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SOME ITERATIVE ALGORITHMS FOR CONSTRAINED CONVEX MINIMIZATION, GENERALIZED MIXED EQUILIBRIUM AND FIXED POINT PROBLEMS

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This paper is dedicated to the memories of Professors Wataru Takahashi and Naoki Shioji.

Abstract. In this paper, we introduce two iterative algorithms (one implicit and one explicit) for finding a common element of the set of solutions to a constrained convex minimization problem for a convex function, the set of solutions to a generalized mixed equilibrium problem and the set of fixed points of a continuous pseudocontractive mapping in Hilbert spaces. Under suitable control conditions, we establish strong convergence of sequence generated by the proposed iterative algorithms to a common element of three sets, which is a solution of a certain variational inequality. As a direct consequence, we obtain the unique minimum-norm common element of three sets.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be self-mapping on C. We denote by $Fix(T)$ the set of fixed points of T.

The constrained convex minimization problem (shortly, CMP) is one of most important problems in nonlinear analysis and optimization theory. The CMP is defined as follows: find $x \in H$ such that

$$f(x) = \min_{y \in C} f(y), \tag{1.1}$$

where $f : C \to \mathbb{R}$ is a real-valued convex function. We denote the set of solutions to the CMP (1.1), that is, the set of minimizers of f, by $S := \arg\min_C f$.

It is well known that the gradient-projection algorithm (shortly, GPA) plays an important role in solving the constrained convex minimization problems. If f is (Fréchet) differentiable, then the GPA generates a sequence $\{x_n\}$ using the following recursive formula:

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 1, \tag{1.2}$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 1, \tag{1.3}$$

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where in both (1.2) and (1.3), the initial guess x_0 is taken from C arbitrarily, the parameters, λ or λ_n, are positive real numbers satisfying ceratin conditions, and P_C is the metric projection from H onto C. The convergence of algorithms (1.2) and (1.3) depends on the behavior of the gradient ∇f. As a matter of fact, it is known that if ∇f is α-strongly monotone and L-Lipschitzian with constants $\alpha,\ L > 0$, then the operator

$$T := P_C(I - \lambda \nabla f) \tag{1.4}$$

is a contractive mapping; hence the sequence $\{x_n\}$ generated by the algorithm (1.2) converges in norm to the unique solution of the CMP (1.1). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{2\alpha}{L^2},$$

then the sequence $\{x_n\}$ generated by the algorithm (1.3) converges in norm to the unique minimizer of (1.1). However, if the gradient ∇f fails to be strongly monotone, the operator T defined by (1.4) would fail to be contractive; consequently, the sequence $\{x_n\}$ generated by the algorithm (1.2) may fail to converges strongly (see [25]). In [25], Xu introduced an alternative operator-oriented approach to the algorithm (1.3); namely, an averaged mapping approach. He gave his averaged mapping approach to the GPA (1.3) and the relaxed gradient-projection algorithm. Combining the hybrid iterative method of Yamada [26] based on viscosity iterative method and the averaged mapping approach to the GPA of Xu [25], Ceng *et al.* [6] considered implicit and explicit iterative algorithms for solving the CMP (1.1).

Let $B : C \to H$ be a nonlinear mapping, let $\varphi : C \to \mathbb{R}$ be a function and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The following generalized mixed equilibrium problem (shortly, GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C \tag{1.5}$$

was introduced by Peng and Yao [19] (also see [16, 30]). The set of solutions to the GMEP(1.5) is denoted by $GMEP(\Theta, \varphi, B)$. The GMEP(1.5) is very general in the sense that it includes, as special cases, fixed point problems, optimization problems, variational inequality problems, minmax problems, Nash equilibrium problems in noncooperative games and others.

If $B = 0$ in GMEP(1.5), then the GMEP(1.5) reduces the following mixed equilibrium problem (shortly, MEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \ge 0, \quad \forall y \in C. \tag{1.6}$$

which was studied by Ceng and Yao [8] (see also [27]).

A fixed point problem (shortly, FPP) is to find a fixed point z of a nonlinear mapping T with property:

$$z \in C,\ Tz = z. \tag{1.7}$$

Fixed point theory is one of the most powerful and important tools of modern mathematics and may be considered a core subject of nonlinear analysis.

As we all know, the convex feasibility problem (shortly, CFP) is the problem of finding a point in the (nonempty) intersection $C = \cap_{i=1}^{m} C_i$ of a finite number of closed convex sets C_i $(i = 1, \cdots, m)$.

Recently, many authors considered iterative methods for finding a common element in solution sets of the CMP (1.1), the GMEP(1.5), the MEP(1.6) and the FPP(1.7) for nonlinear mappings as special cases of the CFP. For instance, we can refer to [8, 11, 12, 28] and the references therein for the MEP(1.6) and the FPP(1.7) for nonlinear mappings. And we can refer to Peng and Yao [19] for the GMEP(1.5) related to an inverse-strongly monotone mapping B and the FPP(1.7) for a nonexpansive mapping T, refer to Jung [13, 15] for the GMEP(1.5) related to a continuous monotone mapping B and the FPP(1.7) for a continuous pseudocontractive mapping T and refer to Ceng et al. [7] for the GMEP(1.5) related to an inverse strongly monotone mapping B and the set of fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{N}$. In particular, Jung [14] considered implicit and explicit iterative algorithms for solving the CMP(1.1) for a convex function f and the MEP(1.6).

In this paper, in order to study the CMP(1.1) combined with the GMEP(1.5) and the FPP(1.7), we introduce implicit and explicit iterative algorithms for finding a common element of the set of solutions to the CMP(1.1) for f, the set of solutions to the GMEP(1.5) related to B and the set of fixed points of T, where $f : C \to \mathbb{R}$ is a real-valued convex function, $B : C \to H$ is a continuous monotone mapping and $T : C \to H$ is a continuous pseudocontractive mapping. Then we establish strong convergence of the sequence generated by the proposed iterative algorithms to a common element of three sets, which is a solution of a certain variational inequality. As a direct consequence, we find the unique solution to the minimum-norm problem:

$$\|x^*\| = \min\{\|x\| : x \in \Omega\},$$

where $\Omega := S \cap GMEP(\Theta, \varphi, B) \cap Fix(T)$. The results in this paper develop, improve upon and complement of the recent results announced by several authors in this direction.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

We recall that

(i) a mapping $V : C \to H$ is said to be *l-Lipschitzian* if there exists a constant $l \geq 0$ such that

$$\|Vx - Vy\| \leq l\|x - y\|, \quad \forall\ x,\ y \in C;$$

(ii) a mapping $T : C \to H$ is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall\ x,\ y \in C;$$

(iii) a mapping $T : C \to H$ is said to be *k-strictly pseudocontractive* ([4]) if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall\ x,\ y \in C;$$

(iv) a mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\ x,\ y \in C$$

where I is the identity mapping.

Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and the class of nonexpansive mappings as a subclass. Moreover, this inclusion is strict (see Example 5.7.1 and Example 5.7.2 in [1]).

Recall ([1, 10]) that the mapping $T : H \to H$ is called *firmly nonexpansive* if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x,\ y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive.

A mapping $T : H \to H$ is said to be an *averaged mapping* if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T = (1 - \alpha)I + \alpha S, \tag{2.1}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged. ([5, 9, 26])

Clearly, a firmly nonexpansive mapping is an $\frac{1}{2}$-averaged mapping.

Proposition 2.1. ([5, 9, 26]) *For given mappings $S,\ T,\ V : H \to H$:*

(a) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(b) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(c) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$, and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(d) *The composite of finitely many averaged mapping is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1,\ \alpha_2 \in (0,1)$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

We recall ([10]) that a nonlinear mapping T whose domain $D(T) \subseteq H$ and $R(T) \subseteq H$ is said to be:

(a) *monotone* if

$$\langle x - y, Tx - Ty \rangle \geq 0, \quad \forall x,\ y \in D(T),$$

(b) *β-strongly monotone* if there exists $\beta > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x,\ y \in D(T),$$

(c) *ν-inverse strongly monotone* (shortly, ν-ism) if there exists $\nu > 0$ such that

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x,\ y \in D(T).$$

It can be easily seen that (i) if T is nonexpansive, then $I - T$ is monotone; (ii) the projection mapping P_C is a 1-ism.

Proposition 2.2. ([5]) *Let $T : H \to H$ be a mapping from H to itself.*

(a) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
(b) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
(c) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

We note that if F is an α-inverse strongly monotone mapping of C into H, then it is obvious that F is $\frac{1}{\alpha}$-Lipschitz continuous, that is, $\|Fx - Fy\| \leq \frac{1}{\alpha}\|x - y\|$ for all $x, y \in C$. Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

In a real Hilbert space H, the following equality holds:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle \tag{2.2}$$

for all $x, y \in H$.

The following lemma is easily proven by property of inner product.

Lemma 2.3. *In a Hilbert space, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \qquad \forall x,\ y \in H.$$

Recall that metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| := d(x, C).$$

Lemma 2.4. ([23]) *For given $x \in H$:*

(a) *$z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C$.*
(b) *$z = P_C x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in C$.*
(c) *$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x,\ y \in H$. Consequently, P_C is nonexpansive and monotone.*

In general, a projection mapping is firmly nonexpansive.

For solving the GMEP(1.5) for a bifunction $\Theta : C \times C \to \mathbb{R}$, let us assume that Θ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x,\ y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

(A4) for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

We can prove the following lemma by using the same method as in [16, 30], and so we omit its proof.

Lemma 2.5. *Let C be a nonempty closed convex subset of H. Let Θ be a bifunction form $C \times C$ to $\mathbb{R}$ satisfies* (A1)–(A4) *and $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Let $B : C \to H$ be a continuous monotone mapping. Then, for $\nu > 0$ and $x \in H$, there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{\nu}\langle y - u, u - x\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $K_\nu : H \to C$ as follows:

$$K_\nu x = \left\{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{\nu}\langle y - u, u - x\rangle \geq 0,\ \forall y \in C\right\}$$

for all $x \in H$ and $\nu > 0$. Then, the following hold:

(1) *For each $x \in H$, $K_\nu x \neq \emptyset$;*
(2) *K_ν is single-valued;*
(3) *K_ν is firmly nonexpansive, that is, for any $x,\ y \in H$,*

$$\|K_\nu x - K_\nu y\|^2 \leq \langle K_\nu x - K_\nu y, x - y\rangle;$$

(4) *$Fix(K_\nu) = GMEP(\Theta, \varphi, B)$;*
(5) *$GMEP(\Theta, \varphi, B)$ is a closed convex subset of C.*

We need the following lemmas for the proof of our main results.

