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REFINED HERMITE-HADAMARD INEQUALITY AND ITS APPLICATION

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Dedicates to the memories of Professor Wataru Takahashi.

Abstract. There are many generalizations of Hermite-Hadamard inequality for convex function f defined on $[a, b]$. Recently we gave several types of refined Hermite-Hadamard inequality and obtained inequalities satisfied by weighted logarithmic mean. In this article we give several relations of upper bounds or lower bounds of refined Hermite-Hadamard inequality. Fuhermore, we apply to different types of inequalities under some conditions.

1. Introduction

A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $[a, b]$ if the inequality

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2} \tag{1.1}$$

holds for all $x, y \in [a, b]$. If inequality (1.1) reverses, then f is said to be concave on $[a, b]$. Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval $[a.b]$. Then

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This double inequality is known in the literature as the Hermite-Hadamard integral inequality for convex functions. It has many applications in more different areas of pure and applied mathematics. Recently we obtained the following two refined Hermite-Hadamard inequalities.

Theorem 1.1 ([13]). *Let $f(x)$ be a convex function on $[a, b]$. Then for any $m, n \in \mathbb{N} \cup \{0\}$*

$$L^{(1)}_{f,n}(a, b) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq L^{(2)}_{f,m}(a, b), \tag{1.3}$$

where $h_n = \frac{b-a}{2^n}$,

$$L^{(1)}_{f,n}(a, b) = \frac{1}{2^n}\sum_{k=1}^{2^n} f(a + (2k-1)h_{n+1})$$

and

$$L^{(2)}_{f,m}(a, b) = \frac{1}{2^{m+1}}\left\{f(a) + f(b) + 2\sum_{k=1}^{2^m-1} f(a + kh_m)\right\}.$$

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Theorem 1.2 ([13]). *Let $f(x)$ be a convex function on $[a,b]$. Then for any $v \in [0,1]$ and $m, n \in \mathbb{N} \cup \{0\}$,*

$$r^{(1)}_{f,v,n}(a,b) \le \frac{1}{b-a}\int_a^b f(t)dt \le r^{(2)}_{f,v,m}(a,b), \tag{1.4}$$

where $h_n = \frac{b-a}{2^n}$,

$$r^{(1)}_{f,v,n}(a,b) = \frac{1}{2^n}\sum_{k=1}^{2^n}\{vf(a+(2k-1)vh_{n+1}) + (1-v)f((1-v)a+vb+(2k-1)(1-v)h_{n+1})\}$$

and

$$r^{(2)}_{f,v,m}(a,b) = \frac{1}{2^{m+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\} + \frac{1}{2^m}\sum_{k=1}^{2^m-1}\{vf(a+kvh_m)+(1-v)f((1-v)a+vb+k(1-v)h_m)\}.$$

In section 2, we examine the relations between $L^{(1)}_{f,n}, L^{(2)}_{f,n}, r^{(1)}_{f.v.n}$ and $r^{(2)}_{f,v,n}$. And we combine Theorem 1.1 and Theorem 1.2. In section 3, we obtain the refined inequality for differentiable convex function f. In section 4, we obtain the refined inequality for twice differentiable but not necessarily convex function g. At last in section 5, we extend the result of Moslehian for operator convex function in the refined operator inequality.

2. Relations between $L^{(1)}_{f,n}(a,b)$, $L^{(2)}_{f,n}(a,b)$, $r^{(1)}_{f,v,n}(a,b)$ and $r^{(2)}_{f,v,n}(a,b)$

We give the following results.

Theorem 2.1. *The following (1), (2), (3) and (4) hold.*

(1) $L^{(1)}_{f.n}(a,b)$ *and* $r^{(1)}_{f,v,n}(a,b)$ *are increasing for n.*
$L^{(2)}_{f,n}(a,b)$ *and* $r^{(2)}_{f,v,n}(a,b)$ *are decreasing for n.*

(2) $r^{(1)}_{f,\frac{1}{2},n}(a,b) = L^{(1)}_{f,n+1}(a,b)$,
$r^{(2)}_{f,\frac{1}{2},n}(a,b) = L^{(2)}_{f,n+1}(a,b)$.

