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ε-SUBDIFFERENTIALS AND RELATED RESULTS FOR QUASICONVEX PROGRAMMING

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Abstract. In this paper, we study ε-subdifferentials and related results for quasiconvex programming. We define two ε-subdifferentials for quasiconvex functions. We investigate some properties of these subdifferentials. We introduce optimality conditions for an ε-minimizer. We investigate characterizations of the solution set. Additionally, we show convergence theorems for a global minimizer.

1. Introduction

In this paper, we consider the following quasiconvex programming problem:

$$\begin{cases} \text{minimize } f(x), \\ \text{subject to } x \in A, \end{cases}$$

where f is an extended real-valued quasiconvex function on $\mathbb{R}^n$, and A is a convex subset of $\mathbb{R}^n$. In the research of optimization, many types of optimality conditions in terms of derivatives have been introduced, see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 18, 19, 20, 23, 25, 26, 28, 29, 30, 31, 32, 33, 34]. In particular, the subdifferential plays an important role in convex programming. It is well known that a feasible solution x_0 is a global minimizer of a convex function f over a closed convex constraint set A if and only if

$$0 \in \partial f(x_0) + N_A(x_0),$$

where $\partial f(x_0)$ is the subdifferential of f at x_0, and $N_A(x_0)$ is the normal cone of A at x_0. The above condition is one of the most well known optimality condition in the research of optimization, and have been studied extensively. Additionally, in convex analysis, the following ε-subdifferential have been investigated:

$$\partial_\varepsilon f(x_0) := \{v \in \mathbb{R}^n : \forall x \in \mathbb{R}^n, f(x) \geq f(x_0) + \langle v, x - x_0 \rangle - \varepsilon\}.$$

By using the ε-subdifferential, characterizations of an ε-solution, duality results, and convergence theorems have been introduced, see [8, 18, 20] and references therein.

Recently, in quasiconvex programming, the authors show the following necessary and sufficient optimality condition for quasiconvex programming in [34]: a feasible solution x_0 is a global minimizer of an upper semicontinuous (usc) quasiconvex function f over a convex constraint set A if and only if

$$0 \in \partial^M f(x_0) + \text{epi}\delta_A^*,$$

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where $\partial^M f(x_0)$ is the Martínez-Legaz subdifferential, and epiδ_A^* is the epigraph of the support function of A. Additionally, the authors show a necessary and sufficient optimality condition for essentially quasiconvex programming in terms of Greenberg-Pierskalla subdifferential in [33]. However, in quasiconvex analysis, there are not so many results of ε-subdifferentials for quasiconvex functions as far as we know. It is expected to study ε-subdifferentials for quasiconvex functions based on recent progress of quasiconvex analysis.

In this paper, we study ε-subdifferentials and related results for quasiconvex programming. We define two ε-subdifferentials for quasiconvex functions. We investigate some properties of these subdifferentials. We introduce optimality conditions for an ε-minimizers. We investigate characterizations of the solution set. Additionally, we show convergence theorems for a global minimizer.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we define two ε-subdifferentials, and investigate some properties of these subdifferentials. In Section 4, We introduce optimality conditions for an ε-minimizer. We investigate characterizations of the solution set. Additionally, we show convergence theorems for a global minimizer.

2. Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Let A be a subset of $\mathbb{R}^n$. The normal cone of A at $x_0 \in A$ is denoted by

$$N_A(x_0) := \{v \in \mathbb{R}^n : \forall x \in A, \langle v, x - x_0\rangle \leq 0\}.$$

The indicator function δ_A of A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & otherwise. \end{cases}$$

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, \infty]$. The epigraph of f is defined as

$$\text{epi}f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\},$$

and f is said to be convex if epi$f$ is convex. The Fenchel conjugate of f, $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined as

$$f^*(v) := \sup_{x \in \mathbb{R}^n} \{\langle v, x\rangle - f(x)\}.$$

The subdifferential of f at x_0 is defined as

$$\partial f(x_0) := \{v \in \mathbb{R}^n : \forall x \in \mathbb{R}^n, f(x) \geq f(x_0) + \langle v, x - x_0\rangle\}.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\text{lev}(f, \diamond, \alpha) := \{x \in \mathbb{R}^n : f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. A function f is said to be quasiconvex if lev$(f, \leq, \alpha)$ is a convex set for all $\alpha \in \mathbb{R}$. A function f is said to be essentially quasiconvex if f is quasiconvex and each local minimizer x of f over $\mathbb{R}^n$ is a global minimizer of f over

$\mathbb{R}^n$. Clearly, all convex functions are essentially quasiconvex. It is known that a real-valued continuous quasiconvex function is essentially quasiconvex if and only if it is semistrictly quasiconvex; see Theorem 3.37 in [1].