Lemma 2.6. ([29]) *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{z \in C : \langle Tz, y - z\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C\right\}.$$

Then the following hold:

(i) *T_r is single-valued;*
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x,\ y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$;*
(iv) *$Fix(T)$ is a closed convex subset of C.*

Lemma 2.7. ([24]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \xi_n)s_n + \xi_n\delta_n, \quad \forall n \geq 1,$$

where $\{\xi\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\xi_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \xi_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} \xi_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8. ([21]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and* $\{\gamma_n\}$ *be a sequence in* $[0,1]$ *which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) y_n$ *for all* $n \geq 1$ *and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

The following lemma can be easily proven, and therefore, we omit the proof.

Lemma 2.9. *Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*, and* $G : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone mapping with constants* κ *and* $\eta > 0$*. Then for* $0 \leq \gamma l < \mu\eta$*,*

$$\langle (\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x,\ y \in C.$$

That is, $\mu G - \gamma V$ *is strongly monotone with constant* $\mu\eta - \gamma l$.

We also need the following lemma (see [26] for the proof).

Lemma 2.10. *Let* $G : H \to H$ *be a* κ*-Lipschizian and* η*-strongly monotone mapping with constants* $\kappa > 0$ *and* $\eta > 0$*. Let* $0 < \mu < \frac{2\eta}{\kappa^2}$ *and* $0 < t \leq 1$*. Then* $I - t\mu G : H \to H$ *is a contraction with contractive constant* $1 - t\tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

Lemma 2.11. ([1])(Demiclosedness principle) *Let* H *be a real Hilbert space, let* C *be a closed convex subset of* H*, and let* $T : C \to C$ *be a nonexpansive mapping. Then* $I - T$ *is demiclosed, that is,*

$$\{x_n\} \subset C, \quad x_n \rightharpoonup x \in C \text{ and } (I - T)x_n \to y \text{ implies that } (I - T)x = y.$$

The following lemma is a variant of a Minty lemma (see [18]).

Lemma 2.12. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Assume that the mapping* $G : C \to H$ *is monotone and weakly continuous along segments, that is,* $G(x + ty) \to G(x)$ *weakly as* $t \to 0$*. Then the variational inequality*

$$\widetilde{x} \in C, \quad \langle G\widetilde{x}, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C,$$

is equivalent to the dual variational inequality

$$\widetilde{x} \in C, \quad \langle Gp, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C.$$

3. Main results

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Assume that $f : C \to \mathbb{R}$ is a real valued convex (Fréchet) differentiable function, and that the gradient ∇f is an L-Lipschitzian mapping with $L \geq 0$. Note that ∇f being L-Lipschitzian implies that the gradient ∇f is $(1/L)$-ism ([2]), which then implies that $\lambda \nabla f$ is $(1/\lambda L)$-ism. So, by Proposition 2.2, $I - \lambda \nabla f$ is $(\lambda L/2)$-averaged. Now since the projection P_C is $(1/2)$-averaged, we see from Proposition 2.1 that the composite $P_C(I - \lambda \nabla f)$ is $((2 + \lambda L)/4)$-averaged for $0 < \lambda < 2/L$. Hence we have that, for each n, $P_C(I - \lambda_n \nabla f)$ is $((2 + \lambda_n L)/4)$-averaged. Therefore, we can write

$$P_C(I - \lambda_n \nabla f) = \frac{2 - \lambda_n L}{4} I + \frac{2 + \lambda_n L}{4} S_n = \alpha_n I + (1 - \alpha_n) S_n,$$

where S_n is nonexpansive and $\alpha_n := \alpha_n(\lambda_n) = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$. It is easy to see that $\lambda_n \to \frac{2}{L} \iff \alpha_n \to 0$.

From now, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subset of H;
- $f : C \to \mathbb{R}$ is a real-valued convex (Fréchet) differentiable function such that the gradient ∇f is an L-Lipschitzian mapping with $L \geq 0$;
- The CMP (1.1) is consistent (that is, the CMP (1.1) is solvable) and $S := \arg\min_C f$ is the solution set of the CMP (1.1) on C;
- $P_C(I - \lambda_n \nabla f) = \frac{2-\lambda_n L}{4} I + \frac{2+\lambda_n L}{4} S_n = \alpha_n I + (1 - \alpha_n) S_n$, where S_n is nonexpansive and $\alpha_n := \alpha_n(\lambda_n) = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$.
- Θ is a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4);
- $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function;
- $B : C \to H$ is a continuous monotone mapping;
- $K_{\nu_n} : H \to C$ is a mapping defined by
$$K_{\nu_n} x = \Big\{ u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{\nu_n} \langle y - u, u - x\rangle \geq 0,\ \forall y \in C \Big\}$$
for all $x \in H$ and for $\nu_n \in (0, \infty)$ and $\liminf_{n\to\infty} \nu_n > 0$;
- $GMEP(\Theta, \varphi, B)$ is the set of solutions to the GMEP (1.5);
- $T : C \to H$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $T_{r_n} : H \to C$ is a mapping defined by
$$T_{r_n} x = \Big\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r_n} \langle y - z, (1 + r_n)z - x\rangle \leq 0, \quad \forall y \in C \Big\}$$
for $r_n \in (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$;
- $V : H \to H$ is l-Lipschitzian mapping with constant $l \in [0, \infty)$;
- $G : H \to H$ is a ρ-Lipschitzian and η-strongly monotone mapping with constants $\rho > 0$ and $\eta > 0$;

- Constants μ, l, τ, and γ satisfy $0 < \mu < \frac{2\eta}{\rho^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\rho^2)}$;
- $\Omega := S \cap GMEP(\Theta, \varphi, B) \cap Fix(T) \neq \emptyset$.

By Lemma 2.5 and Lemma 2.6, K_{ν_n} and T_{r_n} are nonexpansive and $GMEP(\Theta, \varphi, B) = Fix(K_{\nu_n})$ and $Fix(T) = Fix(T_{r_n})$.

First, we introduce the following iterative algorithm which generates a sequence $\{x_n\}$ in an implicit way:

$$\begin{cases} \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad\qquad + \dfrac{1}{\nu_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu G)(\beta_n x_n + (1 - \beta_n) S_n w_n), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\alpha_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$; $\{\beta_n\} \subset (0, 1)$; $\{\nu_n\}$, $\{r_n\} \subset (0, \infty)$; $x_1 \in C$ is an arbitrary initial guess; $u_n = K_{\nu_n} x_n$; and $w_n = T_{r_n} u_n = T_{r_n} K_{\nu_n} x_n$.

Consider the following mapping Q_n on H defined by

$$Q_n x = \alpha_n \gamma V x + (I - \alpha_n \mu G)(\beta_n x + (1 - \beta_n) S_n T_{r_n} K_{\nu_n} x), \quad \forall\, x \in H, \quad n \geq 1.$$

Let $R_n x = \beta_n x + (1 - \beta_n) S_n T_{r_n} K_{\nu_n} x$. Since $S_n T_{r_n} K_{\nu_n}$ is nonexpansive, we have for $x,\ y \in H$,

$$\begin{aligned} \|R_n x - R_n y\| &\leq \beta_n \|x - y\| + (1 - \beta_n)\|S_n T_{r_n} K_{\nu_n} x - S_n T_{r_n} K_{\nu_n} y\| \\ &\leq \beta_n \|x - y\| + (1 - \beta_n)\|x - y\| = \|x - y\|. \end{aligned}$$

Then by Lemma 2.5, Lemma 2.6 and Lemma 2.10, we derive for $x, y \in H$,

$$\begin{aligned} \|Q_n x - Q_n y\| &\leq \alpha_n \gamma \|Vx - Vy\| + \|(I - \alpha_n \mu G) R_n x - (I - \alpha_n \mu G) R_n y\| \\ &\leq \alpha_n \gamma l \|x - y\| + (1 - \alpha_n \tau)\|R_n x - R_n y\| \\ &\leq \alpha_n \gamma l \|x - y\| + (1 - \alpha_n \tau)\|x - y\| \\ &= (1 - \alpha_n(\tau - \gamma l)\|x - y\|. \end{aligned}$$

Since $0 < 1 - \alpha_n(\tau - \gamma l) < 1$, Q_n is a contractive mapping. Therefor, by the Banach contraction principle, Q_n has a unique fixed point $x_n \in H$, which uniquely solves the fixed point equation

$$\begin{aligned} x_n &= \alpha_n \gamma V x_n + (I - \alpha_n \mu G)(\beta_n x_n + (1 - \beta_n) S_n T_{r_n} K_{\nu_n} x_n) \\ &= \alpha_n \gamma V x_n + (I - \alpha_n \mu G)(\beta_n x_n + (1 - \beta_n) S_n w_n). \end{aligned}$$

Now we prove strong convergence of the sequence $\{x_n\}$ and show the existence of $q \in \Omega$, which solves the variational inequality

$$\langle (\mu G - \gamma V) q, p - q\rangle \geq 0, \quad \forall\, p \in \Omega. \tag{3.2}$$

Equivalently, $q = P_\Omega(I - \mu F + \gamma V)q$ (by Lemma 2.4 (a))

Theorem 3.1. *Let $\{x_n\}$ be a sequence defined by* (3.1)*. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\nu_n\}$ and $\{r_n\}$ be satisfy the following condition:*

(i) $\alpha_n \in (0, \frac{1}{2})$ *for each* $\lambda_n \in (0, \frac{2}{L})$,
$\lim_{n\to\infty} \alpha_n = 0$ ($\Longleftrightarrow$ $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$);

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \nu \le \nu_n < \infty$;
(iv) $0 < r \le r_n < \infty$.

