\mathfrak{Su}^{p,q}(\mu) := \{f \in \mathcal{F}_0(X) \colon (f)_{p,q} < \infty\}.$$

The space $\mathfrak{Su}^{p,q}(\mu)$ is called the *Sugeno-Lorentz space* and the prenorm $(\,\cdot\,)_{p,q}$ on $\mathfrak{Su}^{p,q}(\mu)$ is called the *Sugeno-Lorentz prenorm.*

Proposition 3.2. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$.*

(1) *It follows that*

$$(f)_{p,q} = \left(\frac{p}{q}\right)^{1/q} \mathrm{Su}(\mu^{1/p}, |f|).$$

(2) *For any $A \in \mathcal{A}$ and $c \in \mathbb{R}$ it follows that*

$$(c\chi_A)_{p,q} = \left(\frac{p}{q}\right)^{1/q} \min\left\{|c|, \mu(A)^{1/p}\right\}.$$

(3) *For any $f \in \mathfrak{Su}^{p,q}(\mu)$ it follows that $(f)_{p,q} = 0$ if and only if $\mu(\{|f| > c\}) = 0$ for every $c > 0$; they are equivalent to the condition that $\mu(\{|f| > 0\}) = 0$ if μ null-continuous.*

(4) *For any $f \in \mathfrak{Su}^{p,q}(\mu)$ and $c \in \mathbb{R}$ it follows that*

$$(cf)_{p,q} \le \max\{1, |c|\}\,(f)_{p,q}.$$

Hence the prenorm $(\,\cdot\,)_{p,q}$ is truncated subhomogeneous.

(5) *For any $f \in \mathfrak{Su}^{p,q}(\mu)$ and $c > 0$ it follows that*

$$\min\{c^p, \mu(\{|f| > c\})\} \le \left(\frac{q}{p}\right)^{p/q} (f)_{p,q}^p.$$

(6) *For any $f, g \in \mathfrak{Su}^{p,q}(\mu)$, if $|f| \le |g|$ then $(f)_{p,q} \le (g)_{p,q}$.*

(7) *μ is weakly null-additive if and only if $(\,\cdot\,)_{p,q}$ is weakly null-additive.*

(8) *μ is null-additive if and only if $(\,\cdot\,)_{p,q}$ is null-additive.*

(9) *μ is null-additive if and only if it follows that $(f)_{p,q} = (g)_{p,q}$ whenever $f, g \in \mathfrak{Su}^{p,q}(\mu)$ and $f \sim g$.*

(10) *If μ is K-relaxed subadditive for some $K \ge 1$, then $(\,\cdot\,)_{p,q}$ satisfies the $(2K)^{\frac{1}{p}}$-relaxed triangle inequality.*

Proof. Assertions (1)–(9) can be derived in the same manner as [**?**, **?**] or by similar verification.

(10) Since μ is K-relaxed subadditive, $\mu^{1/p}$ is $(2K)^{1/p}$-relaxed subadditive. It thus follows from (1) of Proposition **??** that $(\,\cdot\,)_{p,q}$ satisfies the $(2K)^{1/p}$-triangle inequality. □

From (4) and (10) of Proposition **??** it follows that $\mathfrak{Su}^{p,q}(\mu)$ is a real linear subspace of $\mathcal{F}_0(X)$ if μ is relaxed subadditive.

There is a close relationship between convergence in measure and convergence with respect to $(\,\cdot\,)_{p,q}$. The conclusion of the following proposition can be found in [**?**, **?**], where μ is assumed to be conditionally continuous from above and continuous from below. The same proof works for an arbitrary nonadditive measure; see also [**?**, **?**].

Proposition 3.3. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ and $f \in \mathcal{F}_0(X)$. Then $(f_n - f)_{p,q} \to 0$ if and only if $f_n \xrightarrow{\mu} f$.*

The quotient space

$$Su^{p,q}(\mu) := \{[f] \colon f \in \mathfrak{Su}^{p,q}(\mu)\}$$

is defined by the equivalence relation introduced in Subsection **??**. Given an equivalence class $[f] \in Su^{p,q}(\mu)$, define the prenorm on $Su^{p,q}(\mu)$ by

$$([f])_{p,q} := (f)_{p,q},$$

which is well-defined by (9) of Proposition **??** if μ is null-additive. This prenorm has the same properties as the prenorm on $\mathfrak{Su}^{p,q}(\mu)$ and separates points of $Su^{p,q}(\mu)$, that is, for any $[f] \in Su^{p,q}(\mu)$, if $([f])_{p,q} = 0$ then $[f] = 0$.