In quasiconvex analysis, various types of subdifferentials have been investigated; Greenberg-Pierskalla subdifferential [6, 26, 33], Martínez-Legaz subdifferential [16, 34], Q-subdifferential with a generator [4, 5, 30, 31, 32], Moreau's generalized conjugation [21], and so on; see [3, 9, 12, 14, 15, 17, 21, 22, 23, 24]. In this paper, we investigate the following two subdifferentials. In [6], Greenberg and Pierskalla introduce the Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{GP} f(x_0) := \{v \in \mathbb{R}^n : \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0)\}.$$

In [16], Martínez-Legaz introduces the Martínez-Legaz subdifferential of f at $x_0 \in \mathbb{R}^n$ as follows:

$$\partial^{M} f(x_0) := \{(v,t) \in \mathbb{R}^{n+1} : \inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0), \langle v, x_0\rangle \geq t\}.$$

Martínez-Legaz subdifferential is known as a special case of c-subdifferential in Moreau's generalized conjugation in [21]. In [33, 34], we study necessary and sufficient optimality conditions for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential and Martínez-Legaz subdifferential.

3. ε-SUBDIFFERENTIAL

In this section, we define two ε-subdifferentials, and investigate some properties of these subdifferentials. In particular, we study the closedness of ε-subdifferentials.

At first, we define the following two ε-subdifferentials. Let ε be a nonnegative real number. ε-Greenberg-Pierskalla subdifferential of f at $x_0 \in \mathbb{R}^n$ is defined as follows:

$$\partial^{GP}_{\varepsilon} f(x_0) := \{v \in \mathbb{R}^n : \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0) - \varepsilon\}.$$

ε-Martínez-Legaz subdifferential of f at $x_0 \in \mathbb{R}^n$ is defined as follows:

$$\partial^{M}_{\varepsilon} f(x_0) := \{(v,t) \in \mathbb{R}^{n+1} : \inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0) - \varepsilon, \langle v, x_0\rangle \geq t\}.$$

We define the above ε-subdifferentials in the similar way of the subdifferential and ε-subdifferential for convex functions. We can check the following relation between ε-subdifferentials:

$$\partial^{GP}_{\varepsilon} f(x_0) = \{v \in \mathbb{R}^n : (v, \langle v, x_0\rangle) \in \partial^{M}_{\varepsilon} f(x_0)\}.$$

3.1. ε-Martínez-Legaz subdifferential.

In this section, we study some properties of ε-Martínez-Legaz subdifferential without quasiconvexity of f. At first, we show the following clear, but important equation without proof:

$$\partial^{M} f(x_0) = \partial^{M}_{0} f(x_0) = \bigcap_{\varepsilon > 0} \partial^{M}_{\varepsilon} f(x_0).$$

In the following theorem, we show that $\partial^{M}_{\varepsilon} f(x_0)$ is a convex cone.

Theorem 3.1. *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\varepsilon \geq 0$, and $x_0 \in \mathbb{R}^n$. Then the following statements hold:*

(i) $\partial_\varepsilon^M f(x_0)$ *is convex,*
(ii) *for each* $\alpha > 0$ *and* $(v,t) \in \partial_\varepsilon^M f(x_0)$, $\alpha(v,t) \in \partial_\varepsilon^M f(x_0)$.

Proof. (i) Let (v_1,t_1), $(v_2,t_2) \in \partial_\varepsilon^M f(x_0)$ and $\alpha \in (0,1)$. We can check easily that

$$\langle (1-\alpha)v_1 + \alpha v_2, x_0\rangle \geq (1-\alpha)t_1 + \alpha t_2,$$

and if $\langle (1-\alpha)v_1 + \alpha v_2, x\rangle \geq (1-\alpha)t_1 + \alpha t_2$, then $\langle v_1, x\rangle \geq t_1$ or $\langle v_2, x\rangle \geq t_2$. This shows that

$$\begin{aligned} & \inf\{f(x) : \langle (1-\alpha)v_1 + \alpha v_2, x\rangle \geq (1-\alpha)t_1 + \alpha t_2\} \\ \geq\ & \min\{\inf\{f(x) : \langle v_1, x\rangle \geq t_1\}, \inf\{f(x) : \langle v_2, x\rangle \geq t_2\}\} \\ \geq\ & f(x_0) - \varepsilon \end{aligned}$$

since (v_1,t_1), $(v_2,t_2) \in \partial_\varepsilon^M f(x_0)$. Hence $\partial_\varepsilon^M f(x_0)$ is convex.

(ii) If $\alpha > 0$ and $(v,t) \in \partial_\varepsilon^M f(x_0)$, then

$$\inf\{f(x) : \langle \alpha v, x\rangle \geq \alpha t\} = \inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0) - \varepsilon,$$

and $\langle \alpha v, x_0\rangle \geq \alpha t$. This completes the proof. □

Next, we show characterizations of the closedness of ε-Martínez-Legaz subdifferential.

Theorem 3.2. *Let* f *be a function from* $\mathbb{R}^n$ *to* $\overline{\mathbb{R}}$, $\varepsilon \geq 0$, *and* $x_0 \in \mathbb{R}^n$. *Then the following statements are equivalent:*

(i) $\partial_\varepsilon^M f(x_0)$ *is closed,*
(ii) $\partial_\varepsilon^M f(x_0) = \{(v,t) : \langle v, x_0\rangle \geq t\}$,
(iii) $f(x_0) - \varepsilon \leq \inf_{x\in\mathbb{R}^n} f(x)$.