Then $\{x_n\}$ converges strongly as $\lambda_n \to \frac{2}{L}$ ($\Longleftrightarrow$ $\lim_{n\to\infty} \alpha_n = 0$) to a point $q \in \Omega$, which is the unique solution of the variational inequality (3.2).

Proof. Note that from the condition (i), without loss of generality, we assume that $\alpha_n \tau < 1$ for $n \ge 1$.

First, we can show easily the uniqueness of a solution of the variational inequality (3.2). In fact, noting that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \Longleftrightarrow \kappa \ge \eta$, it follows from Lemma 2.9 that

$$\langle (\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \ge (\mu\eta - \gamma l)\|x - y\|^2.$$

That is, $\mu G - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. So the variational inequality (3.2) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the variational inequality (3.2).

Now, take $p \in \Omega$ and let $u_n = K_{\nu_n} x_n$ and $w_n = T_{r_n} u_n = T_{r_n} K_{\nu_n} x_n$. Then, it follows from Lemma 2.5(4) and Lemma 2.6 (iii) that $p = K_{\nu_n} p$ and $p = T_{r_n} p$. Since $u_n = K_{\nu_n} x_n$, $w_n = T_{r_n} u_n$, and K_{ν_n} and T_{r_n} are nonexpansive, we have

$$\|w_n - p\| \le \|u_n - p\| \le \|x_n - p\|, \quad \forall n \ge 1, \tag{3.3}$$

Since

$$p = P_C(I - \lambda_n \nabla f)p = \alpha_n p + (1 - \alpha_n) S_n p, \quad \forall \lambda_n \in (0, \frac{2}{L}),$$

where $\alpha_n := \alpha_n(\lambda_n) = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$, it is clear that $S_n p = p$ for each $\lambda_n \in (0, \frac{2}{L})$.

From now, put $y_n = \beta_n x_n + (1 - \beta_n) S_n T_{r_n} K_{\nu_n} x_n = \beta_n x_n + (1 - \beta_n) S_n T_{r_n} u_n = \beta_n x_n + (1 - \beta_n) S_n w_n$. We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in \Omega$. Then from (3.3), it follows that

$$\begin{aligned} \|y_n - p\| &\le \beta_n \|x_n - p\| + (1 - \beta_n)\|w_n - p\| \\ &\le \beta_n \|x_n - p\| + (1 - \beta_n)\|w_n - S_n p\| \\ &\le \beta_n \|x_n - p\| + (1 - \beta_n)\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{3.4}$$

Therefore, by (3.4) and Lemma 2.10, we derive

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n - p\| \\ &= \|\alpha_n(\gamma V x_n - \gamma V p) + (I - \alpha_n \mu G) y_n - (I - \alpha_n \mu G) p + \alpha_n(\gamma V p - \mu G p)\| \\ &\le \alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|y_n - p\| + \alpha_n \|\gamma V p - \mu G p\| \\ &\le \alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau)\|x_n - p\| + \alpha_n(\gamma\|Vp\| + \mu\|Gp\|). \end{aligned}$$

and so

$$\|x_n - p\| \le \frac{\gamma\|Vp\| + \mu\|Gp\|}{\tau - \gamma l}.$$

Hence $\{x_n\}$ is bounded. Also, by (3.3), $\{u_n\}$, $\{w_n\}$, $\{S_n w_n\}$, $\{V x_n\}$ and $\{G S_n w_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_n - S_nw_n\| = 0$. Indeed, observing that

$$\begin{aligned}\|x_n - S_nw_n\| &= \|\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n - S_nw_n\| \\ &\le \alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|y_n - S_nw_n\| \\ &= \alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|\beta_nx_n + (1-\beta_n)S_nw_n - S_nw_n\| \\ &= \alpha_n\|\gamma Vx_n - \mu Gy_n\| + \beta_n\|x_n - S_nw_n\|,\end{aligned}$$

by conditions (i) and (ii), we obtain

$$\|x_n - S_nw_n\| \le \frac{\alpha_n}{1-\beta_n}\|\gamma Vx_n - \mu Gy_n\| \to 0 \quad \text{as } n\to\infty.$$

Step 3. We show that $\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|K_{\nu_n}x_n - x_n\| = 0$. Indeed. using Lemma 2.5(c) and (2.2), we have

$$\begin{aligned}\|u_n - p\|^2 &= \|K_{\nu_n}x_n - K_{\nu_n}p\|^2 \\ &\le \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

and so

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 \quad (\le \|x_n - p\|^2). \tag{3.5}$$

Noting that $y_n = \beta_nx_n + (1-\beta_n)S_nw_n$ and $x_n = \alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n$, by (3.3) and (3.5), we induce that

(3.6)

$$\begin{aligned}\|x_n - p\|^2 &= \|\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n - p\|^2 \\ &= \|\alpha_n(\gamma Vx_n - \mu Gy_n) + (y_n - p)\|^2 \\ &= \|\alpha_n(\gamma Vx_n - \mu Gy_n) + \beta_n(x_n - S_nw_n) + (S_nw_n - p)\|^2 \\ &\le [(\|\alpha_n(\gamma Vx_n - \mu Gy_n)\| + \|w_n - p\|) + \beta_n\|x_n - S_nw_n\|]^2 \\ &\le [(\|\alpha_n(\gamma Vx_n - \mu Gy_n)\| + \|u_n - p\|) + \beta_n\|x_n - S_nw_n\|]^2 \\ &= \alpha_n^2\|\gamma Vx_n - \mu Gy_n\|^2 + 2\alpha_n\|\gamma Vx_n - \mu Gy_n\|\|u_n - p\| + \|u_n - p\|^2 \\ &\quad + \beta_n\|x_n - S_nw_n\|2(\alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|u_n - p\|) \\ &\quad + \beta_n^2\|x_n - S_nw_n\|^2 \\ &\le \alpha_n\|\gamma Vx_n - \mu Gy_n\|^2 + \|u_n - p\|^2 + M_n \\ &\le \alpha_n\|\gamma Vx_n - \mu Gy_n\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + M_n,\end{aligned}$$

where

(3.7)

$$\begin{aligned}M_n &= 2\alpha_n\|\gamma Vx_n - \mu Gy_n\|\|u_n - p\| \\ &\quad + \beta_n\|x_n - S_nw_n\|2(\alpha_n\|\gamma Vx_n - \mu Gy_n\| + \|u_n - p\|) + \beta_n^2\|x_n - S_nw_n\|^2.\end{aligned}$$

From (3.6), we obtain

$$\|u_n - x_n\|^2 \le \alpha_n\|\gamma Vx_n - \mu Gy_n\|^2 + M_n.$$

Since $M_n \to 0$ as $n \to \infty$ by condition (i) and Step 2, we get

$$\lim_{n\to\infty} \|u_n - x_n\| = \lim_{n\to\infty} \|K_{\nu_n} x_n - x_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|w_n - u_n\| = \lim_{n\to\infty} \|T_{r_n} u_n - u_n\| = 0$. Again, by Lemma 2.6(ii) and (2.2), we obtain

$$\begin{aligned} \|w_n - p\|^2 &= \|T_{r_n} u_n - T_{r_n} p\|^2 \\ &\le \langle u_n - p, w_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|w_n - p\|^2 - \|u_n - w_n\|^2), \end{aligned}$$

and so

$$\begin{aligned} \|w_n - p\|^2 &\le \|u_n - p\|^2 - \|u_n - w_n\|^2 \\ &\le \|x_n - p\| - \|u_n - w_n\|^2 \quad \text{(by (3.3))} \end{aligned} \tag{3.8}$$

Then, from (3.6) and (3.8), we derive

$$\begin{aligned} \|x_n - p\|^2 &\le \alpha_n \|\gamma V x_n - \mu G y_n\|^2 + \|u_n - p\|^2 + M_n \\ &\le \alpha_n \|\gamma V x_n - \mu G y_n\|^2 + \|x_n - p\|^2 - \|u_n - w_n\|^2 + M_n, \end{aligned}$$

where M_n is of (3.7), and so

$$\|u_n - w_n\|^2 \le \alpha_n \|\gamma V x_n - \mu G y_n\|^2 + M_n.$$

From $\lim_{n\to\infty} M_n = 0$ and condition (i), we obtain

$$\lim_{n\to\infty} \|u_n - w_n\| = \lim_{n\to\infty} \|T_{r_n} u_n - u_n\| = 0.$$

Step 5. We show that $\lim_{n\to\infty} \|x_n - w_n\| = 0$. In fact, by Step 3 and Step 4,

$$\|x_n - w_n\| \le \|x_n - u_n\| + \|u_n - w_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 6. We show that $\lim_{n\to\infty} \|w_n - S_n w_n\| = 0$. From Step 2 and Step 5, it follows that

$$\|w_n - S_n w_n\| \le \|w_n - x_n\| + \|x_n - S_n w_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 7. We show that $\{x_n\}$ converges strongly to $q \in \Omega$ as $n \to \infty$, where q is the unique solution of variational inequality (3.2). To this end, first, observe that

$$\begin{aligned} \|P_C(I - \lambda_n \nabla f) w_n - w_n\| &= \|S_n w_n - (1 - \alpha_n) S_n w_n - w_n\| \\ &= (1 - \alpha_n)\|S_n w_n - w_n\| \\ &\le \|S_n w_n - w_n\|, \end{aligned}$$

where $\alpha_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$. Hence, we have

$$\begin{aligned}
&\left\| P_C\Big(I - \frac{2}{L}\nabla f\Big)w_n - w_n \right\| \\
&\leq \left\| P_C\Big(I - \frac{2}{L}\nabla f\Big)w_n - P_C\Big(I - \lambda_n\nabla f\Big)w_n \right\| + \left\| P_C\Big(I - \lambda_n\nabla f\Big)w_n - w_n \right\| \\
&\leq \left\| \Big(I - \frac{2}{L}\nabla f\Big)w_n - \Big(I - \lambda_n\nabla f\Big)w_n \right\| + \left\| P_C\Big(I - \lambda_n\nabla f\Big)w_n - w_n \right\| \\
&\leq \Big(\frac{2}{L} - \lambda_n\Big)\|\nabla f(w_n)\| + \|S_n w_n - w_n\|.
\end{aligned}$$

From the boundedness of $\{w_n\}$, $\alpha_n \to 0$ ($\Longleftrightarrow \lambda_n \to \frac{2}{L}$) and $\|w_n - S_n w_n\| \to 0$ (by Step 6), we conclude that

$$\lim_{n\to\infty}\left\| w_n - P_C\Big(I - \frac{2}{L}\nabla f\Big)w_n \right\| = 0.$$

Consider a subsequence $\{w_{n_i}\}$ of $\{w_n\}$. Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ which converges weakly to q. Without loss of generality, we can assume that $w_{n_i} \rightharpoonup q$. Since C is closed and convex, C is weakly closed. So, we have $q \in C$. Then, by the same argument as in proofs of Theorem 3.1 in [14, 15], we can show that $q \in \Omega$. For the sake of completeness, we include its proof, which was divided into three steps.

