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IHARA ZETA FUNCTION AND CYCLE FLUCTUATIONS OF RANDOM RAMANUJAN GRAPH GENERATED BY p-ADIC PSEUDORANDOM NUMBERS

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Dedicated to the memories of Prof. Wataru Takahashi and Prof. Naoki Shioji

Abstract. Using the sequences of the pseudorandom numbers generated by the p-adic logistic map, we construct some pseudorandom adjacency matrices and their various nonregular graphs, which are characterized as weak Ramanujan graphs. For these various types of random graphs, which have some clustering properties, we statistically calculate the fitting curves to the distributions of these cycle numbers by using their cycle basis and estimating the numbers of each cycle in these graphs. We can estimate their fluctuation exponents and we call this value the Cycle-Fluctuation Exponent (CFE). Furthermore, using the distributions of poles of the Ihara zeta function, we introduce the Ramanujan radius ratio (RRr) and the the Ramanujan radius standard deviation (RRd). Comparing the value CFE to the other parameters, the small-worldness coefficient (SWC), RRr and RRd and clarifying the relations among these parameters numerically, we characterize the clustering and random properties of our random Ramanujan graphs.

1. Introduction

The Ramanujan graphs, which are known as the the best expander graphs, have excellent properties, sparseness and strong connectivity, as information networks with many applications in computer science and also, they are deeply related to various branches of pure mathematics. In the relation between the number theory and the graph theory it is well known that a regular graph is Ramanujan if and only if the Ihara zeta function of this graph satisfies the graph Riemann Hypothesis. In our previous papers ([9],[10],[11]), using the sequences of the pseudorandom numbers generated by the p-adic logistic map, we constructed some pseudorandom adjacency matrices with their graphs and we numerically investigated these graphs by calculating the eigenvalue distributions whether they have the properties of the 'almost' or 'weak' Ramanujan graphs.

In [12] we numerically estimated the randomness of the generated pseudorandom numbers by the statistical method, comparing the eigevalue distributions of the correlation matrices given by the pseudorandom sequence to the Marchenko-Pastur distribution curve derived from the random matrix theory (cf. [14]). In our previous

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papers we investigated the pseudorandomness or the randomness properties of non-regular almost Ramanujan graphs by comparing the eigenvalue distributions of the adjacency matrices of these graphs to the famous Wigner semi-circle distribution.

In this paper, by calculating the distribution of poles of the Ihara zeta functions we directly investigate whether these almost Ramanujan graphs satisfy the graph Riemann Hypothesis. Comparing to the Ramanujan circle, the radius of which is used as a criteria related to the graph Riemann Hypothesis, we calculate the mean and the standard deviation of the absolute values of the poles and we estimate the difference and the variation from the Ramanujan circle. Here we propose the ratio of this mean value to the radius of the Ramanujan circle, called Ramanujan radius ratio (abbr. RRr), and the variation or the standard deviation of these distributions, called the Ramanujan radius variation and deviation (abbr. RRv and RRd), respectively, as characteristic constants which show levels or some other properties of randomness of these given graphs.

For the intermediate network model between the regular, locally connected, graphs and the random graphs, Watts and Strogatz in [16] proposed the concept of a small-world network by increasing the edges rewiring probability p from $p = 0$, the regular graph, to $p = 1$, the random graph. They characterized the small-world properties as the network model, which has high clustering like a lattice and has short path lengths like a random graph. While the cluster analysis or clustering is very well-known for the recent epidemic or pandemic disease, the clustering states, mainly estimated by the clustering coefficients, give one of the most important indicators, which characterize the small-worldness of graph networks. We calculate the average of clustering coefficients of each nonregular graph to show the relations with the randomness and we estimate the small-worldness coefficients, which also give the level of randomness, of our nonregular graphs.

In order to construct various random matrices we use a simple method, deviding the adjacency matrix into two kinds of submatrices by giving large and small probability values of the edge-connectivity, which generate some clustering parts and the other sparse parts in the graph. We take two modulus $L_1, L_2 : L_1 > L_2$ of the pseudorandom numbers where a given integer L_1 can be called a sparse parameter, since L_1 gives the small probability $1/L_1$ for the occurrence of 1 and the large one $1-(1/L_1)$ for 0- occurrence in adjacency submatrices. On the contrary, by the large probability $1/L_2$ it generates the clustering parts in the graph. In these construction processes we devide the vertices of the graph into 6 groups and, by defining the two probabilities of connectivity between the vertices in one group and those in the other group, we can obtain the various types of random graphs, which have various clustering poperties.

For these random graphs, using the cycle basis and estimating the numbers of each cycle in these graphs, we statistically calculate the fitting curves to the distributions of these cycle numbers and we can estimate their fluctuation exponents. We call this value the Cycle-Fluctuation Exponent (abbr. CFE) on the analogy of the exponent β in $(1/f)^\beta$ fluctuation. Here we show an example in Fig.1,2 where $L_1 = 12, L_2 = 4$,

$|V|$: number of vertices, $|E|$: number of edges. In Figure 1, log-log scale, we use a linear regression.

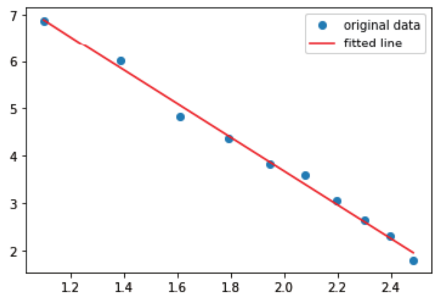

FIG 1. fluct.log fit

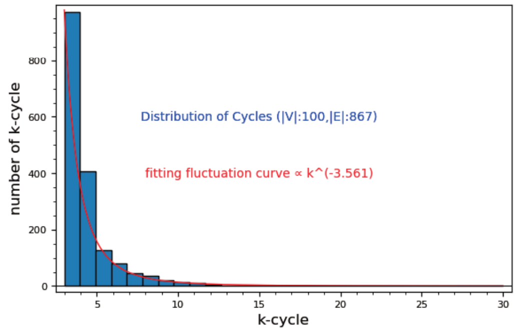

FIG 2. fluct.fitting

In [9] we obtained the following numerical results.

As the randomness of the edge connectivity is uniformly increasing,

(i) the eigenvalue distribution of the adjacency matrix approaches to the Wigner semi-circle,
(ii) the average of clustering coefficients is decreasing,
(iii) the cycle-number band of cycle basis is spreading widely,
(iv) the distribution of poles of the Ihara zeta functions is condensing into the circle given by the graph Riemann Hypothesis,
(v) the small-worldness coefficient is decreasing and converging to 1.

As the randomness and uniformness of the edge connectivity is decreasing, the contrary of each above claim holds.