Proof. Assume that (iii) holds. Then, for each $(v,t) \in \mathbb{R}^{n+1}$ with $\langle v, x_0\rangle \geq t$,

$$\inf\{f(x) : \langle v, x\rangle \geq t\} \geq \inf_{x\in\mathbb{R}^n} f(x) \geq f(x_0) - \varepsilon,$$

that is, $(v,t) \in \partial_\varepsilon^M f(x_0)$. This shows that (ii) holds. It is clear that (ii) implies (i). Assume that (iii) does not hold. Let $(v,t) \in \partial_\varepsilon^M f(x_0)$. By Theorem 3.1, $\frac{1}{k}(v,t) \in \partial_\varepsilon^M f(x_0)$ for each $k \in \mathbb{N}$ and $\frac{1}{k}(v,t)$ converges to $(0,0)$. Since (iii) does not hold,

$$\inf\{f(x) : \langle 0, x\rangle \geq 0\} = \inf_{x\in\mathbb{R}^n} f(x) < f(x_0) - \varepsilon,$$

that is $(0,0) \notin \partial_\varepsilon^M f(x_0)$. This shows that (i) does not hold. This completes the proof. □

By Theorem 3.2, $\partial_\varepsilon^M f(x_0)$ is closed if and only if $f(x_0) - \varepsilon \leq \inf_{x\in\mathbb{R}^n} f(x)$. In other words, $\partial_\varepsilon^M f(x_0)$ is not closed in general. However, the following theorem holds.

Theorem 3.3. *Let* f *be an usc function from* $\mathbb{R}^n$ *to* $\overline{\mathbb{R}}$, $\{\varepsilon_k\} \subset \mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$, *and* $x_0 \in \mathbb{R}^n$. *If* ε_k *converges to* $\varepsilon_0 \geq 0$ *and* $(v_k,t_k) \in \partial_{\varepsilon_k}^M f(x_0)$ *converges to* $(v_0,t_0) \in \mathbb{R}^{n+1}$ *such that* $v_0 \neq 0$, *then* $(v_0,t_0) \in \partial_{\varepsilon_0}^M f(x_0)$.

Proof. Let $(v_k, t_k) \in \partial^M_{\varepsilon_k} f(x_0)$ be a sequence such that (v_k, t_k) converges to (v_0, t_0) with $v_0 \neq 0$, and assume that $(v_0, t_0) \notin \partial^M_{\varepsilon_0} f(x_0)$. We can check easily that $\langle v_0, x_0\rangle \geq t_0$ since $(v_k, t_k) \in \partial^M_{\varepsilon_k} f(x_0)$. By the definition of ε-Martínez-Legaz subdifferential,

$$\inf\{f(x) : \langle v_0, x\rangle \geq t_0\} < f(x_0) - \varepsilon_0.$$

By the above inequality, there exist $\bar{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $\langle v_0, \bar{x}\rangle \geq t_0$ and

$$f(\bar{x}) < \alpha < f(x_0) - \varepsilon_0.$$

By the upper semicontinuity of f, there exists $\delta > 0$ such that

$$f(\bar{x} + \delta v_0) < \alpha < f(x_0) - \varepsilon_0.$$

Clearly, $\langle v_0, \bar{x} + \delta v_0\rangle > t_0$. Since (v_k, t_k) converges to (v_0, t_0) and ε_k converges to ε_0, for sufficiently large $k \in \mathbb{N}$, $\langle v_k, \bar{x} + \delta v_0\rangle > t_k$ and $\alpha < f(x_0) - \varepsilon_k$. Hence,

$$\inf\{f(x) : \langle v_k, x\rangle \geq t_k\} \leq f(\bar{x} + \delta v_0) < \alpha < f(x_0) - \varepsilon_k,$$

that is, $(v_k, t_k) \notin \partial^M_{\varepsilon_k} f(x_0)$. This is a contradiction. Hence, $(v_0, t_0) \in \partial^M_{\varepsilon_0} f(x_0)$. □

By Theorem 3.3, we show the following corollary.

Corollary 3.4. *Let f be an usc function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\varepsilon \geq 0$, and $x_0 \in \mathbb{R}^n$. Then, $\partial^M_{\varepsilon} f(x_0) \cup \{(0, t) : t \leq 0\}$ is closed.*

Proof. Let $(v_k, t_k) \in \partial^M_{\varepsilon} f(x_0)$ be a sequence such that (v_k, t_k) converges to (v_0, t_0). If $v_0 \neq 0$, then $(v_0, t_0) \in \partial^M_{\varepsilon} f(x_0)$ by Theorem 3.3. If $v_0 = 0$, then $0 = \langle v_0, x_0\rangle \geq t_0$ since $\langle v_k, x_0\rangle \geq t_k$. This shows that $(v_0, t_0) \in \{(0, t) : t \leq 0\}$. Additionally, if $(v_k, t_k) \in \{(0, t) : t \leq 0\}$ converges to (v_0, t_0), then $(v_0, t_0) \in \{(0, t) : t \leq 0\}$. This completes the proof. □

By Corollary 3.4, $\partial^M_{\varepsilon} f(x_0)$ is not closed in general, but $\partial^M_{\varepsilon} f(x_0) \cup \{(0, t) : t \leq 0\}$ is closed for an usc function f. Next, we show the following theorem.