(i) We prove that $q \in S$. In fact, by Lemma 2.11, we obtain

$$q = P_C\Big(I - \frac{2}{L}\nabla f\Big)q.$$

This means that $q \in S$.

(ii) We prove that $q \in GMEP(\Theta, \varphi, B)$. Since $u_n = K_{\nu_n} x_n$, by Lemma 2.5, we know that

$$\Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{\nu_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

Then, it follows from (A2) (the monotonicity of Θ) that

$$\langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) + \frac{1}{\nu_n}\langle y - u_n, u_n - x_n\rangle \geq \Theta(y, u_n), \quad \forall y \in C. \tag{3.9}$$

For ϵ with $0 < \epsilon \leq 1$ and $v \in C$, let $v_\epsilon = \epsilon v + (1-\epsilon)q$. Then $v_\epsilon \in C$. So, from (3.9), we obtain

$$\begin{aligned}
\langle Bv_\epsilon, v_\epsilon - u_n\rangle &\geq \langle Bv_\epsilon, v_\epsilon - u_n\rangle - \varphi(v_\epsilon) + \varphi(u_n) \\
&\qquad - \langle Bu_n, v_\epsilon - u_n\rangle - \langle v_\epsilon - u_n, \frac{u_n - z_n}{\nu_n}\rangle + \Theta(v_\epsilon, u_n) \\
&= \langle Bv_\epsilon - Bu_n, v_\epsilon - u_n\rangle - \varphi(v_\epsilon) + \varphi(u_n) \\
&\qquad - \langle v_\epsilon - u_n, \frac{u_n - x_n}{\nu_n}\rangle + \Theta(v_\epsilon, u_n).
\end{aligned}$$

By Step 3 and condition (iii), we get $\frac{\|u_n-x_n\|}{\nu_n} \leq \frac{\|u_n-x_n\|}{\nu} \to 0$ as $n \to \infty$, and by replacing n by n_i and letting $i \to \infty$ along with Step 4, it follows that $u_{n_i} \rightharpoonup q$. Moreover, from the monotonicity of B, we have $\langle Bv_\epsilon - Bu_n, v_\epsilon - u_n\rangle \geq 0$. So, from (A4) and the weak lower semicontinuity of φ, if follows that

$$\langle Bv_\epsilon, v_\epsilon - q\rangle \geq -\varphi(v_\epsilon) + \varphi(q) + \Theta(v_\epsilon, q) \quad \text{as } i \to \infty. \tag{3.10}$$

By (A1), (A4) and (3.10), we also obtain

$$\begin{aligned}
0 &= \Theta(v_\epsilon, v_\epsilon) + \varphi(v_\epsilon) - \varphi(v_\epsilon)\\
&\leq \epsilon\Theta(v_\epsilon, v) + (1-\epsilon)\Theta(v_\epsilon, q) + \epsilon\varphi(v) + (1-\epsilon)\varphi(q) - \varphi(v_\epsilon)\\
&\leq \epsilon[\Theta(v_\epsilon, v) + \varphi(v) - \varphi(v_\epsilon)] + (1-\epsilon)\langle Bv_\epsilon, v_\epsilon - q\rangle\\
&= \epsilon[\Theta(v_\epsilon, v) + \varphi(v) - \varphi(v_\epsilon)] + (1-\epsilon)\epsilon\langle Bv_\epsilon, v - q\rangle,
\end{aligned}$$

and hence

$$0 \leq \Theta(v_\epsilon, v) + \varphi(v) - \varphi(v_\epsilon) + (1-\epsilon)\langle Bv_\epsilon, v - q\rangle. \tag{3.11}$$

Letting $\epsilon \to 0$ in (3.11), we have for each $v \in C$

$$\Theta(q, v) + \langle Bq, v - q\rangle + \varphi(v) - \varphi(q) \geq 0.$$

This implies that $q \in GMEP(\Theta, \varphi, B)$.

(iii) We prove that $q \in Fix(T)$. In fact, noting $w_n = T_{r_n}u_n$, by Lemma 2.6, we induce

$$\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y - w_n, (1+r_n)w_n - u_n\rangle \leq 0, \quad \forall y \in C. \tag{3.12}$$

Put $v_\epsilon = \epsilon v + (1-\epsilon)q$ for $0 < \epsilon \leq 1$ and $v \in C$. Then $v_\epsilon \in C$, and from (3.12) and pseudocontractivity of T, it follows that

$$\begin{aligned}
\langle w_n - v_\epsilon, Tv_\epsilon\rangle \geq\ & \langle w_n - v_\epsilon, Tv_\epsilon\rangle + \langle v_\epsilon - w_n, Tw_n\rangle\\
& - \frac{1}{r_n}\langle v_\epsilon - w_n, (1+r_n)w_n - u_n\rangle\\
=\ & -\langle v_\epsilon - w_n, Tv_\epsilon - Tw_n\rangle - \frac{1}{r_n}\langle v_\epsilon - w_n, w_n - u_n\rangle\\
& -\langle v_\epsilon - w_n, w_n\rangle\\
\geq\ & -\|v_\epsilon - w_n\|^2 - \frac{1}{r_n}\langle v_\epsilon - w_n, w_n - u_n\rangle\\
& -\langle v_\epsilon - w_n, w_n\rangle\\
=\ & -\langle v_\epsilon - w_n, v_\epsilon\rangle - \langle v_\epsilon - w_n, \frac{w_n - u_n}{r_n}\rangle.
\end{aligned} \tag{3.13}$$

Also, by Step 4 and condition (iv), we induce $\frac{\|w_n-u_n\|}{r_n} \leq \frac{\|w_n-u_n\|}{r} \to 0$ as $n \to \infty$. Since $w_{n_i} \rightharpoonup q$ as $i \to \infty$, replacing n by n_i and letting $i \to \infty$, we derive from (3.13) that

$$\langle q - v_\epsilon, Tv_\epsilon\rangle \geq \langle q - v_\epsilon, v_\epsilon\rangle$$

and

$$-\langle v - q, Tv_\epsilon\rangle \geq -\langle v - q, v_\epsilon\rangle, \quad \forall v \in C.$$

Letting $\epsilon \to 0$ and using the fact that T is continuous, we obtain

$$-\langle v-q, Tq\rangle \geq -\langle v-q, q\rangle, \quad \forall v \in C. \tag{3.14}$$

Let $v = Tq$ in (3.14). Then we have $q = Tq$, that is, $q \in Fix(T)$. This along with (i), (ii) and (iii) obtains $q \in \Omega$.

On the other hand, by (3.1) and Lemma 2.10, we derive for $p \in \Omega$

$$\begin{aligned}
&\|x_n - p\|^2 \\
&= \|\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)y_n - p\|^2 \\
&= \|(I-\alpha_n\mu G)y_n - (I-\alpha_n\mu G)p - \alpha_n(\mu G-\gamma V)p + \alpha_n\gamma(Vx_n - Vp)\|^2 \\
&= \|(I-\mu G)y_n - (I-\mu G)p\|^2 \\
&\quad - 2\alpha_n[\langle(\mu G-\gamma V)p, y_n - p\rangle - \alpha_n\langle(\mu G-\gamma V)p, \mu Gy_n - \mu Gp\rangle] \\
&\quad + 2\alpha_n\gamma[\langle Vx_n - Vp, y_n - p\rangle - \alpha_n\langle Vx_n - Vp, \mu Gy_n - \mu Gp\rangle] \\
&\quad - 2\alpha_n^2\gamma\langle(\mu G-\gamma V)p, Vx_n - Vp\rangle \\
&\quad + \alpha_n^2\|(\mu G-\gamma V)p\|^2 + \alpha_n^2\gamma^2\|Vx_n - Vp\|^2 \\
&\leq (1-\alpha_n\tau)^2\|y_n - p\|^2 - 2\alpha_n\langle(\mu G-\gamma V)p, y_n - p\rangle \\
&\quad + 2\alpha_n\gamma l\|x_n - p\|\|y_n - p\| + 2\alpha_n^2\|(\mu G-\gamma V)p\|(\mu\|Gy_n\| + \mu\|Gp\|) \\
&\quad + 2\alpha_n^2\gamma l\|x_n - p\|((\mu\|Gy_n\| + \mu\|Gp\|) + 2\alpha_n^2\gamma l\|(\mu G-\gamma V)p\|\|x_n - p\| \\
&\quad + \alpha_n^2\|(\mu G-\gamma V)p\|^2 + \alpha_n^2\gamma^2 l^2\|x_n - p\|^2 \\
&= (1-2\alpha_n\tau + \alpha_n^2\tau^2)\|y_n - p\|^2 - 2\alpha_n\langle(\mu G-\gamma V)p, y_n - p\rangle \\
&\quad + 2\alpha_n\gamma l\|x_n - p\|\|y_n - p\| + 2\alpha_n^2\|(\mu G-\gamma V)p\|(\mu\|Gy_n\| + \mu\|Gp\|) \\
&\quad + 2\alpha_n^2\gamma l\|x_n - p\|(\mu\|Gy_n\| + \mu\|Gp\|) + 2\alpha_n^2\gamma l\|(\mu G-\gamma V)p\|\|x_n - p\| \\
&\quad + \alpha_n^2(\|(\mu G-\gamma V)p\|^2 + \gamma^2 l^2\|x_n - p\|^2) \\
&\leq (1-2\alpha_n\tau)\|y_n - p\|^2 + 2\alpha_n\langle(\mu G-\gamma V)p, p - y_n\rangle \\
&\quad + \alpha_n\tau l(\|x_n - p\|^2 + \|y_n - p\|^2) + \alpha_n^2 M,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
M = \ &\sup\{\tau^2\|y_n - p\|^2 + 2(\|(\mu G-\gamma V)p\| + \gamma l\|x_n - p\|)(\mu\|Gy_n\| + \mu\|Gp\|) \\
&\quad + 2\gamma l\|(\mu G-\gamma V)p\|\|x_n - p\| + \|(\mu G-\gamma V)p\|^2 + \gamma^2 l^2\|x_n - p\|^2 : n \geq 1\}.
\end{aligned}$$