In the rest of this section, following [**?**], we show that the Sugeno-Lorentz spaces coincide with the space of all Sugeno integrable functions, which is defined by

$$\mathfrak{Su}(\mu) := \{f \in \mathcal{F}_0(X) \colon (f)_1 < \infty\},$$

where $(\,\cdot\,)_1$ is defined by

$$(f)_1 := \mathrm{Su}(\mu, |f|) = (f)_{1,1}$$

for every $f \in \mathcal{F}_0(X)$. This is proved by the fact $\mathrm{Su}(\mu, |f|) < \infty$ if and only if $\mathrm{Su}(\mu^{q/p}, |f|^q) < \infty$, which follows from [**?**, Lemma 9.4]. Consequently, if μ is null-additive, then $Su^{p,q}(\mu)$ also coincides with

$$Su(\mu) := \{[f] \colon f \in \mathfrak{Su}(\mu)\},$$

where $([f])_1 := (f)_1$ for every $[f] \in Su(\mu)$. Furthermore, if μ is finite, then

$$\mathrm{Su}(\mu, |f|) \leq \mu(X) < \infty$$

for every $f \in \mathcal{F}_0(X)$, hence it follows that $\mathfrak{Su}^{p,q}(\mu) = \mathfrak{Su}(\mu) = \mathcal{F}_0(X)$ and $Su^{p,q}(\mu) = Su(\mu) = F_0(X)$. For this reason, in what follows, the spaces $\mathfrak{Su}^{p,q}(\mu)$ and $Su^{p,q}(\mu)$ will be written as $\mathfrak{Su}(\mu)$ and $Su(\mu)$ respectively and the latter spaces are refereed to as the *Sugeno integrable functions space*.

For any $a, b \in [0, \infty]$ and $0 < r < \infty$ the inequalities

$$a \wedge b^r \leq (a \wedge b)^r + a \wedge b, \quad (a \wedge b)^r \leq (a \wedge b^r)^r + a \wedge b^r$$

holds [**?**]. The first inequality yields

$$(f)_{p,q} \leq \left(\frac{p}{q}\right)^{1/q} \{(f)_1^p + (f)_1\} \tag{3}$$

and the second one yields

$$(f)_1^{1/p} \leq \left\{ \left(\frac{q}{p}\right)^{1/q} (f)_{p,q} \right\}^{1/p} + \left(\frac{q}{p}\right)^{1/q} (f)_{p,q}, \tag{4}$$

both of which hold for every $f \in \mathcal{F}_0(X)$. From these inequalities we see that for any sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$, it converges with respect to $(\,\cdot\,)_{p,q}$ if and only if it converges with respect to $(\,\cdot\,)_1$ and that it is Cauchy with respect to $(\,\cdot\,)_{p,q}$ if and only if it is Cauchy with respect to $(\,\cdot\,)_1$.

4. Completeness and separability of the Sugeno-Lorentz spaces

The completeness and separability of function spaces is useful and important in functional analysis. The completeness is especially effective in guaranteeing the limit of a sequence of functions in methods of successive approximation, while the separability enables us to obtain constructive proofs for many theorems that can be turned into algorithms for use in numerical and constructive analysis.

In this section, the completeness of $\mathfrak{Su}(\mu)$ with respect to $(\,\cdot\,)_{p,q}$ will be shown by a Cauchy criterion for functions in $\mathcal{F}_0(X)$. This criterion ensures that any Cauchy in μ-measure sequence in $\mathcal{F}_0(X)$ has a subsequence converging μ-almost uniformly and its proof needs to introduce a new characteristic of nonadditive measures.

Definition 4.1 ([**?**, Definition 3.2])**.** Let $\mu \in \mathcal{M}(X)$. We say that μ satisfies *property (C)* if for any sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, it follows that $\mu\left(\bigcup_{n=k}^{\infty} E_n\right) \to 0$ whenever $\sup_{l\in\mathbb{N}} \mu\left(\bigcup_{n=k}^{k+l} E_n\right) \to 0$.

It is easy to see that every nonadditive measure that is continuous from below satisfies property (C). Other examples of nonadditive measures satisfying property (C) can be found in [**?**, Proposition 3.3].

We can now formulate a Cauchy criterion for functions in $\mathcal{F}_0(X)$.

Theorem 4.2 ([**?**, Theorem 3.4])**.** *Let $\mu \in \mathcal{M}(X)$. If μ satisfies property (C) and the (p.g.p.), then any Cauchy in μ-measure sequence in $\mathcal{F}_0(X)$ has a subsequence converging μ-almost uniformly.*

To prove the completeness of a function space, we must verify that any Cauchy sequence converges and its limit function belongs to the same function space. The following Fatou type lemma is useful for this verification.

Proposition 4.3. *Let $\mu \in \mathcal{M}(X)$. The following assertions are equivalent.*

- (i) *μ is monotone autocontinuous from below.*
- (ii) *For any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(X)$ and $f \in \mathcal{F}_0(X)$, if they satisfy*
  - (a) *$f_n(x) \leq f_{n+1}(x) \leq f(x)$ for every $x \in X$ and $n \in \mathbb{N}$,*
  - (b) *$f_n \xrightarrow{\mu} f$,*

  then $\mu(\{f_n > t\}) \uparrow \mu(\{f > t\})$ for every continuity point t of the function $\mu(\{f > t\})$.
- (iii) *The Sugeno monotone nondecreasing almost uniform convergence theorem holds for μ, that is, for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and $f \in \mathcal{F}_0^+(X)$, if they satisfy*
  - (a) *$f_n(x) \leq f_{n+1}(x) \leq f(x)$ for every $x \in X$ and $n \in \mathbb{N}$,*
  - (b) *$f_n \to f$ μ-a.u.,*