Since our previous results, which show only tendency, increasing or decreasing, of various random properties, contain indistinct and ambiguous expression, the purpose of this paper is to quantify the above properties by introducing the new parameters of the various random graphs. Especially, we compare CFE to the other parameters, the small-worldness coefficient (SWC), the Ramanujan radius ratio (RRr) and the the Ramanujan radius variation or deviation (RRv or RRd) and, clarifying the relations among these parameters, we numerically characterize the clustering and random properties of our weak or almost Ramanujan graphs.

Our plan of this paper is as follows. In section 2 we introduce the notations of the graph theory and the properties of regular Ramanujan graphs and we give the definition of the Ihara zeta function. In section 3 we give the construction process of the various random graphs given by our two types, sparse or clustering, of pseudorandom number generators. In section 4 we give the numerical results on CFE, using cycle bases of our given random graphs and in section 5 we give the definitions of SWC and we numerically estimate these coefficients. In section 6 we calculate RRr and RRd by using the distribution of poles of the Ihara zeta functions and we investigate the relations to the parameters CFE and SWC. In section 7 we give the various definitions of almost Ramanujan graphs and we numerically investigate the relations with the weak graph Riemann Hypothesis by comparing the distribution of the eigenvalues to the Wigner semi-circle. Our concluding remarks are in section 8.

2. Regular Ramanujan graph and Ihara zeta function

Let $X = (V, E)$ be a graph where $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices and E is the set of edges. Let a_{ij} be the number of edges joining v_i to v_j, then the adjacency matrix of the graph X is given by $A = (a_{ij})$. We assume that (i) X is simple; there is at most one edge joining adjacent vertices, $a_{ij} \in \{0, 1\}$ for every i, j, (ii) X has no loops; $a_{ii} = 0$ for every $v_i \in V$ and (iii) X is undirected; A is a $n \times n$ symmetric matrix.

Let $k \geq 2$ be an integer and the graph X be k-regular, that is, for every $v_i \in V$,

$$\sum_{v_j \in V} a_{ij} = k.$$

Since A is an n-by-n symmetric matrix, it had n real eigenvalues, counting multiplicities,

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_{n-1}.$$

The following proposition is easily obtained (cf. [3]).

Proposition 2.1. *Let X be a finite connected k-regular graph with n vertices. Then*

- $\mu_0 = k$;
- $|\mu_i| \leq k, \quad 1 \leq i \leq n-1.$

For a graph $X = (V, E)$ and $F \subset V$, define the boundary ∂F of F by the set of edges with one extremity in F and the other in $V - F$, that is, ∂F is the set of edges connecting F to $V - F$.

Definition 2.2. The expanding constant $h(X)$ of the graph X is defined by

$$h(X) = \inf\{\frac{|\partial F|}{\min\{|F|, |V - F|\}} : \ F \subset V, \ 0 < |F| < +\infty\}.$$

For the relation between the nontrivial eigenvalue $\mu \neq k$ and the expanding constant $h(X)$ Dodziuk has shown the following estimates.

Proposition 2.3 ([4])**.** *Let $X = (V, E)$ be a finite, connected, k-regular simple graph. Let μ_1 be the first nontrivial eigenvalue of X. Then*

$$\frac{k - \mu_1}{2} \leq h(X) \leq \sqrt{2k(k - \mu_1)}.$$

Let $\{X_m\}$ be a family of finite, connected, k-regular graphs with $|V_m| \to +\infty$ as $m \to +\infty$.

$\{X_m\}$ is called a family of expanders if there exists a constant $\varepsilon > 0$ such that

$$h(X_m) \geq \varepsilon, \quad \forall m \geq 1.$$

It follows from Proposition 2.3 that we can easily obtain an equivalent condition for the existence of a family of expanders.

Corollary 2.4. *Let $\{X_m\}$ be a family of finite, connected, k-regular simple graphs with $|V_m| \to \infty$ as $m \to \infty$. Then, $\{X_m\}$ is a family of expanders if and only if there exists a constant $\varepsilon > 0$ such that*

$$k - \mu_1(X_m) \geq \varepsilon, \quad \forall m \geq 1.$$

For the asymptotic behaviors of these eigenvalues the following Alon-Boppana theorem is well known.

Theorem 2.5 ([1]). *Let $\{X_m\}$ be a family of finite, connected, k-regular simple graphs with $|V_m| \to +\infty$ as $m \to +\infty$. Then,*

$$\liminf_{m \to +\infty} \mu_1(X_m) \geq 2\sqrt{k-1}.$$

Here we give the definition of Ramanujan graph.

Definition 2.6. A finite, connected k-regular graph X is a Ramanujan graph if for every nontrivial eigenvalue $\mu(\neq \pm k)$ of X,

$$|\mu| \leq 2\sqrt{k-1}.$$

Since an expander constant of a regular graph is greater than or equal to $(k - \mu_1)/2$, making μ_1 as small as possible gives us good expander graphs. However, by the Alon-Boppana theorem, we cannot do better than

$$\liminf_{m \to +\infty} \mu_1(X_m) \geq 2\sqrt{k-1}.$$

Hence, Ramanujan graphs make good expanders.

For a graph $X = (V, E)$ and a path $C = a_1 a_2 \cdots a_s$ where a_j is an oirented edge of X, we say that it has a backtrack if $a_{j+1} = a_j^{-1}$ for some $j = 1, ..., s-1$ and a tail if $a_s = a_1^{-1}$. The length of C is $s = \nu(C)$ and C is called a closed path or cycle if the starting vertex is the same as the terminal vertex.

The closed path $C = a_1 \cdots a_s$ is called a primitive or prime path if it has no backtrack or tail and $C \neq D^n$, $n \geq 2$. For the closed path $C = a_1 \cdots a_s$, the equivalence class $[C]$ is the following set

$$[C] = \{a_1 \cdots a_s,\ a_2 \cdots a_s a_1,\ ...,\ a_s a_1 \cdots a_{s-1}\}.$$

A prime in the graph X is an equivalent class $[C]$ of prime paths. The length of the path C is denoted by $\nu(C) = s$.

Definition 2.7. The Ihara zeta function for a finite connected graph without 1-degree vertices is defined to be the following function of the complex number u, with $|u|$ sufficiently small,

$$\zeta_X(u) = \prod_{[P]} (1 - u^{\nu(P)})^{-1}$$

where the product is over all the primes $[P]$ in X.

Definition 2.8. The radius of the largest circle of convergence of $\zeta_X(u)$ is denoted by R_X.

When X is a $(q+1)$-regular graph, $R_X = 1/q$.

Definition 2.9. Let X be a connected $(q+1)$-regular graph. We say that the Ihara zeta function $\zeta_X(q^{-s})$ satisfies the Riemann Hypothesis iff, when $0 < \mathrm{Re}\ s < 1$,

$$\zeta_X(q^{-s})^{-1} = 0 \Rightarrow \mathrm{Re}\ s = \frac{1}{2}.$$

If $u = q^{-s}$, Re $s = \frac{1}{2}$ means that $|u| = 1/\sqrt{q}$.

