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GENERIC CONVERGENCE OF POWERS OF GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. In our 2014 work with M. Gabour we introduced a class of generalized nonexpansive self-mappings of a bounded and closed subset of a Banach space and studied, using the Baire category approach, the convergence of iterates of a generic mapping in this class to its unique fixed point. In the present paper we study the generic convergence of powers of generalized nonexpansive mappings to generalized nonexpansive retractions onto the set of their fixed points.

1. Introduction

For nearly sixty years now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 3, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 22, 23, 28, 29] and references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed point problems and variational inequalities, which find important applications in engineering, medical and the natural sciences [4, 8, 9, 26, 27, 28, 29].

In [7] we considered the following class of nonlinear mappings.

Let $(X, \|\cdot\|)$ be a Banach space and let K be a bounded, closed and convex subset of X. Let $f : X \to [0, \infty)$ be a continuous function such that $f(0) = 0$, the set $f(K - K)$ is bounded, and the following three properties hold:

(P1) for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in K$ satisfy $f(x - y) \leq \delta$, then $\|x - y\| \leq \epsilon$;

(P2) for each $\lambda \in (0, 1)$, there is $\phi(\lambda) \in (0, 1)$ such that

$$f(\lambda(x - y)) \leq \phi(\lambda) f(x - y) \text{ for all } x, y \in K;$$

(P3) the function $(x, y) \mapsto f(x - y)$, $x, y \in K$, is uniformly continuous on $K \times K$.

Denote by $\mathcal{A}$ the set of all continuous mappings $A : K \to K$ such that

$$f(Ax - Ay) \leq f(x - y) \text{ for all } x, y \in K. \tag{1.1}$$

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Set

$$\operatorname{diam}(K) := \sup\{\|x - y\| :\ x, y \in K\}. \tag{1.2}$$

We denote the identity operator by A^0 for every mapping $A \in \mathcal{A}$.

For each $A, B \in \mathcal{A}$, set

$$d(A, B) := \sup\{\|Ax - Bx\| :\ x \in K\}.$$

It is clear that $(\mathcal{A}, d)$ is a complete metric space.

In [7] we established the existence of a set $\mathcal{F}$, which is a countable intersection of open and everywhere dense subsets of $\mathcal{A}$, such that each $C \in \mathcal{F}$ has a unique fixed point and all its iterates converge uniformly to this fixed point.

Note that the classical result of De Blasi and Myjak [5] is a particular case of this result where $f = \|\cdot\|$. As a matter of fact, the mappings defined above can be considered generalized nonexpansive mappings with respect to f. Such an approach, where in some problems of functional analysis the norm is replaced by a general function, was used in [20, 21] in the study of generalized best approximation problems.

These generalized nonexpansive mappings were also studied in [24, 25]. In particular, in [25] we constructed an example of a generalized nonexpansive self-mapping of a bounded, closed and convex set in a Hilbert space, which is not nonexpansive in the classical sense.

2. The main result

Assume that the function f is convex, F is a nonempty, closed and convex subset of K, and that $Q : K \to F$ is a generalized nonexpansive retraction, namely,

$$Qx = x,\ x \in F \tag{2.1}$$

and

$$f(Qx - Qy) \le f(x - y) \text{ for all } x, y \in K. \tag{2.2}$$

Denote by $\mathcal{A}^{(F)}$ the set of all mappings $A \in \mathcal{A}$ such that

$$Ax = x,\ x \in F. \tag{2.3}$$

Clearly, $\mathcal{A}^{(F)}$ is a closed subset of the metric space $(\mathcal{A}, d)$. We now consider the complete metric space $(\mathcal{A}^{(F)}, d)$ and establish the following result. Note that in the case where $f = \|\cdot\|$, a prototype of this result was obtained in [17] (see also [23]).

Theorem 2.1. *There exists a set $\mathcal{F} \subset \mathcal{A}^{(F)}$, which is a countable intersection of open and everywhere dense sets in the complete metric space $(\mathcal{A}^{(F)}, d)$, such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. There exists a mapping $P_* : K \to F$ such that

$$\lim_{n\to\infty} B^n x = P_* x$$

for all $x \in K$.

2. For each $\epsilon > 0$, there exist a neighborhood U of B in $\mathcal{A}^{(F)}$ and a natural number N such that for each integer $T \geq N$, each $C_t \in U$, $t = 1, \dots, T$, and each $x \in K$,

$$\|C_T \dots C_1 x - P_* x\| \leq \epsilon.$$

3. Auxiliary results

For each bounded operator $A : K \to X$, we set

$$\|A\| := \sup\{\|Ax\| : x \in K\}.$$

In the sequel we use the following auxiliary result (Lemma 6.9 of [23]).