  then it follows that $\mathrm{Su}(\mu, f_n) \uparrow \mathrm{Su}(\mu, f)$.
- (iv) *The Sugeno Fatou almost uniform convergence lemma holds for μ, that is, for any $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0^+(X)$ and $f \in \mathcal{F}_0^+(X)$, if $f_n \to f$ μ-a.u., then it follows that*

$$\mathrm{Su}(\mu, f) \leq \liminf_{n\to\infty} \mathrm{Su}(\mu, f_n).$$

Proof. (i)⇒(ii) For each $t \geq 0$ and $n \in \mathbb{N}$, let $\varphi_n(t) := \mu(\{f_n > t\})$ and $\varphi(t) := \mu(\{f > t\})$. Let $t_0 \in \mathbb{R}$ be a continuity point of φ. Then condition (a) implies that

$$\varphi_n(t) \leq \varphi_{n+1}(t) \leq \varphi(t)$$

for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$, which yields $\sup_{n\in\mathbb{N}} \varphi_n(t_0) \leq \varphi(t_0)$. It thus suffices to show

$$\varphi(t_0) \leq \sup_{n\in\mathbb{N}} \varphi_n(t_0). \tag{5}$$

To see this, fix $\varepsilon > 0$ and let $A := \{f > t_0 + \varepsilon\}$ and $B_n := \{|f_n - f| > \varepsilon\}$ for every $n \in \mathbb{N}$. Then we have

$$\mu(A \setminus B_n) \leq \mu(\{f_n > t_0\}) \tag{6}$$

for every $n \in \mathbb{N}$. Furthermore, condition (a) implies that $\{B_n\}_{n\in\mathbb{N}}$ is nonincreasing and condition (b) implies that $\mu(B_n) \to 0$. Hence, the monotone autocontinuity of μ from below yields

$$\mu(A) = \sup_{n\in\mathbb{N}} \mu(A \setminus B_n). \tag{7}$$

Consequently, it follows from (**??**) and (**??**) that

$$\varphi(t_0 + \varepsilon) \leq \sup_{n\in\mathbb{N}} \varphi_n(t_0),$$

which implies (**??**) since t_0 is a continuity point of φ.

(ii)⇒(iii) We first show the conclusion in the case where μ is finite. For each $r > 0$, let $g := f \wedge r$ and $g_n := f_n \wedge r$ for every $n \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and take the continuity points $c_1, c_2, \ldots, c_k$ of the function $\mu(\{g > t\})$ such that

- $0 = c_0 < c_1 < c_2 < \cdots < c_{k-1} < c_k < r$,
- $|c_i - c_{i-1}| < 2r/k$ $(i = 1, 2, \ldots, k-1)$ and $|r - c_k| < r/k$.

For each $n \in \mathbb{N}$, let

$$h_{n,k} := \bigvee_{i=1}^{k} c_i \chi_{\{g_n > c_i\}} \quad \text{and} \quad h_k := \bigvee_{i=1}^{k} c_i \chi_{\{g > c_i\}}.$$

Then $0 \leq h_{n,k}(x) \leq r$, $0 \leq h_k(x) \leq r$, $|h_{n,k}(x) - g_n(x)| < 2r/k$, and $|h_k(x) - g(x)| < 2r/k$ for every $x \in X$ and $n \in \mathbb{N}$.

Now, since $\{g_n\}_{n\in\mathbb{N}}$ and g satisfy conditions (a) and (b) of assertion (ii), it follows that

$$\mu(\{g_n > c_i\}) \uparrow \mu(\{g > c_i\})$$

for every $i \in \{1, 2, \ldots, k\}$. Hence, letting $n \to \infty$ yields

$$\mathrm{Su}(\mu, h_{n,k}) = \bigvee_{i=1}^{k} c_i \wedge \mu(\{g_n > c_i\}) \to \bigvee_{i=1}^{k} c_i \wedge \mu(\{g > c_i\}) = \mathrm{Su}(\mu, h_k). \tag{8}$$

Since

$$|\mathrm{Su}(\mu, g) - \mathrm{Su}(\mu, h_k)| \leq \frac{2r}{k}\mu(X)$$

and

$$|\mathrm{Su}(\mu, g_n) - \mathrm{Su}(\mu, h_{n,k})| \le \frac{2r}{k}\mu(X)$$

for every $n \in \mathbb{N}$, it follows from (**??**) that

$$\limsup_{n\to\infty} |\mathrm{Su}(\mu, g_n) - \mathrm{Su}(\mu, g)| \le \frac{4r}{k}\mu(X).$$

Since $k \in \mathbb{N}$ is arbitrarily fixed, letting $k \to \infty$ yields

$$\mathrm{Su}(\mu, g) = \lim_{n\to\infty} \mathrm{Su}(\mu, g_n) = \sup_{n\in\mathbb{N}} \mathrm{Su}(\mu, g_n).$$

Hence, by the upper marginal continuity of the Sugeno integral stated in Subsection **??** we have

$$\begin{aligned}\mathrm{Su}(\mu, f) &= \sup_{r>0} \mathrm{Su}(\mu, f \wedge r) = \sup_{r>0}\sup_{n\in\mathbb{N}} \mathrm{Su}(\mu, f_n \wedge r) \\ &= \sup_{n\in\mathbb{N}}\sup_{r>0} \mathrm{Su}(\mu, f_n \wedge r) = \sup_{n\in\mathbb{N}} \mathrm{Su}(\mu, f_n).\end{aligned}$$