The following theorem shows the deep and significant relation between the Ramanujan graph and the number theory (cf. [15]).

Theorem 2.10. *For a connected $(q+1)$-regular graph X, $\zeta_X(u)$ satisfies the Riemann Hypothesis if and only if the graph X is Ramanujan.*

3. Pseudorandom number generator

In this section we construct the adjacency matrices of nonregular random Ramanujan graphs, using the two sequences of (0,1)-pseudorandom numbers, which have two different probabilities of 1-occurrence and following the method in [10] and [11].

In order to construct various random matrices we use a simple method, deviding the adjacency matrix into two kinds of submatrices by giving large and small probability values of the edge-connectivity, which generate some clustering parts and the other sparse parts in the graph. We take two modulus $L_1, L_2 : L_1 > L_2$ of the pseudorandom numbers where a given integer $L_1 = 12$ can be called a sparse parameter, since L_1 gives the small probability $p_1 = 1/L_1 = 1/12$ for the occurrence of 1 and the probability $1 - (1/L_1) = 11/12$ for 0- occurrence in adjacency submatrices. On the contrary, by the large probability $p_2 = 1/L_2 = 1/4$ it generates the clustering parts in the graph. In our construction process we devide the vertices of the graph into 6 groups, colored by red, yellow and pink, which are warm colors, and blue, green and purple which are cold colors. By defining the two probabilities of connectivity between the vertices in one group and those in the other group, we can obtain the various types of random graphs, which have various clustering properties. To simplify the argument we define the following notations.

$G_r, G_y, G_{pk}, G_b, G_g, G_{pl}$ denote the set of red, yellow, pink, blue, green and purple vertices, respectively, and $p(v_1, v_2)$ denotes the probability of connectivity between the vertices v_1 and v_2.

We consider the following two cases (i), (ii) with the corresponding random graphs (i'), (ii'). Every graph has the same number of vertices $|V| = 150$.

- (i) $p(v_1, v_2) = 1/4$ if v_1 and v_2 in G_t for $t = r, y$ or pk; otherwise, $p(v_1, v_2) = 1/12$.
- (i') The corresponding random graph to (i), which has the same number of edges $|E_1|$ as that of (i) and for every pair of vertices v_1, v_2, $p(v_1, v_2)$ is constant, which satisfies $|E_1| = p(v_1, v_2)\binom{|V|}{2}$.

(ii) $p(v_1, v_2) = 1/4$ if v_1 and v_2 in G_t for $t = r, y$ or pk or if $v_1 \notin G_{pl}$ and $v_2 \in G_{pl}$ or if $v_1 \in G_{pl}$ and $v_2 \notin G_{pl}$;
otherwise, $p(v_1, v_2) = 1/12$.

(ii') The correponding random graph to (ii), which has the same number of edges $|E_2|$ as that of (ii) and for every pair of vertices v_1, v_2, $p(v_1, v_2)$ is constant, which satisfies $|E_2| = p(v_1, v_2)\binom{|V|}{2}$.

First we construct a sequence of pseudorandom numbers by using a p-adic logistic map.

(1): Choose a seed, a p-adic integer number, $\xi = 13^{\frac{1}{103}} \in \mathbb{Z}_p,\ p = 67$.
(2): For large integers $N_1, N_2 : N_1 + N_2 = n^2 \ \ (n = 150)$, construct two sequence $\{\xi_j\}_{j=1}^{N_i},\ i = 1, 2$, by a p-adic logistic map l_p, which is given by

$$l_p(x) = \frac{x^p - x}{p} \text{ for } x \in \mathbb{Z}_p,$$

$$\xi_1 = \xi,\ \xi_2 = l_p(\xi_1), ..., \xi_j = l_p(\xi_{j-1}), ...$$

(see [13]). In [12] we have taken their modulo p: $\xi_{j,p} = \xi_j \pmod p$ and we have shown the randomness of the sequence by RMT test.
(3): Let $M(= 6)$ be the number of groups of vertices and $M^2(= 36)$ be the number of submatrices of the adjacency matrix and $s = 25$ be the dimension of each submatrix.

According to the cases (i) and (ii) we devide the sequence of submatrices to the following two parts, the dense subsequence and the sparse subsequence.
For the two subsequences of submatrices in the case (i)
the number of the dense submatrices is $M/2 = 3$,
the number of the sparse submatrices is $M^2 - M/2 = 33$.
For the two subsequences of submatrices in the case (ii)
the number of the dense submatrices is $2(M - 1) + M/2 = 13$,
the number of the sparse submatrices is $M^2 - \{2(M - 1) + M/2\} = 23$.

According to these partitions we devide the sequence of pseudorandom numbers of (2) to the two parts as follows.

For the two integers $L_1 = 12, L_2 = 4$, which are sparse orders,
in the case of (i), let $N_1 = n^2 - s^2(M/2),\ N_2 = s^2(M/2)$ and then we calculate

$$\xi_{j,L_1} = \xi_j \mod L_1, \quad j = 1, ..., N_1,$$

and

$$\xi_{j,L_2} = \xi_j \mod L_2, \quad j = 1, ..., N_2.$$

In the case of (ii), let $N_1 = n^2 - s^2\{(M - 1) + M/2\},\ N_2 = s^2\{(M - 1) + M/2\}$ (here, note that it is not $\{2(M - 1) + M/2\}$) and then we calculate

$$\xi_{j,L_1} = \xi_j \mod L_1, \quad j = 1, ..., N_1,$$

and

$$\xi_{j,L_2} = \xi_j \mod L_2, \quad j = 1, ..., N_2.$$

(4): Construct two $\{0,1\}$-sequences $\{c_j^{(i)}\}$ from $\{\xi_{j,L_i}\}$, $i=1,2$, defined by

$$c_j^{(i)} = \left\{ \begin{array}{ll} 0 & \text{if } \xi_{j,L_i} \neq 1 \\ 1 & \text{if } \xi_{j,L_i} = 1 \end{array} \right. , \quad i=1,2, \quad j=1,...,N_i.$$

(5): We cut $\{c_j^{(i)}\}$ into $J_i = N_i/s^2$ pieces of equal length s^2, then shape them to two sequences of $s \times s$ square submatrices $\{C_j^{(i)}\} = \{(c_{kl}^{(i)})_j\}, i=1,2, j=1,...,J_i$. Furthermore, from these two sequences of submatrices $\{C_j^{(i)}\}$ we construct $n \times n$ matrices C_I, C_{II}. The dense submatrices ($i=2$) are colored by red to show high contrast to the sparse submatrices ($i=1$).