Theorem 3.5. *Let f and g be functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\varepsilon \geq 0$, and $x_0 \in \mathbb{R}^n$. Then the following statements hold:*

(i) *if $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $\partial^M_{\varepsilon_1} f(x_0) \subset \partial^M_{\varepsilon_2} f(x_0)$,*
(ii) *if $\alpha > 0$, then $\partial^M_{\varepsilon}(\alpha f)(x_0) = \partial^M_{\frac{\varepsilon}{\alpha}} f(x_0)$,*
(iii) *for ε_1, $\varepsilon_2 \geq 0$, $\partial^M_{\varepsilon_1+\varepsilon_2}(f + g)(x_0) \supset \partial^M_{\varepsilon_1} f(x_0) \cap \partial^M_{\varepsilon_2} g(x_0)$.*

Proof. (i) Let $0 \leq \varepsilon_1 \leq \varepsilon_2$ and $(v, t) \in \partial^M_{\varepsilon_1} f(x_0)$. Then $\langle v, x_0\rangle \geq t$, and

$$\inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0) - \varepsilon_1 \geq f(x_0) - \varepsilon_2.$$

This shows that $(v, t) \in \partial^M_{\varepsilon_2} f(x_0)$.

(ii) Let $\alpha > 0$. Then, we can check that

$$\inf\{\alpha f(x) : \langle v, x\rangle \geq t\} \geq \alpha f(x_0) - \varepsilon$$

if and only if

$$\inf\{f(x) : \langle v, x\rangle \geq t\} \geq f(x_0) - \frac{\varepsilon}{\alpha}.$$

This shows that (ii) holds.

(iii) Let ε_1, $\varepsilon_2 \geq 0$, and $(v,t) \in \partial^M_{\varepsilon_1} f(x_0) \cap \partial^M_{\varepsilon_2} g(x_0)$. Then, $\langle v, x_0 \rangle \geq t$, and

$$\inf\{f(x) : \langle v, x \rangle \geq t\} \geq f(x_0) - \varepsilon_1, \text{ and } \inf\{g(x) : \langle v, x \rangle \geq t\} \geq g(x_0) - \varepsilon_2.$$

Hence,

$$\begin{aligned} \inf\{(f+g)(x) : \langle v, x \rangle \geq t\} &\geq \inf\{f(x) : \langle v, x \rangle \geq t\} + \inf\{g(x) : \langle v, x \rangle \geq t\} \\ &\geq f(x_0) + g(x_0) - \varepsilon_1 - \varepsilon_2. \end{aligned}$$

This shows that $(v,t) \in \partial^M_{\varepsilon_1+\varepsilon_2}(f+g)(x_0)$. □

3.2. **ε-Greenberg-Pierskalla subdifferential.** The following theorems are similar to the above results for ε-Martínez-Legaz subdifferential. Hence the proofs will be omitted.

Theorem 3.6. *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\varepsilon \geq 0$, and $x_0 \in \mathbb{R}^n$. Then the following statements hold:*

(i) $\partial^{GP} f(x_0) = \partial^{GP}_0 f(x_0) = \bigcap_{\varepsilon>0} \partial^{GP}_\varepsilon f(x_0)$,
(ii) $\partial^{GP}_\varepsilon f(x_0)$ *is convex,*
(iii) *for each $\alpha > 0$ and $v \in \partial^{GP}_\varepsilon f(x_0)$, $\alpha v \in \partial^{GP}_\varepsilon f(x_0)$,*
(iv) *if $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $\partial^{GP}_{\varepsilon_1} f(x_0) \subset \partial^{GP}_{\varepsilon_2} f(x_0)$,*
(v) *if $\alpha > 0$, then $\partial^{GP}_\varepsilon(\alpha f)(x_0) = \partial^{GP}_{\frac{\varepsilon}{\alpha}} f(x_0)$,*
(vi) *for ε_1, $\varepsilon_2 \geq 0$, $\partial^{GP}_{\varepsilon_1+\varepsilon_2}(f+g)(x_0) \supset \partial^{GP}_{\varepsilon_1} f(x_0) \cap \partial^{GP}_{\varepsilon_2} g(x_0)$.*

Theorem 3.7. *Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\varepsilon \geq 0$, and $x_0 \in \mathbb{R}^n$. Then the following statements are equivalent:*

(i) $\partial^{GP}_\varepsilon f(x_0)$ *is closed,*
(ii) $\partial^{GP}_\varepsilon f(x_0) = \mathbb{R}^n$,
(iii) $f(x_0) - \varepsilon \leq \inf_{x \in \mathbb{R}^n} f(x)$.