We turn to the general case. For any $s > 0$, the nonadditive measure $\mu \wedge s$, which is defined by $(\mu \wedge s)(A) := \mu(A) \wedge s$ for every $A \in \mathcal{A}$, is finite and monotone autocontinuous from below. Hence, by what has been shown above we have

$$\sup_{n\in\mathbb{N}} \mathrm{Su}(\mu \wedge s, f_n) = \mathrm{Su}(\mu \wedge s, f),$$

and hence,

$$\begin{aligned}\mathrm{Su}(\mu, f) &= \sup_{s>0} \mathrm{Su}(\mu \wedge s, f) \\ &= \sup_{s>0}\sup_{n\in\mathbb{N}} \mathrm{Su}(\mu \wedge s, f_n) \\ &= \sup_{n\in\mathbb{N}}\sup_{s>0} \mathrm{Su}(\mu \wedge s, f_n) \\ &= \sup_{n\in\mathbb{N}} \mathrm{Su}(\mu, f_n)\end{aligned}$$

by the measure-truncation of the Sugeno integral stated in Subsection **??**.

(iii)⇒(iv) For each $n \in \mathbb{N}$, let $g_n := \inf_{k\ge n} f_k$. Then, $\{g_n\}_{n\in\mathbb{N}}$ and f satisfy conditions (a) and (b) of assertion (iii). It thus follows that

$$\mathrm{Su}(\mu, f) = \lim_{n\to\infty} \mathrm{Su}(\mu, g_n) \le \lim_{n\to\infty} \inf_{k\ge n} \mathrm{Su}(\mu, f_k) = \liminf_{n\to\infty} \mathrm{Su}(\mu, f_n).$$

(iv)⇒(i) Take $A, B_n \in \mathcal{A}$ and assume that $\{B_n\}_{n\in\mathbb{N}}$ is nonincreasing and $\mu(B_n) \to 0$. For each $r > 0$, let $f_n := r\chi_{A\setminus B_n}$ for every $n \in \mathbb{N}$ and $f := r\chi_A$. Then $f_n \to f$ μ-a.u. Hence, assertion (iv) yields

$$\begin{aligned}r \wedge \mu(A) = \mathrm{Su}(\mu, f) &\le \liminf_{n\to\infty} \mathrm{Su}(\mu, f_n) \\ &= \liminf_{n\to\infty} r \wedge \mu(A \setminus B_n) \\ &\le \limsup_{n\to\infty} r \wedge \mu(A \setminus B_n) \le r \wedge \mu(A),\end{aligned}$$

so that $r\wedge\mu(A\setminus B_n) \to r\wedge\mu(A)$. Since $r > 0$ is arbitrary, we have $\mu(A\setminus B_n) \to \mu(A)$. Therefore μ is monotone autocontinuous from below. □

We are now in a position to show the completeness of the Sugeno-Lorentz spaces.

Theorem 4.4. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Assume that μ is monotone autocontinuous from below and satisfies property (C) and the (p.g.p.). Then $\mathfrak{Su}(\mu)$ and $Su(\mu)$ are quasi-complete with respect to $(\,\cdot\,)_{p,q}$.*

Proof. By inequalities (**??**) and (**??**) given in Section **??**, it suffices to show that $\mathfrak{Su}(\mu)$ and $Su(\mu)$ are quasi-complete with respect to $(\,\cdot\,)_1$. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{Su}(\mu)$ be bounded and Cauchy. By (5) of Proposition **??**, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in μ-measure, so that by Theorem **??** one can find a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ and a function $f \in \mathcal{F}_0(X)$ such that $f_{n_k} \to f$ μ-a.u.

Let $\varepsilon > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy, there is $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$ then

$$\mathrm{Su}(\mu, |f_m - f_n|) = (f_m - f_n)_1 < \varepsilon. \tag{9}$$

Fix $n \in \mathbb{N}$ with $n \geq n_0$. Then, $|f_{n_k} - f_n| \to |f - f_n|$ μ-a.u. if $k \to \infty$. Since μ is monotone autocontinuous from below, it follows from Proposition **??** that

$$\begin{aligned}\mathrm{Su}(\mu, |f - f_n|) &\leq \liminf_{k\to\infty} \mathrm{Su}(\mu, |f_{n_k} - f_n|)\\ &\leq \limsup_{k\to\infty} \mathrm{Su}(\mu, |f_{n_k} - f_n|)\\ &\leq \sup_{k\geq l} \mathrm{Su}(\mu, |f_{n_k} - f_n|) \qquad (10)\end{aligned}$$

for every $l \in \mathbb{N}$. Since $n_k \to \infty$, there is $k_0 \in \mathbb{N}$ such that $n_k \geq n_0$ for every $k \geq k_0$, it thus follows from (**??**) and (**??**) that

$$(f - f_n)_1 \leq \sup_{k\geq k_0} \mathrm{Su}(\mu, |f_{n_k} - f_n|) \leq \varepsilon,$$

which yields $(f - f_n)_1 \to 0$.