Hence by (3.4) and (3.15), we obtain

$$
\begin{aligned}
\|x_n - p\|^2 &\le \frac{1 - 2\alpha_n\tau + \alpha_n\gamma l}{1 - \alpha_n\gamma l}\|y_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma l}\langle(\mu G - \gamma V)p, p - y_n\rangle \\
&\quad + \frac{\alpha_n^2}{1 - \alpha_n\gamma l}M \\
&\le \frac{1 - 2\alpha_n\tau + \alpha_n\gamma l}{1 - \alpha_n\gamma l}\|x_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma l}\langle(\mu G - \gamma V)p, p - y_n\rangle \\
&\quad + \frac{\alpha_n^2}{1 - \alpha_n\gamma l}M.
\end{aligned} \tag{3.16}
$$

Observe that

$$
\begin{aligned}
\langle(\mu G - \gamma V)p, p - y_n\rangle &= \langle(\mu G - \gamma V)p, p - (\beta_n x_n + (1 - \beta_n)S_n w_n)\rangle \\
&= \langle(\mu G - \gamma V)p, p - S_n w_n\rangle + \beta_n\langle(\mu G - \gamma V)p, S_n w_n - x_n\rangle \\
&= \langle(\mu G - \gamma V)p, p - w_n\rangle + \langle(\mu G - \gamma V)p, w_n - S_n w_n\rangle \\
&\quad + \beta_n\langle(\mu G - \gamma V)p, S_n w_n - x_n\rangle \\
&\le \langle(\mu G - \gamma V)p, p - w_n\rangle + \|(\mu G - \gamma V)p\|\|w_n - S_n w_n\| \\
&\quad + \beta_n\|(\mu G - \gamma V)p\|\|S_n w_n - x_n\| \\
&\le \langle(\mu G - \gamma V)p, p - w_n\rangle + L_n,
\end{aligned} \tag{3.17}
$$

where $L_n = \|(\mu G - \gamma V)p\|\|w_n - S_n w_n\| + \beta_n\|(\mu G - \gamma V)p\|\|S_n w_n - x_n\|$. Then, from (3.16) and (3.17), we derive

$$
\|x_n - p\|^2 \le \frac{1}{\tau - \gamma l}\langle\mu G - \gamma V p, p - w_n\rangle + \frac{\alpha_n M}{2(\tau - \gamma l)} + \frac{L_n}{\tau - \gamma l}. \tag{3.18}
$$

Now, replacing n by n_i, we substitute q for p in (3.18) to obtain

$$
\|x_{n_i} - q\|^2 \le \frac{1}{\tau - \gamma l}\langle\mu G - \gamma V q, q - w_{n_i}\rangle + \frac{\alpha_{n_i} M}{2(\tau - \gamma l)} + \frac{L_{n_i}}{\tau - \gamma l}. \tag{3.19}
$$

Note that $w_{n_i} \rightharpoonup q$ as $i \to \infty$ and $\lim_{n\to\infty} L_n = 0$ by Step 2 and Step 6. This fact and the inequality (3.19) along with condition (i) imply that $x_{n_i} \to q$ strongly as $i \to \infty$.

Next, we show that q solves the the variational inequality (3.2). Indeed, taking the limit in (3.18) as $i \to \infty$, we get

$$
\|q - p\|^2 \le \frac{1}{\tau - \gamma l}\langle(\mu G - \gamma V)p, p - q\rangle, \quad \forall p \in \Omega.
$$

In particular, q solves the following variational inequality

$$
q \in \Omega \quad \langle(\mu G - \gamma V)p, p - q\rangle \ge 0, \quad p \in \Omega,
$$

or the equivalent dual variational inequality(Lemma 2.12).

$$
q \in \Omega \quad \langle(\mu G - \gamma V)q, p - q\rangle \ge 0, \quad p \in \Omega. \tag{3.20}
$$

Finally we show that the sequence $\{x_n\}$ converges strongly to q. Indeed, let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ and assume $x_{n_k} \to \widehat{q}$. By the same method as the proof above, we have $\widehat{q} \in \Omega$. Moreover, it follows from (3.20) that

$$\langle (\mu G - \gamma V)q, q - \widehat{q} \rangle \leq 0. \tag{3.21}$$

Interchanging q and $\widehat{q}$, we obtain

$$\langle (\mu G - \gamma V)\widehat{q}, \widehat{q} - q \rangle \leq 0. \tag{3.22}$$

Lemma 2.9 and adding these two inequalities (3.21) and (3.22) yields

$$(\mu\eta - \gamma l)\|q - \widehat{q}\|^2 \leq \langle (\mu G - \gamma V)q - (\mu G - \gamma V)\widehat{q}, q - \widehat{q} \rangle \leq 0.$$

Hence $q = \widehat{q}$. Therefore we conclude that $x_n \to q$ as $n \to \infty$.

The variational inequality (3.2) can be rewritten as

$$\langle (I - \mu G + \gamma V)q - q, q - p \rangle \geq 0, \quad \forall p \in \Omega.$$

By Lemma 2.4(a), this is equivalent to the fixed point equation

$$P_\Omega(I - \mu G + \gamma V)q = q.$$

□

From Theorem 3.1, we deduce the following result.

Corollary 3.2. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \Theta(u_n, y) + \langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) \\ \qquad\qquad + \dfrac{1}{\nu_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_n = (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)S_n T_{r_n} K_{\nu_n} x_n), \quad \forall n \geq 1, \end{cases}$$

Let $\{\alpha_n\}$ $\{\beta_n\}$, $\{\nu_n\}$ and $\{r_n\}$ be sequences satisfying conditions (i), (ii), (iii) *and* (iv) *in Theorem 3.1. Then $\{x_n\}$ converges strongly as $\lambda_n \to \frac{2}{L}$ ($\Longleftrightarrow \lim_{n\to\infty} \alpha_n = 0$) to a point $q \in \Omega$, which solves the following minimum-norm problem: find $x^* \in \Omega$ such that*

$$\|x^*\| = \min\{\|x\| : x \in \Omega\}. \tag{3.23}$$

Proof. Take $G = I$, $\mu = 1$, $\tau = 1$, $V = 0$ and $l = 0$ in Theorem 3.1. Then the variational inequality (3.2) is reduced to the inequality

$$\langle q, p - q \rangle \geq 0, \quad \forall p \in \Omega.$$

This is equivalent to $\|q\|^2 \leq \langle p, q \rangle \|p\|\|q\|$ for all $p \in \Omega$. It turns out that $\|q\| \leq \|p\|$ for all $p \in \Omega$ and q is the minimum-norm point of Ω. □

Now, we propose the following iterative algorithm which generates a sequence $\{x_n\}$ in an explicit way:

$$\begin{cases} \Theta(u_n, y) + \langle Bu_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) \\ \qquad\qquad + \dfrac{1}{\nu_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G)(\beta_n x_n + (1 - \beta_n)S_n w_n), \quad \forall n \geq 1, \end{cases} \tag{3.24}$$

where $\alpha_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$; $\{\beta_n\} \subset (0,1)$; $\{\nu_n\}$, $\{r_n\} \subset (0,\infty)$; $x_1 \in C$ is an arbitrary initial guess; $u_n = K_{\nu_n} x_n$; and $w_n = T_{r_n} u_n = T_{r_n} K_{\nu_n} x_n$.

Theorem 3.3. *Let the sequence $\{x_n\}$ be generated iteratively by the explicit algorithm* (3.24). *Let $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$ and $\{r_n\}$, $\{\nu_n\} \subset (0,\infty)$ satisfy the following conditions:*

- (C1) $\alpha_n \in (0, \frac{1}{2})$ *for each* $\lambda_n \in (0, \frac{2}{L})$,
  $\lim_{n\to\infty} \alpha_n = 0$ ($\Longleftrightarrow$ $\lim_{n\to\infty} \lambda_n = \frac{2}{L}$);
- (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- (C4) $0 < \nu \leq \nu_n < \infty$ *and* $\lim_{n\to\infty} |\nu_{n+1} - \nu_n| = 0$.
- (C5) $0 < r \leq r_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

Then $\{x_n\}$ converges strongly as $\lambda_n \to \frac{2}{L}$ ($\Longleftrightarrow$ $\lim_{n\to\infty} \alpha_n = 0$) to a point $q \in \Omega$, which is the unique solution of the variational inequality (3.2).

Proof. Note that from condition (C1), without loss of generality, we assume that $\alpha_n(\tau - \gamma l) < 1$ for $n \geq 1$. From now, we put $u_n = K_{\nu_n} x_n$, $w_n = T_{r_n} u_n$ and $y_n = \beta_n x_n + (1-\beta_n) S_n T_{r_n} K_{\nu_n} x_n = \beta_n x_n + (1-\beta_n) S_n w_n$ for $n \geq 1$.