Lemma 3.1. *Assume that E is a nonempty uniformly equicontinuous set of operators $A : K \to K$, N is a natural number and ϵ is a positive number. Then there exists a number $\delta > 0$ such that for each sequence $\{A_t\}_{t=1}^N \subset E$, each sequence $\{B_t\}_{t=1}^N$, where the (not necessarily continuous) operators $B_t : K \to K$, $t = 1, \dots N$, satisfy*

$$\|B_t - A_t\| \leq \delta, \quad t = 1, \dots N,$$

and each $x \in K$, the following inequality holds:

$$\|B_N \dots \dot{B}_1 x - A_N \dots A_1 x\| \leq \epsilon.$$

Property (P1), (1.1) and the continuity of f imply the following result.

Lemma 3.2. *The set of operators $\mathcal{A}$ is uniformly equicontinuous.*

4. Proof of Theorems 2.1

Let $A \in \mathcal{A}^{(F)}$ and $\gamma \in (0, 1)$. Define $A_\gamma : K \to K$ by

$$A_\gamma x := (1 - \gamma)Ax + \gamma Qx, \quad x \in K. \tag{4.1}$$

By (2.1), (2.3) and (4.1),

$$A_\gamma x = x, \ x \in F. \tag{4.2}$$

In view of (1.1), (2.1) and (4.1), we have

$$\begin{aligned} f(A_\gamma x - A_\gamma y) &= (f(1 - \gamma)Ax + \gamma Qx - (1 - \gamma)Ay - \gamma Qy) \\ &= f((1 - \gamma)(Ax - Ay) + \gamma(Qx - Qy)) \\ &\leq (1 - \gamma)f(Ax - Ay) + \gamma f(Qx - Qy) \\ &\leq f(x - y). \end{aligned} \tag{4.3}$$

By (4.2) and (4.3),

$$A_\gamma \in \mathcal{A}^{(F)}.$$

In view of (1.2) and (4.1),

$$d(A_\gamma, A) = \sup\{\gamma\|Qx - Ax\| : \ x \in K\} \leq \gamma \mathrm{diam}(K). \tag{4.4}$$

Lemma 4.1. *Let $A \in \mathcal{A}^{(F)}$, $\gamma \in (0, 1)$ and $x \in K$ be given. Then*

$$\inf\{f(A_\gamma x - z) : \ z \in F\} \leq \phi(1 - \gamma) \inf\{f(x - z) : \ z \in F\}. \tag{4.5}$$

Proof. Let $\epsilon > 0$ be given. There exists

$$z_\epsilon \in F \tag{4.6}$$

such that

$$f(x - z_\epsilon) \le \inf\{f(x - z) : \ z \in F\} + \epsilon. \tag{4.7}$$

Property (P2), (1.1), (2.3), (4.1), (4.6) and (4.7) imply that

$$\begin{aligned}\inf\{f(A_\gamma x - z) : \ z \in F\} &\le \inf\{f((1-\gamma)Ax + \gamma Qx - (\gamma Qx + (1-\gamma)z_\epsilon))\\ &= f((1-\gamma)Ax - (1-\gamma)z_\epsilon)\\ &\le \phi(1-\gamma)f(Ax - z_\epsilon) \le \phi(1-\gamma)f(x - z_\epsilon)\\ &\le \phi(1-\gamma)\inf\{f(x-z) : \ z \in F\} + \phi(1-\gamma)\epsilon.\end{aligned}$$

Since ϵ is an arbitrary positive number, inequality(4.5) is indeed true and Lemma 4.1 is proved. $\square$

Let $A \in \mathcal{A}^{(F)}$, $\gamma \in (0,1)$ and let $q \ge 1$ be an integer. Property (P1) implies that there exists a number $\lambda_q \in (0,1)$ such that

$$\text{if } x, y \in K \text{ satisfy } f(x-y) \le 2\lambda_q, \text{ then } \|x - y\| \le (4q)^{-1}. \tag{4.8}$$

Choose an integer $n(\gamma, q) \ge 1$ such that

$$\phi(1-\gamma)^{n(\gamma,q)}[\sup\{f(z - z') : \ z, z' \in K\}] < 4^{-1}\lambda_q. \tag{4.9}$$

Lemma 4.1 and (4.9) imply that for all $x \in K$,

$$\inf\{f(A_\gamma^{n(\gamma,q)}x - z) : \ z \in F\}$$

$$\le \phi(1-\gamma)^{n(\gamma,q)}\inf\{f(x-z) : \ z \in F\} < 4^{-1}\lambda_q. \tag{4.10}$$

Property (P3) implies that there exists $\delta_0 \in (0,1)$ such that the following property holds:

(a) if $z_1, z_2, \xi_1, \xi_2 \in K$ satisfy $\|z_i - \xi_i\| \le \delta_0$, $i = 1, 2$, then

$$|f(z_1 - z_2) - f(\xi_1 - \xi_2)| \le 4^{-1}\lambda_q.$$

Lemmas 3.1 and 3.2 imply that there exists an open neighborhood

$$\mathcal{U}(A, \gamma, q)$$

of A_γ in $\mathcal{A}^{(F)}$ such that the following property holds:

(b) for each $x \in K$ and each $\{B_t\}_{t=1}^{n(\gamma,q)} \in \mathcal{U}(A, \gamma, q)$, we have

$$\|A_\gamma^{n(\gamma,q)}x - B_{n(\gamma,q)} \dots B_1 x\| \le \delta_0.$$

Properties (a) and (b) imply that for each $x \in K$ and each

$$\{B_t\}_{t=1}^{n(\gamma,q)} \in \mathcal{U}(A, \gamma, q),$$

we have

$$\inf\{f(A_\gamma^{n(\gamma,q)}x - z) : \ z \in F\} - \inf\{f(B_{n(\gamma,q)} \dots B_1 x - z) : \ z \in F\}| \le 4^{-1}\lambda_q, \tag{4.11}$$

and in view of (4.10) and (4.8),

$$\inf\{f(B_{n(\gamma,q)}\dots B_1x - z) : \ z \in F\} \le 2^{-1}\lambda_q \tag{4.12}$$

and

$$\inf\{\|B_{n(\gamma,q)}\dots B_1x - z\| : \ z \in F\} \le (4q)^{-1}. \tag{4.13}$$

Define

$$\mathcal{F} := \cap_{p=1}^{\infty} \cup \{\mathcal{U}(A,\gamma,q) : A \in \mathcal{A}^{(F)}, \ \gamma \in (0,1), q = p, p+1, \dots\}. \tag{4.14}$$

Evidently, $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $\mathcal{A}^{(F)}$. Assume that

$$B \in \mathcal{F} \tag{4.15}$$

and that $\epsilon > 0$. Choose a natural number p such that

$$p^{-1} < \epsilon. \tag{4.16}$$

By (4.14) and (4.15), there exist $A \in \mathcal{A}^{(F)}$, $\gamma \in (0,1)$ and an integer $q \ge p$ such that

$$B \in \mathcal{U}(A,\gamma,q). \tag{4.17}$$

Let $x \in K$. In view of (4.12),

$$\inf\{f(B^{n(\gamma,q)}x - z) : \ z \in F\} \le 2^{-1}\lambda_q. \tag{4.18}$$

By (4.18), there exists

$$z_0 \in F \tag{4.19}$$

such that

$$f(B^{n(\gamma,q)}x - z_0) \le \lambda_q. \tag{4.20}$$

It follows from (1.1), (2.3), (4.17), (4.19) and (4.20) that for each integer $T \ge n(\gamma,q)$,

$$f(B^Tx - z_0) \le f(B^{n(\gamma,q)}x - z_0) \le \lambda_q, \tag{4.21}$$

and in view of (4.8) and (4.16),

$$\|B^Tx - z_0\| \le (4q)^{-1} < \epsilon. \tag{4.22}$$

Since ϵ is any positive number, (4.22) implies that $\{B^tx\}_{t=1}^{\infty}$ is a Cauchy sequence, there exists $\lim_{t\to\infty} B^tx$ and

$$\|\lim_{t\to\infty} B^tx - z_0\| \le \epsilon. \tag{4.23}$$

Since ϵ is an arbitrary positive number, (4.19) and (4.23) imply that

$$\lim_{t\to\infty} B^tx \in F. \tag{4.24}$$

Set

$$P_*x := \lim_{t\to\infty} B^tx. \tag{4.25}$$

Clearly, $P_* \in \mathcal{A}^{(F)}$. By (4.22) and (4.25), we have

$$\|P_*x - z_0\| \le (4q)^{-1}, \ x \in K. \tag{4.26}$$

Assume that $x \in K$, $T \geq n(\gamma, q)$ is an integer, and that

$$C_t \in \mathcal{U}(A, \gamma, q),\ t = 1, \dots, T. \tag{4.27}$$

Properties (a) and (b), and (4.27) imply that

$$|f(A_\gamma^{n(\gamma,q)}x - z_0) - f(C_{n(\gamma,q)} \dots C_1 x - z_0)| \leq 4^{-1}\lambda_q$$

and

$$|f(B^{n(\gamma,q)}x - z_0) - f(A_\gamma^{n(\gamma,q)}x - z_0)| \leq 4^{-1}\lambda_q. \tag{4.28}$$

By (4.20) and (4.28),

$$f(C_{n(\gamma,q)} \dots C_1 x - z_0) \leq 2\lambda_q. \tag{4.29}$$

It now follows from (1.1), (2.3), (4.19) and (4.29) that

$$f(C_T \dots C_1 x - z_0) \leq 2\lambda_q.$$

When combined with (4.8), this implies that

$$\|C_T \dots C_1 x - z_0\| \leq (4q)^{-1}.$$

Combining this inequality with (4.16) and (4.26), we arrive at the inequality

$$\|C_T \dots C_1 x - P_* x\| \leq (2q)^{-1} < \epsilon.$$

This completes the proof of Theorem 2.1.