Next we show that $f \in \mathfrak{Su}(\mu)$. Since $\{f_n\}_{n\in\mathbb{N}}$ is bounded with respect to $(\,\cdot\,)_1$ and $|f_{n_k}| \to |f|$ μ-a.u., it follows from Proposition **??** that

$$(f)_1 = \mathrm{Su}(\mu, |f|) \leq \liminf_{k\to\infty} \mathrm{Su}(\mu, |f_{n_k}|) \leq \sup_{n\in\mathbb{N}} (f_n)_1 < \infty.$$

Hence $f \in \mathfrak{Su}(\mu)$. Thus $\mathfrak{Su}(\mu)$ is quasi-complete.

Since every nonadditive measure that is monotone autocontinuous from below is null-additive, the quotient space $Su(\mu)$ and the quotient prenorm $(\,\cdot\,)_{p,q}$ are well-defined and it turns out that $Su(\mu)$ is quasi-complete. □

Corollary 4.5. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Assume that μ is relaxed subadditive, monotone autocontinuous from below, and satisfies property (C). Then $\mathfrak{Su}(\mu)$ and $Su(\mu)$ are complete with respect to $(\,\cdot\,)_{p,q}$. Furthermore, the prenorm $(\,\cdot\,)_{p,q}$ satisfies a relaxed triangle inequality.*

Proof. By assumption, μ is even more null-additive and satisfies the (p.g.p.). Furthermore, by (10) of Proposition **??** the prenorm $(\,\cdot\,)_{p,q}$ satisfies the K-relaxed triangle inequality for some $K > 1$, so that every Cauchy sequence in $\mathfrak{Su}(\mu)$ and $Su(\mu)$ is bounded. The conclusion thus follows from Theorem **??**. □

The following example shows that property (C) cannot be dropped in Theorem **??** and Corollary **??**.

Example 4.6. Let $X := \mathbb{N}$ and $\mathcal{A} := 2^X$. Let $\mu\colon \mathcal{A} \to [0,2]$ be the nonadditive measure defined by

$$\mu(A) := \begin{cases} 0 & \text{if } A = \emptyset, \\ \sum_{i\in A} 1/2^i & \text{if } A \text{ is a nonempty finite subset of } \mathbb{N}, \\ 1 + \sum_{i\in A} 1/2^i & \text{if } A \text{ is an infinite subset of } \mathbb{N}. \end{cases}$$

For each $n \in \mathbb{N}$, let $A_n := \{1, 2, \ldots, n\}$ and $f_n := \chi_{A_n}$. Then by [**?**, Proposition 3.11] μ is subadditive, hence relaxed subadditive, monotone autocontinuous from below, and satisfies the (p.g.p.), while it does not satisfy property (C). Furthermore, $\{f_n\}_{n\in\mathbb{N}}$ does not converge in μ-measure.

Let $0 < p < \infty$ and $0 < q < \infty$. Then it follows from (2) of Proposition **??** that

$$(f_n)_{p,q} = \left(\frac{p}{q}\right)^{1/q} \min\left\{1, \left(\sum_{i=1}^{n} \frac{1}{2^i}\right)^{1/p}\right\}$$

and

$$(f_{n+l} - f_n)_{p,q} = \left(\frac{p}{q}\right)^{1/q} \min\left\{1, \left(\sum_{i=n+1}^{n+l} \frac{1}{2^i}\right)^{1/p}\right\}$$

for every $n, l \in \mathbb{N}$. Hence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{Su}(\mu)$ is bounded and Cauchy. Suppose that $\mathfrak{Su}(\mu)$ is quasi-complete. Then $\{f_n\}_{n\in\mathbb{N}}$ converges, hence converges in μ-measure by Proposition **??**, which is impossible. Therefore, $\mathfrak{Su}(\mu)$ is not quasi-complete.

Next we consider dense subsets and the separability of the Sugeno-Lorentz spaces. Recall that $\mathcal{S}(X)$ is the set of all $\mathcal{A}$-measurable simple functions on X and $S(X) = \{[h]\colon h \in \mathcal{S}(X)\}$.

Theorem 4.7. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Assume that μ is order continuous. Then, $\mathcal{S}(X)$ is a dense subset of $\mathfrak{Su}(\mu)$ with respect to $(\,\cdot\,)_{p,q}$. If μ is additionally assumed to be null-additive, then $S(X)$ is a dense subset of $Su(\mu)$ with respect to $(\,\cdot\,)_{p,q}$.*

Proof. Since any simple function is bounded, we have

$$(h)_{p,q} = \left(\frac{p}{q}\right)^{1/q} \mathrm{Su}(\mu^{1/p}, |h|) \le \left(\frac{p}{q}\right)^{1/q} \sup_{x\in X} |h(x)| < \infty,$$

hence $\mathcal{S}(X)$ is a subset of $\mathfrak{Su}(\mu)$.

Let $f \in \mathfrak{Su}(\mu)$. Then, by [**?**, Theorem 7.2] there is $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(X)$ such that $h_n \xrightarrow{\mu} f$. Hence $(h_n - f)_{p,q} \to 0$ by Proposition **??**. Therefore, $\mathcal{S}(X)$ is dense in $\mathfrak{Su}(\mu)$. The denseness of $S(X)$ in $Su(\mu)$ is now obvious. □

We say that μ has a *countable basis* if there is a countable subset $\mathcal{D}$ of $\mathcal{A}$ such that for any $A \in \mathcal{A}$ and $\varepsilon > 0$ there is $D \in \mathcal{D}$ for which $\mu(A \triangle D) < \varepsilon$.