Now, we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in \Omega$. Then, by Lemma 2.5 (4) and Lemma 2.6 (iii), we see that $p = K_{\nu_n} p$ and $p = T_{r_n} p$. Also, from the proof of Theorem 3.1, we have $p = S_n p$. From $u_n = K_{\nu_n} x_n$ and the fact that K_{ν_n} is nonexpansive, it follows that

$$\|u_n - p\| = \|K_{\nu_n} x_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 1. \tag{3.25}$$

Then, by (3.25), we obtain that

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n x_n + (I-\beta_n) S_n T_{r_n} K_{\nu_n} x_n - p\| \\
&\leq \beta_n \|x_n - p\| + \|(I-\beta_n) S_n T_{r_n} K_{\nu_n} x_n - (I-\beta_n) S_n T_{r_n} K_{\nu_n} p\| \\
&\leq \beta_n \|x_n - p\| + (1-\beta_n) \|T_{r_n} K_{\nu_n} x_n - T_{r_n} K_{\nu_n} p\| \\
&\leq \beta_n \|x_n - p\| + (1-\beta_n) \|T_{r_n} u_n - T_{r_n} p\| \\
&\leq \beta_n \|x_n - p\| + (1-\beta_n) \|u_n - p\| \\
&\leq \beta_n \|x_n - p\| + (1-\beta_n) \|x_n - p\| \\
&= \|x_n - p\|, \quad \forall n \geq 1.
\end{aligned} \tag{3.26}$$

Thus, noting Lemma 2.10 and (3.26), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n \|\gamma V x_n - \gamma V p\| + \|(I - \alpha_n \mu G) y_n - (I - \alpha_n \mu G) p\| \\
&\quad + \alpha_n \|\gamma V p - \mu G p\| \\
&\leq \alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau) \|y_n - p\| + \alpha_n \|\gamma V p - \mu G p\| \\
&\leq \alpha_n \gamma l \|x_n - p\| + (1 - \alpha_n \tau) \|x_n - p\| + \alpha_n \|\gamma V p - \mu G p\| \\
&= (1 - (\tau - \gamma l)\alpha_n) \|x_n - p\| + \alpha_n \|\gamma V p - \mu G p\|
\end{aligned} \tag{3.27}$$

By induction, it follows from (3.27) that

$$\|x_n - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau - \gamma l}\right\}, \quad \forall n \geq 1.$$

Therefore $\{x_n\}$ is bounded, and so $\{y_n\}$, $\{u_n\} = \{K_{\nu_n} x_n\}$, $\{Vx_n\}$, $\{Gy_n\}$ and $\{S_n w_n\}$ are bounded. Moreover, since $\|T_{r_n} u_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|$, $\{w_n\}$ $= \{T_{r_n} u_n\}$ is bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. To this end, let $w_n = T_{r_n} u_n$. Since ∇f is $\frac{1}{L}$-ism, $P_C(I - \lambda_n \nabla f)$ is nonexpansive. So, it follows that for any given $p \in S$,

$$\begin{aligned}
\|P_C(I - \lambda_{n+1}\nabla f)w_n\| &\leq \|P_C(I - \lambda_{n+1}\nabla f)w_n - p\| + \|p\| \\
&\leq \|P_C(I - \lambda_{n+1}\nabla f)w_n - P_C(I - \lambda_{n+1}\nabla f)p\| + \|p\| \\
&\leq \|w_n - p\| + \|p\| \\
&\leq \|w_n\| + 2\|p\|.
\end{aligned}$$

This together with the boundedness of $\{w_n\}$ implies that $\{P_C(I - \lambda_{n+1}\nabla f)w_n\}$ is bounded. Also, observe that

$$\begin{aligned}
\|S_{n+1}w_n - S_n w_n\| &= \Bigg\| \frac{4P_C(I - \lambda_{n+1}\nabla f) - (2 - \lambda_{n+1}L)I}{2 + \lambda_{n+1}L} w_n \\
&\qquad - \frac{4P_C(I - \lambda_n\nabla f) - (2 - \lambda_n L)I}{2 + \lambda_n L} w_n \Bigg\| \\
&\leq \left\| \frac{4P_C(I - \lambda_{n+1}\nabla f)}{2 + \lambda_{n+1}L} w_n - \frac{4P_C(I - \lambda_n\nabla f)}{2 + \lambda_n L} w_n \right\| \\
&\quad + \left\| \frac{2 - \lambda_n L}{2 + \lambda_n L} w_n - \frac{2 - \lambda_{n+1}L}{2 + \lambda_{n+1}L} w_n \right\| \\
&= \Bigg\| \frac{4(2 + \lambda_n L)P_C(I - \lambda_{n+1}\nabla f)w_n}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \\
&\qquad - \frac{4(2 + \lambda_{n+1}L)P_C(I - \lambda_n\nabla f)w_n}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \Bigg\| \\
&\quad + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \|w_n\| \\
&= \Bigg\| \frac{4L(\lambda_n - \lambda_{n+1})P_C(I - \lambda_{n+1}\nabla f)w_n}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \\
&\quad + \frac{4(2 + \lambda_{n+1}L)(P_C(I - \lambda_{n+1}\nabla f)w_n - P_C(I - \lambda_n\nabla f)w_n)}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \Bigg\| \\
&\quad + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \|w_n\| \\
&\leq \frac{4L|\lambda_n - \lambda_{n+1}| \|P_C(I - \lambda_{n+1}\nabla f)w_n\|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)} \\
&\quad + \frac{4(2 + \lambda_{n+1}L)\|P_C(I - \lambda_{n+1}\nabla f)w_n - P_C(I - \lambda_n\nabla f)w_n\|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)}
\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
&+\frac{4L|\lambda_{n+1}-\lambda_n|}{(2+\lambda_{n+1}L)(2+\lambda_n L)}\|w_n\| \\
&\leq |\lambda_{n+1}-\lambda_n|[L\|P_C(I-\lambda_{n+1}\nabla f)w_n\|+4\|\nabla f(w_n)\|+L\|w_n\|] \\
&\leq M_1|\lambda_{n+1}-\lambda_n|,
\end{aligned}
$$

where some constant $M_1 > 0$ such that

$$M_1 \geq L\|P_C(I-\lambda_{n+1}\nabla f)w_n\|+4\|\nabla f(w_n)\|+L\|w_n\|, \quad \forall n \geq 1.$$

So, by (3.28), we have that

$$
\begin{aligned}
\|S_{n+1}w_{n+1}-S_n w_n\| &\leq \|S_{n+1}w_{n+1}-S_{n+1}w_n\|+\|S_{n+1}w_n-S_n w_n\| \\
&\leq \|w_{n+1}-w_n\|+M_1|\lambda_{n+1}-\lambda_n| \\
&\leq \|w_{n+1}-w_n\|+\frac{4M_1}{L}(\alpha_{n+1}+\alpha_n).
\end{aligned} \tag{3.29}
$$

On the other hand, from $u_n = K_{\nu_n}x_n$ and $u_{n+1} = K_{\nu_{n+1}}x_{n+1}$, it follows that

$$
\begin{aligned}
&\Theta(u_n,y)+\langle Bu_n, y-u_n\rangle+\varphi(y)-\varphi(u_n) \\
&\qquad +\frac{1}{\nu_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C,
\end{aligned} \tag{3.30}
$$

and

$$
\begin{aligned}
&\Theta(u_{n+1},y)+\langle Bu_{n+1}, y-u_{n+1}\rangle+\varphi(y)-\varphi(u_{n+1}) \\
&\qquad +\frac{1}{\nu_{n+1}}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0, \quad \forall y \in C.
\end{aligned} \tag{3.31}
$$

Substituting $y = u_{n+1}$ into (3.30) and $y = u_n$ into (3.31), we obtain

$$
\begin{aligned}
&\Theta(u_n,u_{n+1})+\langle Bu_n, u_{n+1}-u_n\rangle+\varphi(u_{n+1})-\varphi(u_n) \\
&\qquad +\frac{1}{\nu_n}\langle u_{n+1}-u_n, u_n-x_n\rangle \geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\Theta(u_{n+1},u_n)+\langle Bu_{n+1}, u_n-u_{n+1}\rangle+\varphi(u_n)-\varphi(u_{n+1}) \\
&\qquad +\frac{1}{\nu_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0.
\end{aligned}
$$

By (A2), we have

$$\langle u_{n+1}-u_n, Bu_n-Bu_{n+1}+\frac{u_n-x_n}{\nu_n}-\frac{u_{n+1}-x_{n+1}}{\nu_{n+1}}\rangle \geq 0,$$

and then

$$\langle u_{n+1}-u_n, \nu_n(Bu_n-Bu_{n+1})+u_n-x_n-\frac{\nu_n}{\nu_{n+1}}(u_n-x_n)\rangle \geq 0.$$

So, it follows that

$$
\begin{aligned}
&\langle u_{n+1}-u_n, u_n-u_{n+1}\rangle+\nu_n\langle u_{n+1}-u_n, Bu_n-Bu_{n+1}\rangle \\
&+\langle u_{n+1}-u_n, x_{n+1}-x_n\rangle+\left(1-\frac{\nu_n}{\nu_{n+1}}\right)\langle u_{n+1}-u_n, u_{n+1}-x_{n+1}\rangle \geq 0.
\end{aligned} \tag{3.32}
$$

Then, from (3.32), condition (C4) and the fact that $\langle u_{n+1}-u_n, Bu_n - Bu_{n+1}\rangle \le 0$, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\le \langle u_{n+1}-u_n, x_{n+1}-x_n\rangle + \left(1-\frac{\nu_n}{\nu_{n+1}}\right)\langle u_{n+1}-u_n, u_{n+1}-x_{n+1}\rangle \\ &\le \|u_{n+1}-u_n\|\left[\|x_{n+1}-x_n\| + \left|1-\frac{\nu_n}{\nu_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\right],\end{aligned}$$

which implies that

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \frac{1}{\nu_{n+1}}|\nu_{n+1}-\nu_n|\|u_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \frac{1}{\nu}|\nu_{n+1}-\nu_n|M_2,\end{aligned} \tag{3.33}$$

where $M_2 = \sup\{\|u_n - x_n\| : n \ge 1\}$.