In the case of (i) we define

$$C_I = \begin{pmatrix} C_1^{(2)} & C_2^{(1)} & \cdots & \cdots & \cdots & C_6^{(1)} \\ C_7^{(1)} & C_8^{(1)} & \cdots & \cdots & \cdots & C_{12}^{(1)} \\ C_{13}^{(1)} & C_{14}^{(1)} & C_{15}^{(2)} & \cdots & \cdots & C_{18}^{(1)} \\ \cdots & \cdots & \cdots & C_{22}^{(1)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & C_{29}^{(2)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & C_{36}^{(1)} \end{pmatrix}$$

and in the case of (ii) we define

$$C_{II} = \begin{pmatrix} C_1^{(2)} & C_2^{(1)} & \cdots & \cdots & \cdots & C_6^{(2)} \\ C_7^{(1)} & C_8^{(1)} & \cdots & \cdots & \cdots & C_{12}^{(2)} \\ C_{13}^{(1)} & C_{14}^{(1)} & C_{15}^{(2)} & \cdots & \cdots & C_{18}^{(2)} \\ \cdots & \cdots & \cdots & C_{22}^{(1)} & \cdots & C_{24}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & C_{29}^{(2)} & C_{30}^{(2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & C_{36}^{(1)} \end{pmatrix}.$$

Next we construct the adjacency symmetric matrices from C_I, C_{II}.

(6): Construct upper triangle matrices S_i of $C_i, i=I,II$ which have the upper triangle parts of $C_i, i=I,II$ with 0 diagonal elements, respectively.

(7): Calculate $T_i = S_i +^t S_i, i=I,II$, which are the adjacency matrices.

(8): When the graphs given by the adjacency matrices have tails, 1-degree vertices, we apply the sagemath command, ".remove_tails". Then we can obtain the two simple graphs $\mathrm{G_I}, \mathrm{G_{II}}$ from these adjacency matrices T_I, T_{II}, respectively.

(9): By using the numbers of vertices and edges of these two graphs and applying the sagemath command "graphs.RandomGNM($|V|, |E|$)" where $|V| = 150$ and $|E|$ is the number of edges of each graph we construct the two random graphs $\mathrm{RG_I}, \mathrm{RG_{II}}$, which have the same numbers of edges as those of $\mathrm{G_I}, \mathrm{G_{II}}$, respectively.

(10): We numerically estimate the various properties, the qualities and the quantities of these graphs $\mathrm{G_I}, \mathrm{RG_I}, \mathrm{G_{II}}, \mathrm{RG_{II}}$.

4. Fluctuation of k-cycles

A cycle basis of an undirected graph is a set of simple cycles that forms a basis of the cycle space of the graph. There are so many various definitions of cycle basis, but here we use a fundamental cycle basis formed from the minimum spanning tree of the given graph. Here we calculate the distributions of number of k-cycle, which has k vertices, by using the cycle basis. For the various random graphs constructed in section 3, using the cycle basis and estimating the numbers of each cycle in these graphs, we statistically calculate the fitting curves to the distributions of these cycle numbers and we estimate their fluctuation exponents. We call this value the Cycle-Fluctuation Exponent (abbr. CFE) on the analogy of the exponent β in $(1/f)^\beta$ fluctuation.

By using the sagemath command "graphs.cycle_basis", we can easily count the numbers of each k-cycle in the list of the cycle basis and draw their histograms, which show the distributions of the numbers of k-cycles from $k = 3$ to around 20. Since we consider that the stability of the cyclic structure is most important, that is, each selected k-cycle cannot be constructed by some symmetric differences of $k'(< k)$-cycles in the cycle basis, we use the numbers of the 3-cycles not only in the basis, but considering all 3-cycles of the graph, and the k-cycles in the cycle basis for $k \geq 4$. Under this revision for the 3-cycles we obtain the following numerical results.

[$\mathbf{G_I}$]

Fig.3 and Fig.4: By using the sagemath command "min_spanning_tree" we draw the spanning tree and the graph of all edges with $|E| = 1117$. Since the warm colored vertices have higher connectivity between the same color vertices, we can admit the clusters of the red, the pink and the yellow vertices, while the cold colored vertices are dissipative in the minimum spanning trees. In the other random cases all vertices are completely mixed in $\mathrm{RG_I}$ (Fig.7), $\mathrm{RG_{II}}$(Fig.15) and we can find the clusters of the purple and the other color vertices in $\mathrm{G_{II}}$ (Fig.11), which have higher connectivity between them, while the purple vertices themselves are dissipative.

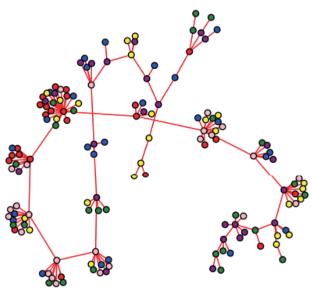

FIG 3. $\mathrm{G_I}$: Spantree

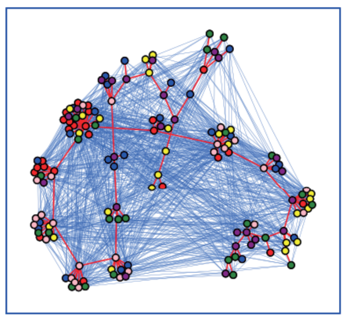

FIG 4. $\mathrm{G_I}$: $|E| = 1117$

Fig.5 and Fig.6: The fitting line on the log-log scale by linear regression and the numbers of k-cycle in the cycle-basis with its fitting curve: $f(k) \propto (1/k)^\beta$, CFE: $\beta = 1.8928$

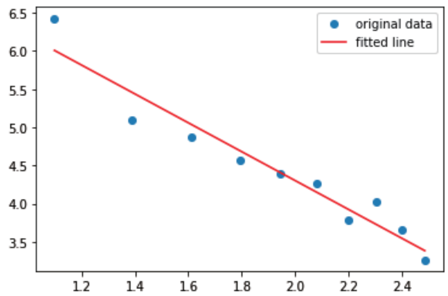

FIG 5. $\mathrm{G_I}$: LinRegress

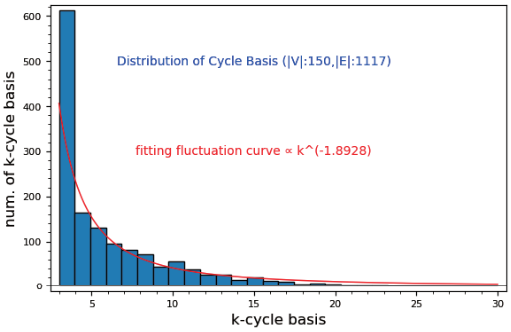

FIG 6. $\mathrm{G_I}$: CFE

[$\mathbf{RG_I}$]

Fig.7 and Fig.8: The spanning tree and the graph of all edges with $|E| = 1117$.