(3) $r^{(1)}_{f,v,n}(a,b) \le L^{(1)}_{f,n+1}(a,b)$ *does not hold.*
$L^{(2)}_{f,n+1} \le r^{(2)}_{f,v,n}(a,b)$ *does not hold.*
$r^{(1)}_{f,v,n}(a,b) \ge L^{(1)}_{f,n}(a,b)$ *does not hold.*
$L^{(2)}_{f,n}(a,b) \ge r^{(2)}_{f,v,n}(a,b)$ *does not hold.*

(4) $L^{(2)}_{f,n+1}(a,b) = \frac{1}{2}(L^{(1)}_{f,n}(a,b) + L^{(2)}_{f,n}(a,b))$,
$r^{(2)}_{f,v,n+1}(a,b) = \frac{1}{2}(r^{(1)}_{f,v,n}(a,b) + r^{(2)}_{f,v,n}(a,b))$.

Proof. (1) They are shown in Proposition 2.2 and Theorem 3.1 in [13].

(2) We show the first relation.

$$
\begin{aligned}
&L^{(1)}_{f,n+1}(a,b)\\
&= \frac{1}{2^{n+1}}\sum_{k=1}^{2^{n+1}} f((1-\frac{2k-1}{2^{n+2}})a+\frac{2k-1}{2^{n+2}}b)\\
&= \frac{1}{2^{n+1}}\Bigg[f\left(\left(1-\frac{1}{2^{n+2}}\right)a+\frac{1}{2^{n+2}}b\right)+f\left(\left(1-\frac{3}{2^{n+2}}\right)a+\frac{3}{2^{n+2}}b\right)\\
&\qquad +f\left(\left(1-\frac{5}{2^{n+2}}\right)a+\frac{5}{2^{n+2}}b\right)\\
&\qquad +\cdots+f\left(\left(1-\frac{2^{n+1}-1}{2^{n+2}}\right)a+\frac{2^{n+1}-1}{2^{n+2}})b\right)\\
&\qquad +f\left(\left(1-\frac{2^{n+1}+1}{2^{n+2}}\right)a+\frac{2^{n+1}+1}{2^{n+2}}b\right)\\
&\qquad +f\left(\left(1-\frac{2^{n+1}+3}{2^{n+2}}\right)a+\frac{2^{n+1}+3}{2^{n+2}})b\right)\\
&\qquad +\cdots+f\left(\frac{5}{2^{n+2}}a+\frac{2^{n+2}-5}{2^{n+2}}b\right)+f\left(\frac{3}{2^{n+2}}a+\frac{2^{n+2}-3}{2^{n+2}}b\right)\\
&\qquad +f\left(\frac{1}{2^{n+2}}a+\frac{2^{n+2}-1}{2^{n+2}}b\right)\Bigg]\\
&= \frac{1}{2^{n+1}}\Bigg[\sum_{k=1}^{2^n} f\left(\left(1-\frac{2k-1}{2^{n+2}}\right)a+\frac{2k-1}{2^{n+2}}b\right)\\
&\qquad +\sum_{k=2^n+1}^{2^{n+1}} f\left(\left(1-\frac{2k-1}{2^{n+2}}\right)a+\frac{2k-1}{2^{n+2}}b\right)\Bigg]\\
&= \frac{1}{2^{n+1}}\Bigg[\sum_{k=1}^{2^n} f\left(\left(1-\frac{2k-1}{2^{n+2}}\right)a+\frac{2k-1}{2^{n+2}}b\right)\\
&\qquad ++\sum_{k=1}^{2^n} f\left(\left(\frac{1}{2}-\frac{2k-1}{2^{n+2}}\right)a+\left(\frac{1}{2}+\frac{2k-1}{2^{n+2}}\right)b\right)\Bigg]\\
&= \frac{1}{2^{n+1}}\sum_{k=1}^{2^n}\left\{f(a+(2k-1)h_{n+2})+f\left(\frac{a+b}{2}+(2k-1)h_{n+2}\right)\right\}\\
&= r^{(1)}_{f,\frac{1}{2},n}(a,b).
\end{aligned}
$$

Next we show the second relation.