Theorem 3.8. *Let f be an usc function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\{\varepsilon_k\} \subset \mathbb{R}_+$, $\varepsilon \geq 0$, and $x_0 \in \mathbb{R}^n$. Then the following statements hold:*

(i) *if ε_k converges to $\varepsilon_0 \geq 0$ and $v_k \in \partial^{GP}_{\varepsilon_k} f(x_0)$ converges to $v_0 \in \mathbb{R}^n \setminus \{0\}$, then $v_0 \in \partial^{GP}_{\varepsilon_0} f(x_0)$,*
(ii) $\partial^{GP}_\varepsilon f(x_0) \cup \{0\}$ *is closed.*

4. Optimality conditions and convergence theorems

In this section, we study optimality conditions and convergence theorems for quasiconvex programming. We show necessary and sufficient optimality conditions for an ε-minimizer of quasiconvex programming. We investigate characterizations of the solution set. Additionally, we show convergence theorems in terms of ε-subdifferentials.

4.1. **Optimality conditions.** An element x_0 of A is said to be a global ε-minimizer of f over A if $f(x_0)-\varepsilon \le \inf_{x\in A} f(x)$. In the following theorem, we show necessary and sufficient optimality conditions for an ε-minimizer of an unconstrained problem.

Theorem 4.1. *Let f be an usc function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, $\varepsilon \ge 0$, and $x_0 \in \mathbb{R}^n$. Then the following statements are equivalent:*

(i) $0 \in \partial_\varepsilon^{GP} f(x_0)$,
(ii) $(0,0) \in \partial_\varepsilon^{M} f(x_0)$,
(iii) $f(x_0)-\varepsilon \le \inf_{x\in\mathbb{R}^n} f(x)$.

Proof. By Theorem 3.7, (iii) implies (i). Since $\partial_\varepsilon^{GP} f(x_0) = \{v \in \mathbb{R}^n : (v, \langle v, x_0\rangle) \in \partial_\varepsilon^{M} f(x_0)\}$, (i) implies (ii). By the statement (ii),

$$\inf_{x\in\mathbb{R}^n} f(x) = \inf\{f(x) : \langle 0, x\rangle \ge 0\} \ge f(x_0)-\varepsilon.$$

This completes the proof. □

Next, we investigate a necessary and sufficient optimality condition for an ε-minimizer of constrained quasiconvex programming.

Theorem 4.2. *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, $\varepsilon \ge 0$, and $x_0 \in A$. Then, x_0 is a global ε-minimizer of f over A if and only if*

$$0 \in \partial_\varepsilon^{M} f(x_0) + \mathrm{epi}\delta_A^*.$$

Proof. Assume that x_0 is a global ε-minimizer of f over A, that is $f(x_0)-\varepsilon \le \inf_{x\in A} f(x)$. If $f(x_0)-\varepsilon \le \inf_{x\in\mathbb{R}^n} f(x)$, then $(0,0) \in \partial_\varepsilon^{M} f(x_0)$ by Theorem 3.2. Hence $0 \in \partial_\varepsilon^{M} f(x_0) + \mathrm{epi}\delta_A^*$. Assume that $f(x_0)-\varepsilon > \inf_{x\in\mathbb{R}^n} f(x)$. Then, A is nonempty convex, $\mathrm{lev}(f,<,f(x_0)-\varepsilon)$ is nonempty open convex, and $A \cap \mathrm{lev}(f,<,f(x_0)-\varepsilon)$ is empty. By the separation theorem, there exist $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that for each $x \in A$ and $y \in \mathrm{lev}(f,<,f(x_0)-\varepsilon)$,

$$\langle v, x\rangle \ge t > \langle v, y\rangle .$$

We can check easily that $-(v,t) \in \mathrm{epi}\delta_A^*$ and $\langle v, x_0\rangle \ge t$. Additionally,

$$\inf\{f(y) : \langle v, y\rangle \ge t\} \ge f(x_0)-\varepsilon.$$

This shows that $(v,t) \in \partial_\varepsilon^{M} f(x_0)$. Hence, $0 \in \partial_\varepsilon^{M} f(x_0) + \mathrm{epi}\delta_A^*$.

Conversely, assume that $0 \in \partial_\varepsilon^{M} f(x_0)+\mathrm{epi}\delta_A^*$. Then, there exists $(v,t) \in \partial_\varepsilon^{M} f(x_0)$ such that $-(v,t) \in \mathrm{epi}\delta_A^*$. Since $-(v,t) \in \mathrm{epi}\delta_A^*$, $\langle v, x\rangle \ge t$ for each $x \in A$. Additionally,

$$\inf_{x\in A} f(x) \ge \inf\{f(x) : \langle v, x\rangle \ge t\} \ge f(x_0)-\varepsilon$$

since $(v,t) \in \partial_\varepsilon^{M} f(x_0)$. This completes the proof. □

4.2. **Characterizations of the solution set.** In this section, we show characterizations of the set of ε-minimizers in terms of ε-Martínez-Legaz subdifferential.