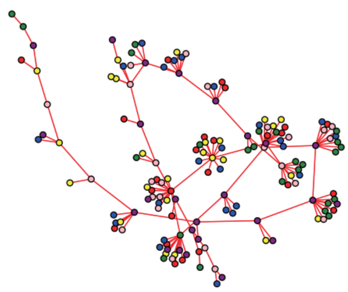

FIG 7. $\mathrm{RG_I}$:Spantree

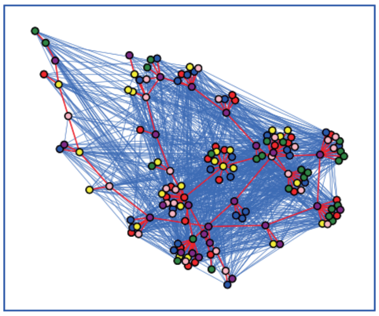

FIG 8. $\mathrm{RG_I}$: $|E| = 1117$

Fig.9 and Fig.10: The fitting line on the log-log scale by linear regression and the numbers of k-cycle in the cycle-basis with its fitting curve: $f(k) \propto (1/k)^{\beta}$, CFE: $\beta = 1.7852$

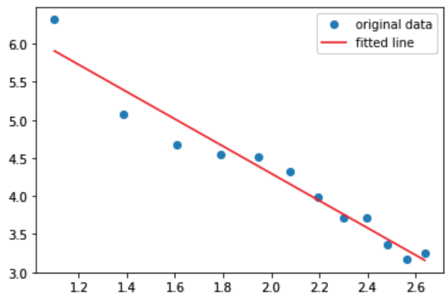

FIG 9. $\mathrm{RG_I}$: LinRegress

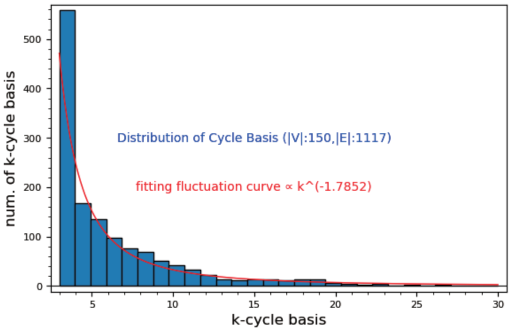

FIG 10. $\mathrm{RG_I}$: CFE

[$\mathbf{G_{II}}$]

Fig.11 and Fig.12: The spanning tree and the graph of all edges with $|E| = 1604$.

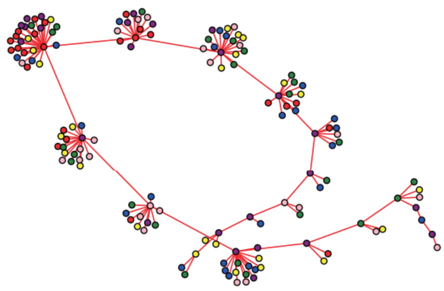

FIG 11. G_{II} : Spantree

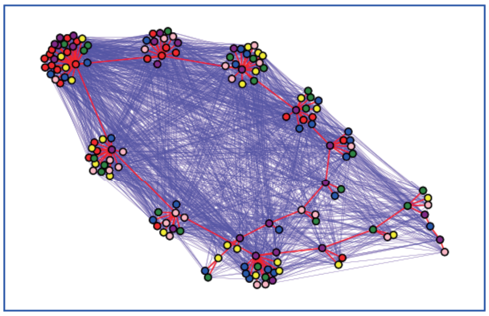

FIG 12. G_{II} : $|E| = 1604$

Fig.13 and Fig.14: The fitting line on the log-log scale by linear regression and the numbers of k-cycle in the cycle-basis with its fitting curve: $f(k) \propto (1/k)^{\beta}$, CFE: $\beta = 3.5682$

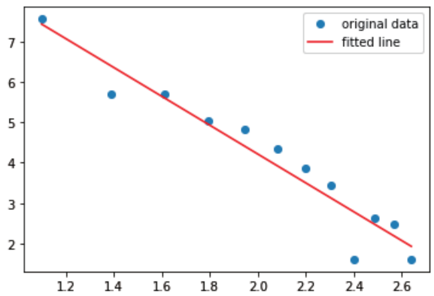

FIG 13. G_{II}: LinRegress

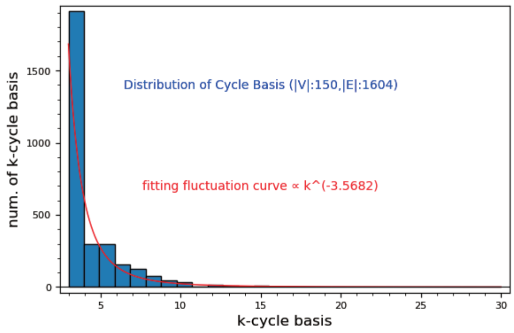

FIG 14. G_{II}: CFE

[$\mathbf{RG_{II}}$]

Fig.15 and Fig.16: The spanning tree and the graph of all edges with $|E| = 1604$.

Fig.17 and Fig.18: The fitting line on the log-log scale by linear regression and the numbers of k-cycle in the cycle-basis with its fitting curve: $f(k) \propto (1/k)^{\beta}$, CFE: $\beta = 2.6543$

5. Small-worldness coefficient

For a vertex v, which has its degree value k, in a graph $G(V, E)$, the (local) clustering coefficient $C(v)$ is defined by

$$C(v) = \frac{2\lambda(v)}{k(k-1)}$$

where $\lambda(v)$ is the the number of triangles on v, that is, the number of subgraphs of G with 3 edges and 3 vertices, one of which is v and $k(k-1)/2$ is the number