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References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. **3** (1922), 133–181.

[2] A. Betiuk-Pilarska and T. Domínguez Benavides, *Fixed points for nonexpansive mappings and generalized nonexpansive mappings on Banach lattices*, Pure Appl. Func. Anal. **1** (2016), 343–359.

[3] D. Butnariu, S. Reich and A. J. Zaslavski, *Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, 2006, pp. 11–32.

[4] Y. Censor and M. Zaknoon, *Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review*, Pure Appl. Func. Anal. **3** (2018), 565–586.

[5] F. S. de Blasi and J. Myjak, *Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach*, C. R. Acad. Sci. Paris **283** (1976), 185–187.

[6] F. S. de Blasi, J. Myjak, S. Reich and A. J. Zaslavski, *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set-Valued and Variational Analysis **17** (2009), 97–112.

[7] M. Gabour, S. Reich and A. J. Zaslavski, *A generic fixed point theorem*, Indian J. Math. **56** (2014), 25–32.

[8] A. Gibali, *A new split inverse problem and an application to least intensity feasible solutions*, Pure Appl. Funct. Anal. **2** (2017), 243–258.

[9] A. Gibali, S. Reich and R. Zalas, *Outer approximation methods for solving variational inequalities in Hilbert space*, Optimization **66** (2017), 417–437.

[10] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.

[11] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, New York and Basel, 1984.

[12] J. Jachymski, *Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points*, Pure Appl. Funct. Anal. **2** (2017), 657–666.

[13] W. A. Kirk, *Contraction mappings and extensions*, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 1–34.

[14] R. Kubota, W. Takahashi and Y. Takeuchi, *Extensions of Browder's demiclosedness principle and Reich's lemma and their applications*, Pure Appl. Func. Anal. **1** (2016), 63–84.

[15] E. Pustylnyk, S. Reich and A. J. Zaslavski, *Convergence to compact sets of inexact orbits of nonexpansive mappings in Banach and metric spaces*, Fixed Point Theory and Applications **2008** (2008), 1–10.

[16] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459–465.

[17] S. Reich and A. J. Zaslavski, *Convergence of generic infinite products of nonexpansive and uniformly continuous operators*, Nonlinear Analysis **36** (1999), 1049–1065.

[18] S. Reich and A. J. Zaslavski, *Well-posedness of fixed point problems*, Far East J. Math. Sci., Special Volume (Functional Analysis and Its Applications), Part III (2001), 393–401.

[19] S. Reich and A. J. Zaslavski, *Generic aspects of metric fixed point theory*, Handbook of Metric Fixed Point Theory, Kluwer, Dordrecht, 2001, pp. 557–575.

[20] S. Reich and A. J. Zaslavski, *Well-posedness of generalized best approximation problems*, Nonlinear Func. Anal. Appl.**7** (2002), 115–128.

[21] S. Reich and A. J. Zaslavski, *Porous sets and generalized best approximation problems*, Nonlinear Anal. Forum **9** (2004), 135–152.

[22] S. Reich and A. J. Zaslavski, *Convergence to attractors under perturbations*, Commun. Math. Anal. **10** (2011), 57–63.

[23] S. Reich and A. J. Zaslavski, *Genericity in nonlinear analysis*, Developments in Mathematics, **34**, Springer, New York, 2014.

[24] S. Reich and A. J. Zaslavski, *Contractivity and genericity results for a class of nonlinear mappings*, J. Nonlinear Convex Anal. **16** (2015), 1113–1122.

[25] S. Reich and A. J. Zaslavski, *On a class of generalized nonexpansive mappings*, Mathematics, **2020**, 8, 1085, https://doi.org/10.3390/math8071085.

[26] W. Takahashi, *The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces*, Pure Appl. Funct. Anal. **2** (2017), 685–699.

[27] W. Takahashi, *A general iterative method for split common fixed point problems in Hilbert spaces and applications*, Pure Appl. Funct. Anal. **3** (2018), 349–369.

[28] A. J. Zaslavski, *Approximate solutions of common fixed point problems*, Springer Optimization and Its Applications, Springer, Cham, 2016.

[29] A. J. Zaslavski, *Algorithms for solving common fixed point problems*, Springer Optimization and Its Applications, Springer, Cham, 2018.

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