Theorem 4.3. *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, ε, $\varepsilon_0 \geq 0$, $x_0 \in A$ is a ε_0-minimizer of f over A, and Then, $S_\varepsilon \subset C_1^\varepsilon \subset C_2^\varepsilon \subset S_{\varepsilon+\varepsilon_0}$, where*

(i) $S_\varepsilon = \{x \in A : f(x) - \varepsilon \leq \inf_{y\in A} f(y)\}$,
(ii) $C_1^\varepsilon = \{x \in A : \partial^M_{\varepsilon+\varepsilon_0} f(x_0) \cap \partial^M_\varepsilon f(x) \neq \emptyset\}$,
(iii) $C_2^\varepsilon = \{x \in A : \exists (v,t) \in \partial^M_\varepsilon f(x) \ s.t. \ \langle v, x_0\rangle \geq t\}$,

Additionally, if $x_0 \in A$ is a global minimizer of f over A, then $S_\varepsilon = C_1^\varepsilon = C_2^\varepsilon$.

Proof. Let $x \in S_\varepsilon$, then by Theorem 4.2,

$$0 \in \partial^M_\varepsilon f(x) + \mathrm{epi}\delta_A^*,$$

that is, there exists $(v,t) \in \partial^M_\varepsilon f(x)$ such that $-(v,t) \in \mathrm{epi}\delta_A^*$. Hence, $\langle v, x_0\rangle \geq t$ and

$$\inf\{f(y) : \langle v, y\rangle \geq t\} \geq f(x) - \varepsilon \geq f(x_0) - \varepsilon - \varepsilon_0,$$

since $x_0 \in S_{\varepsilon_0}$. This shows that $(v,t) \in \partial^M_{\varepsilon+\varepsilon_0} f(x_0)$ and $S_\varepsilon \subset C_1^\varepsilon$. Let $x \in C_1^\varepsilon$, then there exists $(v,t) \in \partial^M_\varepsilon f(x)$ such that $(v,t) \in \partial^M_{\varepsilon+\varepsilon_0} f(x_0)$. By the definition of ε-Martínez-Legaz subdifferential, $\langle v, x_0\rangle \geq t$. Hence $C_1^\varepsilon \subset C_2^\varepsilon$. Let $x \in C_2^\varepsilon$, then there exists $(v,t) \in \partial^M_\varepsilon f(x)$ such that $\langle v, x_0\rangle \geq t$. Therefore,

$$\inf_{y\in A} f(y) \geq f(x_0) - \varepsilon_0 \geq \inf\{f(y) : \langle v, y\rangle \geq t\} - \varepsilon_0 \geq f(x) - \varepsilon - \varepsilon_0,$$

that is, $x \in S_{\varepsilon+\varepsilon_0}$.

Assume that $x_0 \in A$ is a global minimizer of f over A, that is, $x \in S_0$. By the similar way in the first half of the proof, we can show that $C_2^\varepsilon \subset S_\varepsilon$. This completes the proof. □

As seen in Theorem 4.3, we can characterize the set of all ε-minimizers S_ε by only one global-minimizer x_0 and ε-Martínez-Legaz subdifferential.

4.3. **Convergence theorem.** In this section, we show convergence theorems in terms of ε-subdifferentials. At first, we show the following convergence theorem in terms of ε-Martínez-Legaz subdifferential.

Theorem 4.4. *Let f be a continuous quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a convex subset of $\mathbb{R}^n$. Assume that $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, and for each $k \in \mathbb{N}$,*

(i) $x_k \in A$,
(ii) $\varepsilon_k > 0$,
(iii) $(v_k, t_k) \in \partial^M_{\varepsilon_k} f(x_k)$,
(iv) $\|v_k\| = 1$,
(v) $d((v_k,t_k), -\mathrm{epi}\delta_A^*) := \inf\{\|(v_k,t_k) - (w_k,s_k)\| : (w_k,s_k) \in -\mathrm{epi}\delta_A^*\} < \varepsilon_k$.

If x_k converges to $x_0 \in A$ and ε_k converges to 0, then x_0 is a global minimizer of f over A.

Proof. Since $\|v_k\| = 1$ for each $k \in \mathbb{N}$, we assume that v_k converges to some v_0 without loss of generality. By the definition of ε-Martínez-Legaz subdifferential, $\langle v_k, x_k \rangle \geq t_k$ for each $k \in \mathbb{N}$. Since (v_k, x_k) converges to (v_0, x_0), $\{t_k\}$ is bounded from above. Now, we show that $\{t_k\}$ is bounded from below. If not so, we can assume that $\lim_{k\to\infty} t_k = -\infty$. By the assumption, $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, there exists $z \in \mathbb{R}^n$ such that

$$\inf_{x\in A} f(x) > f(z) \geq \inf_{x\in\mathbb{R}^n} f(x).$$

Since $\langle v_k, z\rangle$ converges to $\langle v_0, z\rangle$ and $\lim_{k\to\infty} t_k = -\infty$, $\langle v_k, z\rangle \geq t_k$ for sufficiently large k. Additionally,

$$f(z) \geq \inf\{f(x) : \langle v_k, x\rangle \geq t_k\} \geq f(x_k) - \varepsilon_k \geq \inf_{x\in A} f(x) - \varepsilon_k.$$