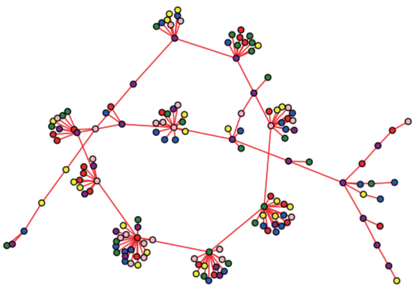

FIG 15. $\mathrm{RG_{II}}$: Spantree

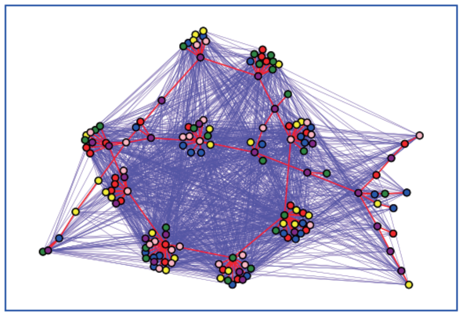

FIG 16. $\mathrm{RG_{II}}$: $|E| = 1604$

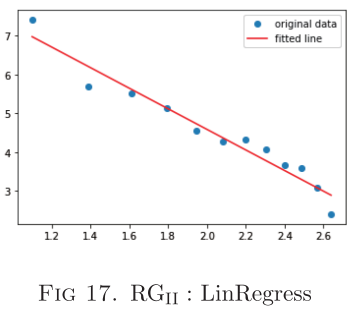

FIG 17. $\mathrm{RG_{II}}$: LinRegress

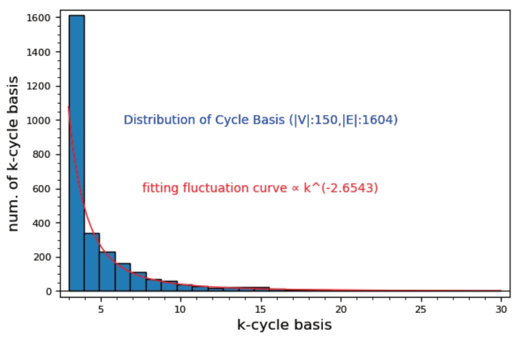

FIG 18. $\mathrm{RG_{II}}$: CFE

of triples on $v \in G$, that is, the number of subgraphs with 2 edges and 3 vertices, one of which is v and such that v is incident to both edges. The average of the clustering coefficients C is given by

$$C = \frac{1}{|V|} \sum_{v \in V} C(v).$$

The small-worldness coefficient σ is defined by

$$\sigma = \frac{(C/C_r)}{(L/L_r)}$$

where C and L are the average clustering coefficient and the average path length of the given graph and C_r and L_r are those of an equivalent random graph, respectively. It is calculated by comparing clustering and path length of a given graph to an equivalent random graph and it is known that if $\sigma > 1 : C \gg C_r, L \approx L_r$, the network, given by this graph, has small-worldness and, as $\sigma \downarrow 1$, the randomness of this network is increasing.

While the cluster analysis or clustering is very well-known for the recent epidemic or pandemic disease, the clustering states, mainly estimated by the clustering coefficients, give one of the most important indicators, which characterize the small-worldness of graph networks. We calculate the average of clustering coefficients of each nonregular graph to show the relations with the randomness and we estimate the small-worldness coefficients, which also give the level of randomness, of our nonregular graphs.

Since the random values of the clustering coefficient C_r and the average path length L_r have rather large variance and instability, we take their mean values from the 1000 times trials. Since we use the pseudo-random process given by the p-adic chaotic system, the constants C, L are invariant if the seeds and the other parameters are not changed.

Here we give the table of the small-worldness coefficients σ with the constants, C, L, C_r, L_r in our random nonregular graphs $\mathrm{G_I}, \mathrm{RG_I}, \mathrm{G_{II}}, \mathrm{RG_{II}}$ and also, with the exponent CFE to compare the small-worldness coefficients to the distributions of k-cycle numbers of the cycle-basis.

Gr \Const	$\lvert E\rvert$	C_r	L_r	C	L	σ	CFE
$\mathrm{G_I}$	1117	0.100	2.103	0.105	2.108	1.048	1.893
$\mathrm{RG_I}$	1117	0.100	2.103	0.101	2.106	1.006	1.785
$\mathrm{G_{II}}$	1604	0.144	1.896	0.168	1.904	1.164	3.568
$\mathrm{RG_{II}}$	1604	0.144	1.895	0.144	1.894	1.013	2.654

While the small-world coefficients are essentially related to the triangles, that is, 3-cycles of the graph, CFE are calculated by k-cycle basis for all $k \geq 3$ and the diffrence values of CFE between $\mathrm{G_I}$ and $\mathrm{G_{II}}$ are greater than those of SWC. We can say that CFE contains more information on clustering or dissipative structures of graphs than SWC.

6. Graph Riemann hypothesis and poles of Ihara zeta functions

In Section 2 for the regular Ramanujan graph case we see that the poles of the Ihara zeta function $\zeta_X(u)$, $u = q^{-s} = R_X^s$, satisfies Graph Riemann Hypothesis, that is, they are just on the circle $|u| = \sqrt{R_X}$, Re $s = 1/2$, when $0 <$ Re $s < 1$. The value R_X, defined in section 2 as the radius of the largest circle of convergence of $\zeta_X(u)$, plays an important role in the nonregular graph. We call $\sqrt{R_X}$ Ramanujan radius and the circle with its radius $\sqrt{R_X}$ Ramanujan circle.

In this section we numerically calculate the distribution of poles of $\zeta_X(u)$ for our nonregular pseudorandom graphs, using the determinant formula given by Bass in [2]. We use the same notations and the definitions as those in Section 2, but here we consider the nonregular graphs.

Theorem 6.1 ([2]). *Let A be the adjacency matrix of X and Q the diagonal matrix with jth diagonal entry q_j such that q_j+1 is the degree of the jth vertex of X. Then we have the Ihara three-term determinant formula*

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1}\det(I - Au + Qu^2)$$

where r is the rank of th fundamental group of X; $r-1 = |E| - |V|$.

Following the results obtained by Kotani and Sunagawa in [7] on nonregular graphs, we investigate the 'weak Graph Riemann Hypothesis'.

Theorem 6.2 ([7])**.** *Suppose that a graph X has vertices with maximum degree $q+1$ and minimum degree $p+1$. Then*

(1) *Every pole of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$ and*

$$q^{-1} \leq R_X \leq p^{-1}. \tag{6.1}$$

(2) *Every non-real pole of $\zeta_X(u)$ satisfies the inequality*

$$\frac{1}{\sqrt{q}} \leq |u| \leq \frac{1}{\sqrt{p}}. \tag{6.2}$$

In [15] Terras defined the graph theory Riemann Hypothesis (abbr. GRH) by the following pole free region of $\zeta_X(u)$,

$$R_X < |u| < \sqrt{R_X}.$$

and the weak graph theory Riemann Hypothesis (abbr. wGRH) by the following pole free region of $\zeta_X(u)$,

$$R_X < |u| < \frac{1}{\sqrt{q}}.$$

In our previous paper [9] we have already shown that the almost all poles of Ihara zeta functions of our random graphs defined by p-adic pseudorandom numbers satisfy wGRH.

In this paper, by calculating the distribution of poles of the Ihara zeta functions we directly investigate whether these almost Ramanujan graphs satisfy the graph Riemann Hypothesis. Comparing to the Ramanujan circle, the radius of which is used as a criteria related to the graph Riemann Hypothesis, we calculate the mean, denoted by M_z, and the standard deviation, denoted by σ_z, of the absolute values of the poles and we estimate the difference and the variation from the Ramanujan circle. Here we propose the ratio of this mean value to the radius of the Ramanujan circle, called Ramanujan radius ratio, denoted by RRr $(= M_z/\sqrt{R_X})$, and the deviation of these distributions, called the Ramanujan radius deviation, denoted by RRd $(= \sigma_z/\sqrt{R_X})$, as characteristic constants which show levels or some properties of randomness of these given graphs.