Theorem 4.8. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Assume that μ is order continuous and satisfies the (p.q.p.). Assume that μ has a countable basis. Then there is a countable subset $\mathcal{E}$ of $\mathfrak{Su}(\mu)$ such that for any $f \in \mathfrak{Su}(\mu)$ and $\varepsilon > 0$ there is $h \in \mathcal{E}$ such that $(f - h)_{p,q} < \varepsilon$. Hence $\mathfrak{Su}(\mu)$ is separable with respect to $(\,\cdot\,)_{p,q}$. If μ is additionally assumed to be null-additive, then $Su(\mu)$ is separable with respect to $(\,\cdot\,)_{p,q}$.*

Proof. Let $\mathcal{D}$ be a countable basis of μ. We may assume that $\mathcal{D}$ is a field of sets in $\mathcal{A}$; see [**?**, Lemma III.8.4]. Let $\mathcal{E}$ be the set of all linear combinations of the characteristic functions of sets in $\mathcal{D}$ with rational coefficients. Then $\mathcal{E}$ is a countable subset of $\mathfrak{Su}(\mu)$.

Let $f \in \mathfrak{Su}(\mu)$. By the proof of [**?**, Theorem 7.7] there is $\{h_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(X)$ such that $h_n \xrightarrow{\mu} f$. Hence $(h_n - f)_{p,q} \to 0$ by Proposition **??**. Therefore $\mathfrak{Su}(\mu)$ is separable. The separability of $Su(\mu)$ is now obvious. □

5. The Shilkret-Lorentz spaces

In this section, a type of the Lorentz spaces is defined by using the Shilkret integral instead of the Lebesgue integral and its completeness and separability are discussed.

Definition 5.1. Let $\mu \in \mathcal{M}(X)$. Define the function $\langle\,\cdot\,\rangle_{p,q} \colon \mathcal{F}_0(X) \to [0,\infty]$ by

$$\langle f \rangle_{p,q} := \left(\frac{p}{q}\right)^{1/q} \mathrm{Sh}(\mu^{q/p}, |f|^q)^{1/q}$$

for every $f \in \mathcal{F}_0(X)$ and let

$$\mathfrak{Sh}^{p,q}(\mu) := \{f \in \mathcal{F}_0(X) \colon \langle f \rangle_{p,q} < \infty\}.$$

The space $\mathfrak{Sh}^{p,q}(\mu)$ is called the *Shilkret-Lorentz space* and the prenorm $\langle\,\cdot\,\rangle_{p,q}$ on $\mathfrak{Sh}^{p,q}(\mu)$ is called the *Shilkret-Lorentz prenorm.*

The following proposition can be proved in the same way as Proposition **??**.

Proposition 5.2. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$.*

(1) *It follows that*

$$\langle f \rangle_{p,q} = \left(\frac{p}{q}\right)^{1/q} \mathrm{Sh}(\mu^{1/p}, |f|).$$

(2) *For any $A \in \mathcal{A}$ and $c \in \mathbb{R}$ it follows that*

$$\langle c\chi_A \rangle_{p,q} = \left(\frac{p}{q}\right)^{1/q} |c|\, \mu(A)^{1/p}.$$

(3) *For any $f \in \mathfrak{Sh}^{p,q}(\mu)$ it follows that $\langle f \rangle_{p,q} = 0$ if and only if $\mu(\{|f| > c\}) = 0$ for every $c > 0$; they are equivalent to the condition that $\mu(\{|f| > 0\}) = 0$ if μ null-continuous.*

(4) *For any $f \in \mathfrak{Sh}^{p,q}(\mu)$ and $c \in \mathbb{R}$ it follows that $\langle cf \rangle_{p,q} = |c|\langle f \rangle_{p,q}$. Hence the prenorm $\langle\,\cdot\,\rangle_{p,q}$ is homogeneous.*

(5) *For any* $f \in \mathfrak{Sh}^{p,q}(\mu)$ *and* $c > 0$ *it follows that*

$$\mu(\{|f| > c\}) \leq \frac{1}{c^p}\left(\frac{q}{p}\right)^{p/q} \langle f \rangle_{p,q}^p.$$

(6) *For any* $f, g \in \mathfrak{Sh}^{p,q}(\mu)$, *if* $|f| \leq |g|$ *then* $\langle f \rangle_{p,q} \leq \langle g \rangle_{p,q}$.

(7) μ *is weakly null-additive if and only if* $\langle \cdot \rangle_{p,q}$ *is weakly null-additive.*

(8) μ *is null-additive if and only if* $\langle \cdot \rangle_{p,q}$ *is null-additive.*

(9) μ *is null-additive if and only if it follows that* $\langle f \rangle_{p,q} = \langle g \rangle_{p,q}$ *whenever* $f, g \in \mathfrak{Sh}^{p,q}(\mu)$ *and* $f \sim g$.