On another hand, let $w_{n+1} = T_{r_{n+1}}u_{n+1}$ and $w_n = T_{r_n}u_n$. Then we get

$$\langle y - w_n, Tw_n\rangle - \frac{1}{r_n}\langle y-w_n, (1+r_n)w_n - u_n\rangle \le 0, \quad \forall y \in C, \tag{3.34}$$

and

$$\langle y - w_{n+1}, Tw_{n+1}\rangle - \frac{1}{r_{n+1}}\langle y-w_{n+1}, (1+r_{n+1})w_{n+1} - u_{n+1}\rangle \le 0, \quad \forall y \in C. \tag{3.35}$$

Putting $y = w_{n+1}$ in (3.33) and $y = w_n$ in (3.34), we obtain

$$\langle w_{n+1}-w_n, Tw_n\rangle - \frac{1}{r_n}\langle w_{n+1}-w_n, (1+r_n)w_n - u_n\rangle \le 0, \tag{3.36}$$

and

$$\langle w_n - w_{n+1}, Tw_{n+1}\rangle - \frac{1}{r_{n+1}}\langle w_n - w_{n+1}, (1+r_{n+1})w_{n+1} - u_{n+1}\rangle \le 0. \tag{3.37}$$

Adding up (3.36) and (3.37), we have

$$\begin{aligned}&\langle w_{n+1}-w_n, Tw_n - Tw_{n+1}\rangle \\ &\qquad - \langle w_{n+1}-w_n, \frac{(1+r_n)w_n - u_n}{r_n} - \frac{(1+r_{n+1})w_{n+1}-u_{n+1}}{r_{n+1}}\rangle \le 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\langle w_{n+1}-w_n, (w_{n+1}-Tw_{n+1}) - (w_n - Tw_n)\rangle \\ &\qquad\qquad - \langle w_{n+1}-w_n, \frac{w_n - u_n}{r_n} - \frac{w_{n+1}-u_{n+1}}{r_{n+1}}\rangle \le 0.\end{aligned}$$

Now, using the fact that T is pseudocontractive, we induce

$$\langle w_{n+1}-w_n, \frac{w_n-u_n}{r_n} - \frac{w_{n+1}-u_{n+1}}{r_{n+1}}\rangle \ge 0,$$

and hence

$$\langle w_{n+1}-w_n, w_n - w_{n+1} + w_{n+1} - u_n - \frac{r_n}{r_{n+1}}(w_{n+1}-u_{n+1})\rangle \ge 0. \tag{3.38}$$

By (3.38) and condition (C5), we have

$$\begin{aligned}\|w_{n+1}-w_n\|^2 &\le \langle w_{n+1}-w_n, u_{n+1}-u_n+\left(1-\frac{r_n}{r_{n+1}}\right)(w_{n+1}-u_{n+1})\rangle\\ &\le \|w_{n+1}-w_n\|\left[\|u_{n+1}-u_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|w_{n+1}-u_{n+1}\|\right],\end{aligned}$$

which implies

$$\|w_{n+1}-w_n\| \le \|u_{n+1}-u_n\|+\frac{1}{r}|r_{n+1}-r_n|M_3, \tag{3.39}$$

where $M_3=\sup\{\|w_n-u_n\|: n\ge 1\}$. From (3.33) and (3.39), it follows that

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|T_{r_{n+1}}u_{n+1}-T_{r_n}u_n\|\\ &\le \|u_{n+1}-u_n\|+\frac{1}{r}|r_{n+1}-r_n|M_3\\ &\le \|x_{n+1}-x_n\|+\frac{1}{\nu}|\nu_{n+1}-\nu_n|M_2+\frac{1}{r}|r_{n+1}-r_n|M_3.\end{aligned} \tag{3.40}$$

Now, define

$$x_{n+1}=\beta_n x_n+(1-\beta_n)k_n, \quad \forall n\ge 1.$$

Then, from the definition of k_n, we obtain

$$\begin{aligned}&k_{n+1}-k_n\\ &=\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\ &=\frac{\alpha_{n+1}\gamma Vx_{n+1}+(I-\alpha_{n+1}\mu G)y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}\\ &\quad-\frac{\alpha_n\gamma Vx_n+(I-\alpha_n G)y_n-\beta_n x_n}{1-\beta_n}\\ &=\frac{\alpha_{n+1}\gamma Vx_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma Vx_n}{1-\beta_n}-\frac{(I-\alpha_n\mu G)(\beta_n x_n+(1-\beta_n)S_n w_n)-\beta_n x_n}{1-\beta_n x_n}\\ &\quad+\frac{(I-\alpha_{n+1}\mu G)(\beta_{n+1}x_{n+1}+(1-\beta_{n+1})S_{n+1}w_{n+1})-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}x_{n+1}}\\ &=\frac{\alpha_{n+1}\gamma Vx_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma Vx_n}{1-\beta_n}-\frac{\beta_n x_n+(1-\beta_n)S_n w_n-\beta_n x_n}{1-\beta_n}+\frac{\alpha_n\mu Gy_n}{1-\beta_n}\\ &\quad+\frac{\beta_{n+1}x_{n+1}+(1-\beta_{n+1})S_{n+1}w_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}\mu G)y_{n+1}}{1-\beta_{n+1}}\\ &=\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma Vx_{n+1}-\mu Gy_{n+1})-\frac{\alpha_n}{1-\beta_n}(\gamma Vx_n-\mu Gy_n)\\ &\quad+S_{n+1}w_{n+1}-S_n w_n\end{aligned}$$

So, it follows from (3.29) and (3.40)that

$$\begin{aligned}
\|k_{n+1}-k_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|Vx_{n+1}\|+\mu\|Gy_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\gamma\|Vx_n\|+\mu\|Gy_n\|) + \|S_{n+1}w_{n+1}-S_nw_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|Vx_{n+1}\|+\mu\|Gy_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\gamma\|Vx_n\|+\mu\|Gy_n\|) + \|w_{n+1}-w_n\| \\
&\quad + \frac{4M_1}{L}(\alpha_{n+1}+\alpha_n) \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\|Vx_{n+1}\|+\mu\|Gy_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\gamma\|Vx_n\|+\mu\|Gy_n\|) \\
&\quad + \|x_{n+1}-x_n\| + \frac{4M_1}{L}(\alpha_{n+1}+\alpha_n) \\
&\quad + \frac{1}{\nu}|\nu_{n+1}-\nu_n|M_2 + \frac{1}{r}|r_{n+1}-r_n|M_3 \\
&\le \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}+\frac{\alpha_n}{1-\beta_n}\right)M_4 + \|x_{n+1}-x_n\| \\
&\quad + \frac{4M_1}{L}(\alpha_{n+1}+\alpha_n) \\
&\quad + \frac{1}{\nu}|\nu_{n+1}-\nu_n|M_2 + \frac{1}{r}|r_{n+1}-r_n|M_3.
\end{aligned}$$

where $M_4 = \sup\{\gamma\|Vx_n\|+\mu\|Gy_n\| : n \ge 1\}$. This implies that

$$\begin{aligned}
&\|k_{n+1}-k_n\| - \|x_{n+1}-x_n\| \\
&\le \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}+\frac{\alpha_n}{1-\beta_n}\right)M_4 + +\frac{4M_1}{L}(\alpha_{n+1}+\alpha_n) \\
&+ \frac{1}{\nu}|\nu_{n+1}-\nu_n|M_2 + \frac{1}{r}|r_{n+1}-r_n|M_3.
\end{aligned} \tag{3.41}$$

Thus, by conditions (C1), (C3), (C4) and (C5), from (3.41) we induce

$$\limsup_{n\to\infty}(\|k_{n+1}-k_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence, by Lemma 2.8,

$$\lim_{n\to\infty}\|k_n-x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|k_n-x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - S_n w_n\| = 0$. Noting that $x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n$ and $y_n = \beta_n x_n + (1-\beta_n) S_n w_n$, we have

$$\begin{aligned}
\|x_n - S_n w_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_n w_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma V x_n - \mu G y_n\| \\
&\quad + \|y_n - S_n w_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma V x_n - \mu G y_n\| \\
&\quad + \|\beta_n x_n + (1-\beta_n) S_n w_n - S_n w_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n\|\gamma V x_n - \mu G y_n\| \\
&\quad + \beta_n\|x_n - S_n w_n\|,
\end{aligned}$$

that is,

$$\|x_n - S_n w_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma V x_n - \mu G y_n\|.$$

From the conditions (C1), (C3) and Step 2, it follows that

$$\lim_{n\to\infty}\|x_n - S_n w_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\|x_n - K_{\nu_n} x_n\| = 0$. To this end, let $p \in \Omega$. Using $u_n = K_{\nu_n} x_n$, $K_{\nu_n} p = p$ and firmly nonexpansivity of K_{ν_n} (Lemma 2.5 (3), (4)), we derive from (2.2) that

$$\begin{aligned}
\|u_n - p\|^2 &= \|K_{\nu_n} x_n - p\|^2 \\
&\le \langle K_{\nu_n} x_n - K_{\nu_n} p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2).
\end{aligned}$$

This implies

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|v_n - x_n\|^2. \tag{3.42}$$

Again, noting that $x_{n+1} = \alpha_n \gamma V x_n + (I - \alpha_n \mu G) y_n$, $y_n = \beta_n x_n + (1-\beta_n) S_n w_n$ and $w_n = T_{r_n} u_n$, from (3.42) we induce that

(3.43)

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma V x_n - \mu G y_n) + (y_n - p)\|^2 \\
&= \|\alpha_n(\gamma V x_n - \mu G y_n) + \beta_n(x_n - S_n w_n) + (S_n w_n - p)\|^2 \\
&\le [(\|\alpha_n(\gamma V x_n - \mu G y_n)\| + \|w_n - p\|) + \beta_n\|x_n - S_n w_n\|]^2 \\
&= \alpha_n^2\|\gamma V x_n - \mu G y_n\|^2 + 2\alpha_n\|\gamma V x_n - \mu G y_n\|\|w_n - p\| + \|w_n - p\|^2 \\
&\quad + \beta_n\|x_n - S_n w_n\| 2(\alpha_n\|\gamma V x_n - \mu G y_n\| + \|w_n - p\|) \\
&\quad + \beta_n^2\|x_n - S_n w_n\|^2 \\
&\le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|w_n - p\|^2 + M_n \\
&\le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|u_n - p\|^2 + M_n \\
&\le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + (\|x_n - p\|^2 - \|x_n - u_n\|^2) + M_n
\end{aligned}$$

where

$$\begin{aligned} M_n = {} & \beta_n\|x_n - S_n w_n\| 2(\alpha_n\|\gamma V x_n - \mu G y_n\| + \|w_n - p\|) \\ & + \beta_n^2\|x_n - S_n w_n\|^2 + 2\alpha_n\|\gamma V x_n - \mu G y_n\|\|w_n - p\|. \end{aligned} \tag{3.44}$$