We plot the five circles colored by green, purple, red, brown and blue as follows.

- The green circle is $|u| = R_X$.
- The purple circle is $|u| = 1/\sqrt{q}$.
- The red circle is $|u| = \sqrt{R_X}$.
- The brown circle is $|u| = M_z$.
- The blue circle is $|u| = 1/\sqrt{p}$.

The following inequalities

$$R_X < \frac{1}{\sqrt{q}} < \sqrt{R_X} < M_z < \frac{1}{\sqrt{p}}$$

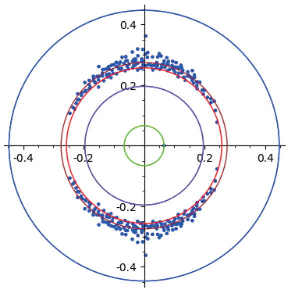

FIG 19. $\mathrm{G_I}$: ZetaPolDist.

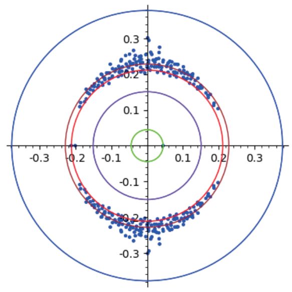

FIG 20. $\mathrm{RG_I}$: ZetaPolDist

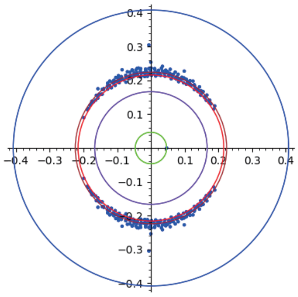

FIG 21. $\mathrm{G_{II}}$: ZetaPolDist.

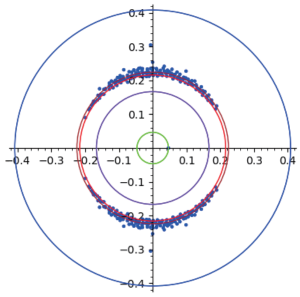

FIG 22. $\mathrm{RG_{II}}$: ZetaPolDist

hold.

- $\sqrt{R_X}$: Ramanujan radius
- M_z: mean of the absolute values of the poles
- $\mathrm{RRr} := M_z/\sqrt{R_X}$, Ramanujan radius ratio
- σ_z: standard deviation of the absolute values of the poles
- $\mathrm{RRd} := \sigma_z/\sqrt{R_X}$, Ramanujan radius deviation

Gr \Const	$\|E\|$	$\sqrt{R_X}$	M_z	RRr	σ_z	RRd	CFE
$\mathrm{G_I}$	1117	0.257	0.275	1.069	0.02	0.078	1.893
$\mathrm{RG_I}$	1117	0.259	0.274	1.056	0.018	0.068	1.785
$\mathrm{G_{II}}$	1604	0.211	0.228	1.08	0.019	0.091	3.568
$\mathrm{RG_{II}}$	1604	0.216	0.224	1.039	0.015	0.068	2.654

We can see that the values $\sqrt{R_X}, M_z$ depend on the numbers of edges and the larger values RRr and RRd of $\mathrm{G_{II}}$ than those of the others show some clustering and deviating properties in the graph.

While the small worldness coefficients depend on only 3-cycles, RRr and RRd are derived from the Ihara zeta function, which are defined by using all cycles in the graph and CFE are directly given from all cycles of the cycle basis. These three constants, RRr, RRd and CFE, should contain information on not only clustering properties but more complex structural properties derived from their cyclic structure in each graph. The mathematical analysis on the relations between the distribution of poles of Ihara zeta function and the numbers distribution of k-cycles of cycle basis will clarify this information and will generate the most fruitful results on structure and functional connectivity problems of graph networks.

7. Almost Ramanujan graph

In section 3 we construct the adjacency matrices T_I, T_{II} for the nonregular graphs $\mathrm{G_I}, \mathrm{G_{II}}$, using the sparse matrix C_I, C_{II}, respectively. The various types of almost Ramanujan graphs by using the upper bounds of nontrivial eigenvalues of their adjacency matrices can be defined (cf. [15]). Corresponding to the regular graph case we can consider that the trivial eigenvalue in the nonregular case is the one which has the largest absolute value.

We say that a non-regular graph X is a naive Ramanujan graph if

$$|\mu| \leq 2\sqrt{\sigma_X - 1}$$

for every nontrivial eigenvalue $\mu : |\mu| \lneqq \sigma_X$ where σ_X is the largest absolute value of eigenvalues of the adjacency matrix A,

$$\sigma_X = \max\{|\mu| : \mu \in \mathrm{Spectrum} A\}.$$

The degree of a vertex v_i in the graph X is the number of edges joining v_i,

$$\sum_{v_j \in V} a_{ij}, \quad a_{i,j} \in \{0, 1\}.$$

Let $\overline{d_X}$ be the average degree of the vertices of X. We say that a non-regular graph X is a weak Ramanujan graph if

$$|\mu| \leq 2\sqrt{\overline{d_X} - 1}$$

for every nontrivial eigenvalue $\mu : |\mu| \lneqq \sigma_X$.

In our numerical experiments, using the maximal degree D_X, we say that a non-regular graph X is a mild Ramanujan graph if the following inequality hold

$$|\mu| \leq 2\sqrt{D_X - 1}$$

for every nontrivial eigenvalue $\mu : |\mu| \lneqq \sigma_X$.

In the histograms showing the distributions of the eigenvalues we plot these upper bound values colored as follows:

$2\sqrt{D_X - 1}$: mild Ramanujan bounds (green diamond marker ♦)

$> 2\sqrt{\sigma_X - 1}$: naive Ramanujan bounds (red diamond marker ♦)

$> 2\sqrt{\overline{d_X} - 1}$: weak Ramanujan bounds (blue diamond marker ♦).

According to Random Matrix Theory, Wigner semi-circle law describes the asymptotic behavior of eigenvalue distributions of large symmetric random matrices. The Wigner semi-circle distribution is the probability distribution supported on the interval $[-R, R]$ the graph of whose probability density function $f(x)$ defined by

$$f(x) = \frac{2}{\pi R^2}\sqrt{R^2 - x^2}$$

for $-R \leq x \leq R$ and $f(x) = 0$ if $|x| > R$.

If the eigenvalue distributions of our adjacency matrices are matching with this semi-circle, we can see sufficiently high randomness of our pseudorandom data. In these numerical results the eigenvalues distribution of T_{II}, which is the adjacency matrix of G_{II}, shows some nonuniform randomness, not matching with the Wigner semi-circle.