$$
\begin{aligned}
&L^{(2)}_{f,n+1}(a,b)\\
&= \frac{1}{2^{n+2}}\left\{ f(a)+f(b)+2\sum_{k=1}^{2^{n+1}-1} f(a+kh_{n+1})\right\}\\
&= \frac{1}{2^{n+2}}\left\{ f(a)+f(b)+2\sum_{k=1}^{2^{n+1}-1} f\left(\Big(1-\frac{k}{2^{n+1}}\Big)a+\frac{k}{2^{n+1}}b\right)\right\}\\
&= \frac{1}{2^{n+1}}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{2^{n+1}-1}{2^{n+1}}a+\frac{1}{2^{n+1}}b\right)+f\left(\frac{2^{n+1}-2}{2^{n+1}}a+\frac{2}{2^{n+1}}b\right)\right.\\
&\quad +f\left(\frac{2^{n+1}-3}{2^{n+1}}a+\frac{3}{2^{n+1}}b\right)+\cdots+f\left(\frac{2^{n+1}-2^n+1}{2^{n+1}}a+\frac{2^n-1}{2^{n+1}}b\right)\\
&\quad +f\left(\Big(1-\frac{2^n}{2^{n+1}}\Big)a+\frac{2^n}{2^{n+1}}b\right)+f\left(\frac{2^n-1}{2^{n+1}}a+\Big(\frac{2^n-1}{2^{n+1}}+\frac{1}{2^n}\Big)b\right)\\
&\quad \left.+f\left(\frac{2^n-2}{2^{n+1}}a+\Big(\frac{2^n-2}{2^{n+1}}+\frac{2}{2^n}\Big)b\right)+\cdots+f\left(\frac{1}{2^{n+1}}a+\Big(\frac{1}{2^{n+1}}+\frac{2^n-1}{2^n}\Big)b\right)\right]\\
&= \frac{1}{2^{n+1}}\left[\frac{f(a)+f(b)}{2}+f\Big(\frac{a+b}{2}\Big)\right.\\
&\quad \left.+2\sum_{k=1}^{2^n-1}\left\{\frac{1}{2}f\left(\Big(1-\frac{k}{2^{n+1}}\Big)a+\frac{k}{2^{n+1}}b\right)+\frac{1}{2}f\left(\frac{2^n-k}{2^{n+1}}a+\Big(\frac{2^n-k}{2^{n+1}}+\frac{k}{2^n}\Big)b\right)\right\}\right]\\
&= r^{(2)}_{f,\frac{1}{2},n}(a,b).
\end{aligned}
$$

(3) Let $f(x)=x^3$, $a=0,b=1$ and $v=\frac{13}{25}=0.52$. Then

$$r^{(1)}_{f,v,0}(a,b)=0.219848,\ L^{(1)}_{f,1}(a,b)=0.21875.$$

So $r^{(1)}_{f,v,0}(a,b)>L^{(1)}_{f,1}(a,b)$. And

$$r^{(2)}_{f,v,0}(a,b)=0.310304,\ L^{(2)}_{f,1}(a,b)=0.3125.$$

So $r^{(2)}_{f,v,0}(a,b)<L^{(2)}_{f,1}(a,b)$. Let $f(x)=|x-\frac{1}{2}|$, $a=0,b=1$ and $v=\frac{1}{3}$. Then

$$r^{(1)}_{f,v,1}(a,b)=0.22222\cdots,\ L^{(1)}_{f,1}(a,b)=0.25.$$

So $r^{(1)}_{f,v,1}(a,b)<L^{(1)}_{f,1}(a,b)$. And

$$r^{(2)}_{f,v,1}(a,b)=0.2777\cdots,\ L^{(2)}_{f,1}(a,b)=0.25.$$

So $L^{(2)}_{f,1}(a,b)<r^{(2)}_{f,v,1}(a,b)$.

(4)

$$
\begin{aligned}
&L_{f,n}^{(1)}(a,b)+L_{f,n}^{(2)}(a,b)\\
&=\frac{1}{2^n}\left\{\frac{f(a)+f(b)}{2}+\sum_{k=1}^{2^n}f(a+(2k-1)h_{n+1})+\sum_{k=1}^{2^n-1}f(a+kh_n)\right\}\\
&=\frac{1}{2^n}\left\{\frac{f(a)+f(b)}{2}+\sum_{k=1}^{2^n}f(a+(2k-1)h_{n+1})+\sum_{k=1}^{2^n-1}f(a+2kh_{n+1})\right\}\\
&=\frac{1}{2^n}\left\{\frac{f(a)+f(b)}{2}+f(a+h_{n+1})+f(a+3h_{n+1})\right.\\
&\qquad+\cdots+f(a+(2^{n+1}-1)h_{n+1})\\
&\qquad\qquad+f(a+2h_{n+1})+f(a+4h_{n+1})+f(a+6h_{n+1})\\
&\qquad\qquad\quad+\cdots+f(a+(2^{n+1}-2)h_{n+1})\}\\
&=\frac{1}{2^n}\left\{\frac{f(a)+f(b)}{2}+\sum_{k=1}^{2^{n+1}-1}f(a+kh_{n+1})\right\}\\
&=2L_{f,n+1}^{(2)}(a,b).
\end{aligned}
$$