(10) *If* μ *is* K*-relaxed subadditive for some* $K \geq 1$, *then* $\langle \cdot \rangle_{p,q}$ *satisfies the* $2^{1+\frac{1}{p}} K^{\frac{1}{p}}$*-relaxed triangle inequality.*

If μ is relaxed subadditive, then it follows from (4) and (10) of Proposition **??** that $\mathfrak{Sh}^{p,q}(\mu)$ is a real linear subspace of $\mathcal{F}_0(X)$ and the prenorm $\langle \cdot \rangle_{p,q}$ on $\mathfrak{Sh}^{p,q}(\mu)$ is a quasi-seminorm

The quotient space

$$Sh^{p,q}(\mu) := \{[f] \colon f \in \mathfrak{Sh}^{p,q}(\mu)\}$$

is defined by the equivalence relation introduced in Subsection **??**. Given an equivalence class $[f] \in Sh^{p,q}(\mu)$, define the prenorm on $Sh^{p,q}(\mu)$ by

$$\langle [f] \rangle_{p,q} := \langle f \rangle_{p,q},$$

which is well-defined by (9) of Proposition **??** if μ is null-additive. This prenorm has the same properties as the prenorm on $\mathfrak{Sh}^{p,q}(\mu)$ and separates points of $Sh^{p,q}(\mu)$, that is, for any $[f] \in Sh^{p,q}(\mu)$, if $\langle [f] \rangle_{p,q} = 0$ then $[f] = 0$.

We first show that the Shilkret-Lorentz spaces coincide with the Lorentz spaces of weak type discussed in [**?**] with equivalence of prenorms. Recall that the Lorentz space of weak type is defined by

$$\mathcal{L}^{p,\infty}(\mu) := \{f \in \mathcal{F}_0(X) \colon \|f\|_{p,\infty} < \infty\},$$

where $\|\cdot\|_{p,\infty}$ is the Lorentz prenorm of weak type defined by

$$\|f\|_{p,\infty} := \mathrm{Sh}(\mu^{1/p}, |f|)$$

for every $f \in \mathcal{L}^{p,\infty}(\mu)$. If μ is null-additive, then the quotient space

$$L^{p,\infty}(\mu) := \{[f] \colon f \in \mathcal{L}^{p,\infty}(\mu)\}$$

is defined by the equivalence relation introduced in Subsection **??** and the prenorm on $L^{p,\infty}(\mu)$ is defined by

$$\|[f]\|_{p,\infty} := \|f\|_{p,\infty}$$

for every $[f] \in L^{p,\infty}(\mu)$. Then it follows from (1) of Proposition **??** that

$$\langle f \rangle_{p,q} = \left(\frac{p}{q}\right)^{1/q} \mathrm{Sh}(\mu^{1/p}, |f|) = \left(\frac{p}{q}\right)^{1/q} \|f\|_{p,\infty}, \tag{11}$$

which shows that $\mathfrak{Sh}^{p,q}(\mu) = \mathcal{L}^{p,\infty}(\mu)$ and $Sh^{p,q}(\mu) = L^{p,\infty}(\mu)$ with equivalence of prenorms. Consequently, the following results immediately follow from [?, Theorem 7.4 and Corollary 7.5]. Recall that every nonadditive measure that is monotone autocontinuous from below is null-additive.

Theorem 5.3. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Assume that μ is monotone autocontinuous from below and satisfies property (C) and the (p.g.p.). Then $\mathfrak{Sh}^{p,q}(\mu)$ and $Sh^{p,q}(\mu)$ are quasi-complete with respect to $\langle \cdot \rangle_{p,q}$.*

Corollary 5.4. *Let $\mu \in \mathcal{M}(X)$. Let $0 < p < \infty$ and $0 < q < \infty$. Assume that μ is relaxed subadditive, monotone autocontinuous from below, and satisfies property (C). Then $\mathfrak{Sh}^{p,q}(\mu)$ is complete with respect to the quasi-seminorm $\langle \cdot \rangle_{p,q}$ and $Sh^{p,q}(\mu)$ is complete with respect to the quasi-norm $\langle \cdot \rangle_{p,q}$.*

The following example shows that property (C) cannot be dropped in Theorem **??** and Corollary **??**.

Example 5.5. Let $X := \mathbb{N}$ and $\mathcal{A} := 2^X$. Let μ be the nonadditive measure given in Example **??**. For each $n \in \mathbb{N}$, let $A_n := \{1, 2, \ldots, n\}$ and $f_n := \chi_{A_n}$. As already mentioned in Example **??**, μ is subadditive, hence relaxed subadditive, monotone autocontinuous from below, and satisfies the (p.g.p.), while it does not satisfy property (C). Furthermore, $\{f_n\}_{n\in\mathbb{N}}$ does not converge in μ-measure.

Let $0 < p < \infty$ and $0 < q < \infty$. Then it follows from (2) of Proposition **??** that

$$\langle f_n \rangle_{p,q} = \left(\frac{p}{q}\right)^{1/q} \left(\sum_{i=1}^{n} \frac{1}{2^i}\right)^{1/p}$$

and

$$\langle f_{n+l} - f_n \rangle_{p,q} = \left(\frac{p}{q}\right)^{1/q} \left(\sum_{i=n+1}^{n+l} \frac{1}{2^i}\right)^{1/p}$$

for every $n, l \in \mathbb{N}$, hence the sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{Sh}^{p,q}(\mu)$ is bounded and Cauchy. Suppose that $\mathfrak{Sh}^{p,q}(\mu)$ is quasi-complete. Then, $\{f_n\}_{n\in\mathbb{N}}$ converges with respect to $\langle \cdot \rangle_{p,q}$, hence it converges in μ-measure by (5) of Proposition **??**, which is impossible. Therefore $\mathfrak{Sh}^{p,q}(\mu)$ is not quasi-complete.