All absolute values of nontrivial eigenvalues are smaller than the upper bound of the mild Ramanujan bound and almost all nontrivial eigenvalues are also under the upper bounds of the naive and weak Ramanujan bounds. For the definitions of almost Ramanujan graphs the weak (blue marker) and the naive (red marker) Ramanujan bounds are just fitting to the upper bounds of the nontrivial eigenvalues while the mild Ramanujan bounds given by the green diamond marker are too mild.

All non-real poles in our fourcases satisfy the inequalities $1/\sqrt{q} \leq |u| \leq 1/\sqrt{p}$, given by Kotani-Sunagawa and also satisfy wGRH and approximately satisfy GRH, the definitions of which were given by Terras.

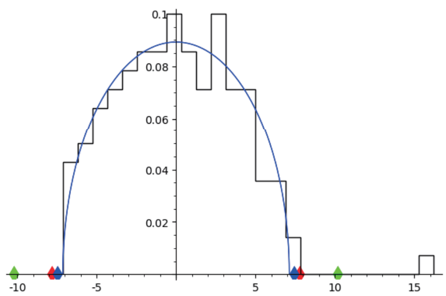

Fig 23. G_{I} : Eigenval.Dist.

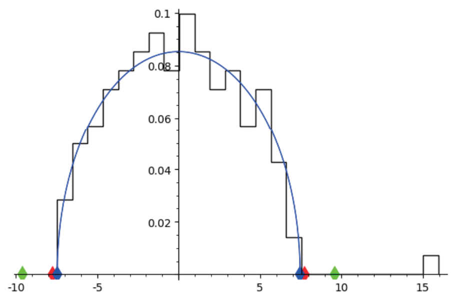

Fig 24. RG_{I} : Eigenval.Dist

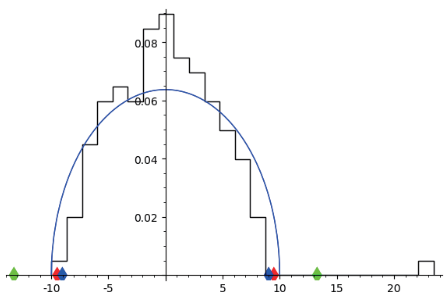

Fig 25. G_{II} : Eigenval.Dist

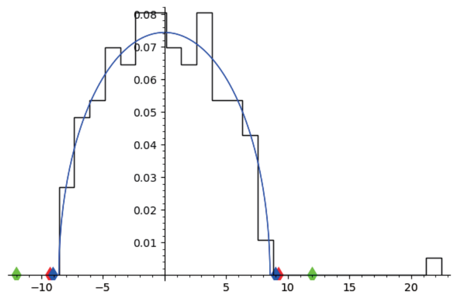

Fig 26. $\mathrm{RG}_{\mathrm{II}}$: Eigenval.Dist

8. Indra'net: Guidance from Taiwan and Wakayama

The most important objects of our random graphs in this paper are the spanning trees, which are the fundamental tools to obtain cycle-basis. When we prepare and plot these tree-graphs, we find the following two similar graphs drawn by Bunzo

Hayata (Fig.27 in [5]) and Kumagusu Minakata (Fig. 28 referred in [8]). Bunzo Hayata was a famous botanist, especially, for his taxonomic work in Taiwan. About his image graph in [5] he described that

> $\cdots$. The universe is like a boundless net with innumerable millions of crystalline beads, each on a mesh of a different colour, each reflecting the images of other beads, and each consequently presenting different hues, according to the position of the observer. The beads present different hues, according as they are observed from this point or that. It is, however, only in their phenomena that they are different; in their real entities, they are all and ever the same crystalline beads. $\cdots$ I have been influenced by a suggestion from the Indra - nets, an allegory found in one of the Buddhist scriptures, which is called the Mahavai- pulyabuddhaganda vyūha-sūtra (Kegonkyô) $\cdots$

A great scientist Kumagusu Minakata was born in Wakayama, 1883. He is well-known all over the world as a biologist, naturalist, ethnologist, folklorist, environmentalist, philosopher... . In 1903 he explained Figure 28 of the mandala system in his letter to Dogi Horyu (cf. [8]).

> Curiously enough, this universe is composed of innumerable lines of logic and causality coming from all the directions as in this model (although this model could not be drawn but on a plane surface, it actually should be supposed as three dimensional with depth besides length and width). As Confucians say, the heaven's way is logic (logic means causality here). It is therefore possible to reach to everywhere and do everything starting from any point if you make a complete pursuit of it.

The common keyword of their description on the complex networks is Indra's net in "Kegonkyo". About Indra's net was also described in a short story with the same title by Kenji Miyazawa [6].

> ...
>
> The heavenly children jumped up and down in rapture, running over the silica sand of the pure-blue lake of True Enlightenment. Then suddenly one of the children bumped into me, and jumping back, screamed out while pointing up to the sky.
>
> " Look, look, look at Indra's net! "
>
> I looked up at the sky. The zenith was now azure blue, and from it to the four corners of the pale edges of the sky, Indra's spectral net vibrated radiantly as if burning, its fibers more fine than a spider's web, its construction more elaborate than that of hypha, all blending together transparently, purely, in a billion intermingled arts.
>
> ...

Kenji also described Indra'net as a fractal net in his other short story by showing the infinite numbers of knots of precious stones which have the fractal property such that each precious stone contains and reflects the whole net-world. This fractal net must be the main mathematical object in our forthcoming papers where we can more essentially use the theory of p-adic numbers, the set of which contains the fractal properties.

Professor Wataru Takahashi made his greatest contribution to the mathematical society in not only Japan but especially in Taiwan and all over the world.

Professor Naoki Shioji, who was born in Wakayama, also made his greatest contribution to the mathematical society, especially to the PDE society all over the world.

We shall never forget your greatest achievements in mathematics and also, your kindness and guidance showing us what to study in future.

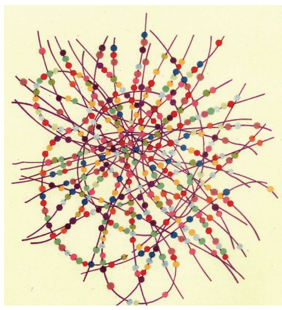

Fig 27. by B.Hayata

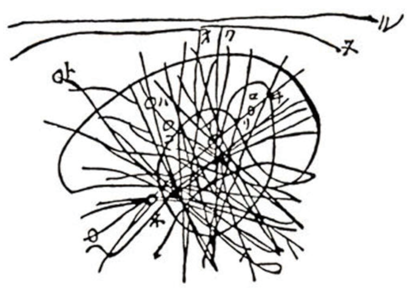

Fig 28. by K.Minakata

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