Then we have $L_{f,n+1}^{(2)}(a,b)=\frac{1}{2}\{L_{f,n}^{(1)}(a,b)+L_{f,n}^{(2)}(a,b)\}$. Similarly,

$$
\begin{aligned}
&r_{f,v,n}^{(1)}(a,b)+r_{f,v,n}^{(2)}(a,b)\\
&=\frac{1}{2^{n+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\}\\
&\quad+\frac{1}{2^n}\sum_{k=1}^{2^n}\{vf(a+(2k-1)vh_{n+1})\\
&\qquad\qquad+(1-v)f((1-v)a+vb+(2k-1)(1-v)h_{n+1})\}\\
&\quad+\frac{1}{2^n}\sum_{k=1}^{2^n-1}\{vf(a+kvh_n)+(1-v)f((1-v)a+vb+k(1-v)h_n)\}\\
&=\frac{1}{2^{n+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\}\\
&\quad+\frac{1}{2^n}\sum_{k=1}^{2^n}\{vf(a+(2k-1)vh_{n+1})\\
&\qquad\qquad+(1-v)f((1-v)a+vb+(2k-1)(1-v)h_{n+1})\}\\
&\quad+\frac{1}{2^n}\sum_{k=1}^{2^n-1}\{vf(a+2kvh_{n+1})+(1-v)f((1-v)a+vb+2k(1-v)h_{n+1})\}\\
&=\frac{1}{2^{n+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2^n}\{vf(a+vh_{n+1})+(1-v)f((1-v)a+vb+(1-v)h_{n+1})\\
&+vf(a+3vh_{n+1})+(1-v)f((1-v)a+vb+3(1-v)h_{n+1})\\
&+vf(a+5vh_{n+1})+(1-v)f((1-v)a+vb+5(1-v)h_{n+1})\\
&+\cdots\\
&+vf(a+(2^{n+1}-1)vh_{n+1})\\
&\qquad\qquad+(1-v)f((1-v)a+vb+(2^{n+1}-1)(1-v)h_{n+1})\\
&+vf(a+2vh_{n+1})+(1-v)f((1-v)a+vb+2(1-v)h_{n+1})\\
&+vf(a+4vh_{n+1})+(1-v)f((1-v)a+vb+4(1-v)h_{n+1})\\
&+vf(a+6vh_{n+1})+(1-v)f((1-v)a+vb+6(1-v)h_{n+1})\\
&+\cdots\\
&+vf(a+(2^{n+1}-2)vh_{n+1})\\
&\qquad\qquad+(1-v)f((1-v)a+vb+(2^{n+1}-2)(1-v)h_{n+1})\}\\
=&\frac{1}{2^{n+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\}\\
&+\frac{1}{2^n}\sum_{k=1}^{2^{n+1}-1}\{vf(a+kvh_{n+1})+(1-v)f((1-v)a+vb+k(1-v)h_{n+1})\}\\
=&2r^{(2)}_{f,v,n+1}(a,b)
\end{aligned}
$$

Then we have $r^{(2)}_{f,v,n+1}(a,b)=\frac{1}{2}\{r^{(1)}_{f,v,n}(a,b)+r^{(2)}_{f,v,n}(a,b)\}$. □

Finally we get the result by combining Theorem 1.1 and Theorem 1.2.

Theorem 2.2. *For any* $0\le\alpha,\beta\le 1$, $0\le v\le 1$ *and* $m_1,m_2,n_1,n_2\in\mathbb{N}\cup\{0\}$,

$$
\begin{aligned}
\alpha L^{(1)}_{f,m_1}(a,b)+(1-\alpha)r^{(1)}_{f,v,m_2}(a,b)&\le\int_0^1 f((1-t)a+tb)dt\\
&=\frac{1}{b-a}\int_a^b f(t)dt\\
&\le\beta L^{(2)}_{f,n_1}(a,b)+(1-\beta)r^{(2)}_{f,v,n_2}(a,b).
\end{aligned}
$$

3. Case of differential convex functions

In the case of differentiable convex function f we give another Hermite-Hadamard inequality different from (1.2).