This contradicts to ε_k converges to 0. Hence, $\{t_k\}$ is bounded. We assume that t_k converges to some t_0 without loss of generality. By the closedness of $\mathrm{epi}\delta_A^*$ and the assumption (v), $(v_0, t_0) \in -\mathrm{epi}\delta_A^*$. Finally, we show $(v_0, t_0) \in \partial^M f(x_0)$. Since $\langle v_k, x_k\rangle \geq t_k$ for each $k \in \mathbb{N}$, $\langle v_0, x_0\rangle \geq t_0$. Assume that $(v_0, t_0) \notin \partial^M f(x_0)$, then

$$\inf\{f(y) : \langle v_0, y\rangle \geq t_0\} < f(x_0),$$

and there exists $z_0 \in \mathbb{R}^n$ such that $\langle v_0, z_0\rangle \geq t_0$ and $f(z_0) < f(x_0)$. Since f is continuous, (actually we need the upper semicontinuity of f), there exists $r_0 > 0$ such that $f(z_0 + r_0 v_0) < f(x_0)$. Since $\langle v_0, z_0 + r_0 v_0\rangle > t_0$, there exists $K \in \mathbb{N}$ such that for each $k \geq K$, $\langle v_k, z_0 + r_0 v_0\rangle > t_k$. This shows that for each $k \geq K$

$$f(x_0) > f(z_0 + r_0 v_0) \geq \inf\{f(y) : \langle v_k, y\rangle \geq t_k\} \geq f(x_k) - \varepsilon_k.$$

By the continuity of f, (actually we need the lower semicontinuity of f)

$$f(x_0) = \lim_{k\to\infty} (f(x_k) - \varepsilon_k) \leq f(z_0 + r_0 v_0) < f(x_0).$$

This is a contradiction. Hence, $(v_0, t_0) \in \partial^M f(x_0)$. By Theorem 4.2, $\inf_{x\in A} f(x) = f(x_0)$, that is, x_0 is a global minimizer of f over A. □

Next, we show the following convergence theorem in terms of ε-Greenberg-Pierskalla subdifferential as a corollary of Theorem 4.4.

Corollary 4.5. *Let f be a continuous quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, and A a convex subset of $\mathbb{R}^n$. Assume that $\inf_{x\in A} f(x) > \inf_{x\in\mathbb{R}^n} f(x)$, and for each $k \in \mathbb{N}$,*

- (i) $x_k \in A$,
- (ii) $\varepsilon_k > 0$,
- (iii) $v_k \in \partial^{GP}_{\varepsilon_k} f(x_k)$,
- (iv) $\|v_k\| = 1$,
- (v) $d(v_k, -N_A(x_k)) < \varepsilon_k$.

If x_k converges to $x_0 \in A$ and ε_k converges to 0, then x_0 is a global minimizer of f over A.

Proof. We can check easily that a sequence $\{(v_k, \langle v_k, x_k\rangle)\}$ satisfies the assumptions in Theorem 4.4. By Theorem 4.4, x_0 is a global minimizer of f over A. □

5. Discussion

In this section, we discuss our results. We compare our results with previous ones and we show some examples.

5.1. **Remark of Theorem 3.5.** In Lemma 4.2 of [27], we show the following similar result of the statement (iii) in Theorem 3.5:

$$\partial^M (f+g)(x_0) \supset \partial^M f(x_0) \cap \partial^M g(x_0).$$

In [27], we show a characterization of quasiconvexity of $f+g$ in terms of the above inclusion. The statement (iii) in Theorem 3.5 is a generalized result of Lemma 4.2 of [27].

5.2. **Remark of Theorem 4.2.** In [33, 34], we show necessary and sufficient optimality conditions in terms of subdifferentials. We can prove the following result as a corollary of Theorem 4.2.

Corollary 5.1 ([34]). *Let f be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x\in A} f(x)$,
(ii) $0 \in \partial^M f(x_0) + \mathrm{epi}\delta_A^*$.

On the other hand, the following condition is not a necessary optimality condition but a sufficient optimality condition:

$$0 \in \partial_\varepsilon^{GP} f(x_0) + N_A(x_0).$$

Actually, if $0 \in \partial_\varepsilon^{GP} f(x_0) + N_A(x_0)$, then there exists $v \in \partial_\varepsilon^{GP} f(x_0)$ such that $-v \in N_A(x_0)$. Since $-v \in N_A(x_0)$, $\langle v, x\rangle \geq \langle v, x_0\rangle$ for each $x \in A$. Hence,

$$\inf_{x\in A} f(x) \geq \inf\{f(x) : \langle v, x\rangle \geq \langle v, x_0\rangle\} \geq f(x_0) - \varepsilon,$$

that is, x_0 is a global ε-minimizer of f over A. However, the condition is not a necessary optimality condition in general, see the the following example.