The last example shows that the set $\mathcal{S}(X)$ is not dense in $\mathfrak{Sh}^{p,q}(\mu)$.

Example 5.6. Let $X := (0, 1]$ and $\mathcal{A}$ be the σ-field of all Borel subsets of X. Let λ be the Lebesgue measure on $\mathbb{R}$. Let $0 < p < \infty$ and $0 < q < \infty$. Let $\mu := \lambda^p$ and let $f(x) := 1/x$ for every $x \in X$. Then, μ is order continuous, $f \in \mathfrak{Sh}^{p,q}(\mu)$, and $\mathcal{S}(X) \subset \mathfrak{Sh}^{p,q}(\mu)$. Nevertheless, it follows that

$$\langle f - h \rangle_{p,q} \geq \left(\frac{p}{q}\right)^{1/q}$$

for every $h \in \mathcal{S}(X)$. Thus $\mathcal{S}(X)$ is not dense in $\mathfrak{Sh}^{p,q}(\mu)$.

Proof. It follows from (**??**) and [?, Proposition 7.7]. □

6. Summary of results

In this paper, given a nonadditive measure μ, the Sugeno-Lorentz spaces $\mathfrak{Su}^{p,q}(\mu)$, the Shilkret-Lorentz spaces $\mathfrak{Sh}^{p,q}(\mu)$, and their quotient spaces are defined by using the Sugeno and Shilkret integrals. The completeness and separability of those spaces are discussed in terms of the characteristic of μ. Some of our results are as follows.

- The Sugeno-Lorentz spaces $(\mathfrak{Su}^{p,q}(\mu), (\,\cdot\,)_{p,q})$ are equal to the Sugeno integrable functions space $(\mathfrak{Su}(\mu), (\,\cdot\,)_1)$ and their prenorms satisfy inequalities (**??**) and (**??**).
- The Shilkret-Lorentz spaces $(\mathfrak{Sh}^{p,q}(\mu), \langle\,\cdot\,\rangle_{p,q})$ are equal to the Lorentz spaces of weak type $(\mathcal{L}^{p,\infty}(\mu), \|\cdot\|_{p,\infty})$ and their prenorms satisfy equation (**??**).

Consequently, the study of the Sugeno-Lorentz spaces $\mathfrak{Su}^{p,q}(\mu)$ and the Shilkret-Lorentz spaces $\mathfrak{Sh}^{p,q}(\mu)$ can be reduced to the study of the Sugeno integrable functions space $\mathfrak{Su}(\mu)$ and the Lorentz spaces of weak type $\mathcal{L}^{p,\infty}(\mu)$, respectively.

Regarding the completeness, the following results are shown.

- Assume that μ is monotone autocontinuous from below and satisfies property (C) and the (p.g.p.). Then
  - the spaces $\mathfrak{Su}(\mu)$ and $Su(\mu)$ are quasi-complete with respect to $(\,\cdot\,)_{p,q}$.
  - the spaces $\mathfrak{Sh}^{p,q}(\mu)$ and $Sh^{p,q}(\mu)$ are quasi-complete with respect to $\langle\,\cdot\,\rangle_{p,q}$.
- Assume that μ is relaxed subadditive, monotone autocontinuous from below, and satisfies property (C). Then
  - the spaces $\mathfrak{Su}(\mu)$ and $Su(\mu)$ are complete with respect to the prenorm $(\,\cdot\,)_{p,q}$ that satisfies a relaxed triangle inequality.
  - the space $\mathfrak{Sh}^{p,q}(\mu)$ is complete with respect to the quasi-seminorm $\langle\,\cdot\,\rangle_{p,q}$ and the space $Sh^{p,q}(\mu)$ is complete with respect to the quasi-norm $\langle\,\cdot\,\rangle_{p,q}$.

All the results listed above hold for every subadditive nonadditive measure that is continuous from below since such a nonadditive measure is relaxed subadditive, monotone autocontinuous from below, null-additive, weakly null-additive, and satisfies property (C) and the (p.g.p.).

Regarding dense subsets and the separability, the following results are shown.

- Assume that μ is order continuous. Then $\mathcal{S}(X)$ is dense in $\mathfrak{Su}(\mu)$ with respect to $(\,\cdot\,)_{p,q}$. If μ is additionally assumed to be null-additive, then $S(X)$ is dense in $Su(\mu)$ with respect to $(\,\cdot\,)_{p,q}$.
- Assume that μ is order continuous and satisfies the (p.g.p.). Assume that μ has a countable basis. Then $\mathfrak{Su}(\mu)$ is separable with respect to $(\,\cdot\,)_{p,q}$. If μ is additionally assumed to be null-additive, then $Su(\mu)$ is separable with respect to $(\,\cdot\,)_{p,q}$.

It is possible to advance the study to topological and the topological linear properties of the Sugeno-Lorentz spaces and the Shilkret-Lorentz spaces, but we will not develop them here.

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