Thus, by (3.43), we obtain

$$\begin{aligned} \|u_n - x_n\|^2 & \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + M_n \\ & \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + M_n. \end{aligned} \tag{3.45}$$

Noting $\lim_{n\to\infty} M_n = 0$ by condition (C1) and Step 3, we derive from (3.45), condition (C1) and Step 2 that

$$\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|K_{\nu_n}x_n - x_n\| = 0.$$

Step 5. We show that $\lim_{n\to\infty}\|w_n - u_n\| = \lim_{n\to\infty}\|T_{r_n}u_n - u_n\| = 0$. Indeed, using $w_n = T_{r_n}u_n$, $p = T_{r_n}p$ for $p \in \Omega$ and firmly nonexpansivity of T_{r_n} (Lemma 2.6 (ii), (iii)), we observe that

$$\begin{aligned} \|w_n - p\|^2 & = \|T_{r_n}u_n - T_{r_n}p\|^2 \\ & \le \langle T_{r_n}u_n - T_{r_n}p, u_n - p\rangle \\ & = \langle w_n - p, u_n - p\rangle \\ & = \frac{1}{2}(\|w_n - p\|^2 + \|u_n - p\|^2 - \|u_n - w_n\|^2). \end{aligned}$$

This implies that

$$\|w_n - p\|^2 \le \|u_n - p\|^2 - \|u_n - w_n\|^2 \le \|x_n - p\|^2 - \|u_n - w_n\|^2. \tag{3.46}$$

Again, from (3.43) and (3.46), we compute

$$\begin{aligned} \|x_{n+1} - p\|^2 & \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|w_n - p\|^2 + M_n \\ & \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + (\|x_n - p\|^2 - \|u_n - w_n\|^2) + M_n, \end{aligned} \tag{3.47}$$

where M_n is of (3.44). So, we get

$$\begin{aligned} \|u_n - w_n\|^2 & \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + M_n \\ & \le \alpha_n\|\gamma V x_n - \mu G y_n\|^2 + \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + M_n. \end{aligned} \tag{3.48}$$

From condition (C1), Step 2, $\lim_{n\to\infty} M_n = 0$ and (3.48), we obtain

$$\lim_{n\to\infty}\|u_n - w_n\| = \lim_{n\to\infty}\|u_n - T_{r_n}u_n\| = 0.$$

Step 6. We show that $\lim_{n\to\infty}\|x_n - w_n\| = 0$. Indeed, from Step 4 and Step 5, it follows that

$$\|x_n - w_n\| \le \|x_n - u_n\| + \|u_n - w_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 7. We show that $\lim_{n\to\infty}\|w_n - S_n w_n\| = 0$. In fact, by Step 3 and Step 6, we obtain

$$\|w_n - S_n w_n\| \le \|w_n - x_n\| + \|x_n - S_n w_n\| \to 0 \ \text{ as } n \to \infty.$$

Step 8. We show that $\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, x_n-q\rangle \le 0$, where q is the unique solution of the variational inequality (3.2). To this end, first we prove that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle \le 0.$$

Since $\{w_n\}$ is bounded, we can choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle = \lim_{i\to\infty}\langle(\gamma V-\mu G)q, w_{n_i}-q\rangle. \tag{3.49}$$

Without loss of generality, we may assume that $\{w_{n_i}\}$ converges weakly to $z\in C$. Then, by the same argument as in (i), (ii) and (iii) in proof of Theorem 3.1 along with Step 5, Step 6 and Step 7, we obtain $z\in\Omega$, Hence, from (3.49), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle &= \lim_{i\to\infty}\langle(\gamma V-\mu G)q, w_{n_i}-q\rangle\\ &= \langle(\gamma V-\mu G)q, z-q\rangle \le 0.\end{aligned} \tag{3.50}$$

Since $\lim_{n\to\infty}\|x_n-w_n\|=0$ by Step 6, from (3.50), we conclude that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, x_n-q\rangle\\ \le\ &\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, x_n-w_n\rangle + \limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle\\ \le\ &\limsup_{n\to\infty}\|(\gamma V-\mu G)q\|\|x_n-w_n\| + \limsup_{n\to\infty}\langle(\gamma V-\mu G)q, w_n-q\rangle \le\ 0.\end{aligned}$$

Step 9. We show that $\lim_{n\to\infty}\|x_n-q\|=0$, where q is the unique solution of the variational inequality (3.2). Indeed, from (3.24), Lemma 2.3 and Lemma 2.8, we have

$$\begin{aligned}&\|x_{n+1}-q\|^2\\ =\ &\|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)y_n-p\|^2\\ =\ &\|\alpha_n(\gamma Vx_n-\gamma Vq)+(I-\alpha_n\mu G)y_n-(I-\alpha_n\mu G)q+\alpha_n(\gamma Vq-\mu Gq)\|^2\\ \le\ &[\alpha_n\gamma l\|x_n-q\|+(1-\alpha_n\tau)\|y_n-q\|]^2+2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ \le\ &[\alpha_n\gamma l\|x_n-q\|+(1-\alpha_n\tau)(\beta_n\|x_n-q\|+(1-\beta_n)\|S_nw_n-q\|)]^2\\ &+2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ \le\ &[\alpha_n\gamma l\|x_n-q\|+(1-\alpha_n\tau)(\beta_n\|x_n-q\|+(1-\beta_n)\|w_n-q\|)]^2\\ &+2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ \le\ &[\alpha_n\gamma l\|x_n-q\|+(1-\alpha_n\tau)\|x_n-q\|]^2+2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ \le\ &(1-(\tau-\gamma l)\alpha_n)\|x_n-q\|^2+2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle\\ =\ &(1-(\tau-\gamma l)\alpha_n)\|x_n-q\|^2+2\alpha_n(\tau-\gamma l)\frac{\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle}{\tau-\gamma l}\\ =\ &(1-\xi_n)\|x_n-q\|^2+\xi_n\delta_n\end{aligned} \tag{3.51}$$

where $\xi_n=(\tau-\gamma l)\alpha_n$ and $\delta_n=\frac{2\alpha_n\langle(\gamma V-\mu G)q, x_{n+1}-q\rangle}{\tau-\gamma l}$. From the conditions (C1) and (C2), and Step 9, it is easily seen that $\xi_n\to 0$, $\sum_{n=1}^{\infty}\xi_n=\infty$, and $\limsup_{n\to\infty}\delta_n\le 0$. Hence, by applying Lemma 2.7 to (3.51), we conclude $x_n\to q$ as $n\to\infty$.

In addition, from Step 4, and Step 6, we derive that $u_n \to q$, and $w_n \to q$ as $n \to \infty$. This completes the proof. $\square$

Taking $G = I$, $\mu = 1$ $\tau = 1$, $V = 0$, and $l = 0$ in Theorem 3.3, we obtain immediately the following result.

Corollary 3.4. *Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ \qquad\qquad\qquad + \dfrac{1}{\nu_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = (1-\alpha_n)(\beta_n x_n + (1-\beta_n)S_n w_n), \quad \forall n \geq 1, \end{cases}$$

where $\alpha_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$; $\{\beta_n\} \subset (0,1)$; $\{\nu_n\}$, $\{r_n\} \subset (0,\infty)$; $x_1 \in C$ is an arbitrary initial guess; $u_n = K_{\nu_n}x_n$; and $w_n = T_{r_n}u_n = T_{r_n}K_{\nu_n}x_n$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\nu_n\}$ and $\{r_n\}$ be sequences satisfying conditions (C1), (C2), (C3), (C4) *and* (C5) *in Theorem 3.3. Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which solves the minimum-norm problem* (3.23).

Remark 3.5. Here some special cases of the the GMEP(1.5) are stated as follows:

1) If $\varphi = 0$, then the GMEP(1.5) reduces the following generalized equilibrium problem (shortly, GEP) of finding $x \in C$ such that

$$\Theta(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{3.52}$$

which was studied by Takahashi and Takahashi [22].

2) If $\Theta = 0$ and $\varphi = 0$, then the GMEP(1.5) reduces the following variational inequality problem (shortly, VIP) of finding $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{3.53}$$

which was studied by Stampacchia [17, 20].

3) If $B = 0$ and $\varphi = 0$ then the GMEP(1.5) reduces the following equilibrium problem (shortly, EP) of finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C, \tag{3.54}$$

which was studied by Blum and Oettli [3].

Applying Theorem 3.1, Theorem 3.3, Corollary 3.2 and Corollary 3.4, we can also establish new corresponding results for the CMP(1.1) combined with the GEP(3.52) and the FPP(1.7), or VIP(3.53) and the FPP(1.7), or the EP(3.54) and the FPP(1.7).

Remark 3.6. 1) As new results for solving constrained convex minimization problem combined with the GMEP(1.5) related to a continuous monotone mapping B and The FPP (1.7) for a continuous pseudocontractive mapping T, Theorem 3.1 and Theorem 3.3 improve, develop and complement the corresponding results, which were obtained by several authors in references. In particular, Theorem 3.1 and Theorem 3.3 improve and develop the corresponding results in [6, 14, 25] in following aspect:

(a) The MEP (1.6) in [14] is extended to the case of the GMEP(1.5).

(b) The FPP (1.7) for a continuous pseudocontractive mapping T in comparison with [14] is considered.
(c) The GMEP (1.5) and the FPP (1.7) for a continuous pseudocontractive mapping T in comparison with in [6, 25] are studied.
(d) Our conditions in Theorem 3.3 dispense with condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n = 1$ in comparison with Theorem 4.2 in [6].

2) We point out that Corollary 3.2 and Corollary 3.4 for finding the minimum-norm element of $S \cap GMEP(\Theta, \varphi, B) \cap Fix(T)$ are also new results.

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