Example 1. Let $f(x) = x$, $A = [0,1]$, and $\varepsilon = \frac{1}{4}$. Then, f is essentially quasiconvex and $x_0 = \frac{1}{4}$ is a global $\frac{1}{4}$-minimizer of f over A. However,

$$0 \notin (0,\infty) + \{0\} = \partial_\varepsilon^{GP} f(x_0) + N_A(x_0).$$

On the other hand,

$$(0,0) = (1,0) + (-1,0) \in \partial_\varepsilon^M f(x_0) + \mathrm{epi}\delta_A^*.$$

Hence, we can apply Theorem 4.2.

Hence, we cannot prove the following result in [33] by using Theorem 4.2 and our result in this paper.

Theorem 5.2 ([33]). *Let f be an usc essentially quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, A a convex subset of $\mathbb{R}^n$, and $x_0 \in A$. Then, the following statements are equivalent:*

(i) $f(x_0) = \min_{x\in A} f(x)$,
(ii) $0 \in \partial^{GP} f(x_0) + N_A(x_0)$.

5.3. **Remark of Theorem 4.3.** In [34], we show characterizations of the solution set for quasiconvex programming in terms of Martínez-Legaz subdifferential. We can prove the following result as a corollary of Theorem 4.3.

Corollary 5.3 ([34])**.** *Let f be an usc quasiconvex function, A a nonempty convex subset of $\mathbb{R}^n$, and x_0 is a global minimizer of f over A. Then, the following sets are equal:*

(i) $S = \{x \in A \mid f(x) = \min_{y \in A} f(y)\}$,
(ii) $S_2' = \{x \in A \mid \partial^M f(x_0) \cap \partial^M f(x) \neq \emptyset\}$,
(iii) $S_6' = \{x \in A \mid \exists (v,t) \in \partial^M f(x) \ s.t. \ \langle v, x_0 \rangle \geq t\}$.

Proof. Let $\varepsilon = 0$, then $S = S_\varepsilon = S_{2\varepsilon}$, $S_2' = C_1^\varepsilon$, and $S_6' = C_2^\varepsilon$. This completes the proof by Theorem 4.3. □

On the other hand, we can not prove characterizations of the solution set in [33] by our results in the paper for the same reason in Section 5.2.

5.4. **Remark of Theorem 4.4.** In Theorem 4.4 and Corollary 4.5, we show convergence theorems in terms of ε-subdifferentials. In the following example, we show an application of these results.

Example 2. Let $A = [1,2] \times [1,5]$, and f the following function on $\mathbb{R}^2$:

$$f(x_1, x_2) = -x_1 x_2.$$

f is known as a Cobb-Douglas type function. Clearly, f is continuous quasiconvex, but not convex, and $\inf_{x \in A} f(x) > \inf_{x \in \mathbb{R}^2} f(x)$. For each $k \in \mathbb{N}$, let

(i) $x_k = \left(2, 5 - \frac{1}{k}\right) \in A$,
(ii) $v_k = \frac{\nabla f(x_k)}{\|\nabla f(x_k)\|} = \frac{(-(x_k)_2, -(x_k)_1)}{\|x_k\|} \in \partial^{GP} f(x_k)$,

Then, x_k converges to $x_0 = (2,5)$ and $d((v_k, \langle v_k, x_k \rangle), -\mathrm{epi}\delta_A^*)$ converges to 0. Let $\varepsilon_k = d((v_k, \langle v_k, x_k \rangle), -\mathrm{epi}\delta_A^*) + \frac{1}{k}$, then $v_k \in \partial^{GP}_{\varepsilon_k} f(x_k)$ and $(v_k, \langle v_k, x_k \rangle) \in \partial^M_{\varepsilon_k} f(x_k)$ since $v_k \in \partial^{GP} f(x_k)$. By Theorem 4.4 or Corollary 4.5, $x_0 = (2,5)$ is a global minimizer of f over A.

6. Conclusion

In this paper, we study ε-subdifferentials and related results for quasiconvex programming. We define ε-Greenberg-Pierskalla subdifferential and ε-Martínez-Legaz subdifferential, and show some properties of these subdifferentials in Section 3. In particular, $\partial^M_\varepsilon f(x_0)$ and $\partial^{GP}_\varepsilon f(x_0)$ are not closed in general, but $\partial^M_\varepsilon f(x_0) \cup \{(0,t) : t \leq 0\}$ and $\partial^{GP}_\varepsilon f(x_0) \cup \{0\}$ are closed. In Theorem 4.2, we introduce a necessary and sufficient optimality condition for an ε-minimizer in terms of ε-Martínez-Legaz subdifferential. In Theorem 4.3, we show characterizations of the solution set. Additionally, in Theorem 4.4 and Corollary 4.5, we show convergence theorems for a global minimizer in terms of ε-subdifferentials.

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