Theorem 3.1. *Let $f(x)$ be a differentiable convex function and $v \in [0,1]$. Then the following holds.*

$$\begin{aligned} f((1-v)a+vb)+\frac{1}{2}f^{'}((1-v)a+vb)(1-2v)(b-a) &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\} \end{aligned} \tag{3.1}$$

Proof. Let $P(a,0), Q(b,0), R((1-v)a+vb,0), A(a,f(a)), B(b,f(b)), C((1-v)a+vb, f((1-v)a+vb))$. Since the upper bound of $\int_a^b f(x)dx$ is sum of area of the trapezoid $APRC$ and area of the trapezoid $CRQB$, we get

$$\begin{aligned} \int_a^b f(x)dx &\leq \frac{1}{2}v(b-a)(f(a)+f((1-v)a+vb)) \\ &\quad + \frac{1}{2}(1-v)(b-a)(f(b)+f((1-v)a+vb)) \\ &= \frac{1}{2}(b-a)\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\} \end{aligned}$$

Next we get the lower bound of $\int_a^b f(x)dx$.The tangent line in C is given by

$$y-f((1-v)a+vb)=f^{'}((1-v)a+vb)(x-(1-v)a-vb).$$

When $x=a$, we put

$$\begin{aligned} y_a &= f((1-v)a+vb)+f^{'}((1-v)a+vb)(a-(1-v)a-vb) \\ &= f((1-v)a+vb)+f^{'}((1-v)a+vb)v(a-b). \end{aligned}$$

And when $x=b$, we put

$$\begin{aligned} y_b &= f((1-v)a+vb)+f^{'}((1-v)a+vb)(b-(1-v)a-vb) \\ &= f((1-v)a+vb)+f^{'}((1-v)a+vb)(1-v)(b-a). \end{aligned}$$

Let $E(a,y_a), F(b,y_b)$. Since the lower bound of $\int_a^b f(x)dx$ is area of the trapezoid $EPQF$, we get

$$\begin{aligned} \int_a^b f(x)dx &\geq \frac{1}{2}(b-a)(y_a+y_b) \\ &= \frac{1}{2}(b-a)\{2f((1-v)a+vb)+f^{'}((1-v)a+vb)(1-2v)(b-a)\}. \end{aligned}$$

Then we have the result. □

We give other type of inequality.

Theorem 3.2. *Let $f(x)$ be a differentiable convex function and $v \in [0,1]$. Then the following holds.*

$$\begin{aligned} &vf((1-v^2)a+v^2b)+\frac{1}{2}v^2f^{'}((1-v^2)a+v^2b)(b-a)(1-2v) \\ &\quad + (1-v)f((1-v)^2a+v(2-v)b) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}(1-v)^2(1-2v)f^{'}((1-v)^2a+v(2-v)b)(b-a)\\
&\quad\le \frac{1}{b-a}\int_a^b f(x)dx\\
&\quad\le \frac{1}{2}\{v^2f(a)+v(1-v)f((1-v)a+vb)+vf((1-v^2)a+v^2b)\}\\
&\qquad+\frac{1}{2}\{v(1-v)f((1-v)a+vb)\\
&\qquad+(1-v)^2f(b)+(1-v)f((1-v)^2a+v(2-v)b)\}.
\end{aligned}
$$

Proof. Applying the inequality (3.1) on the interval $[a,(1-v)a+vb]$, we have

$$
\begin{aligned}
&f((1-v^2)a+v^2b)+\frac{1}{2}f^{'}((1-v^2)a+v^2b)v(b-a)(1-2v)\\
&\quad\le \frac{1}{v(b-a)}\int_a^{(1-v)a+vb} f(x)dx \qquad (3.2)\\
&\quad\le \frac{1}{2}\{vf(a)+(1-v)f((1-v)a+vb)+f((1-v^2)a+v^2b)\}
\end{aligned}
$$

Similarly we apply (3.1) on the $[(1-v)a+vb,b]$, we have

$$
\begin{aligned}
&f((1-v)^2a+v(2-v)b)+\frac{1}{2}f^{'}((1-v)^2a+v(2-v)b)(1-v)(1-2v)(b-a)\\
&\quad\le \frac{1}{(1-v)(b-a)}\int_{(1-v)a+vb}^b f(x)dx \qquad (3.3)\\
&\quad\le \frac{1}{2}\{vf((1-v)a+vb)+(1-v)f(b)+f((1-v)^2a+v(2-v)b)\}
\end{aligned}
$$

Multiplying v and $(1-v)$ to the both sides (3.2) and (3.3) respectively and summing each side, we obtain the result. □

4. Case of twice differentiable but not necessary convex functions

We begin with an improved variant of the refined Hermite-Hadamard inequality.

Theorem 4.1. *Let g be a twice differentiable function on $[a,b]$ and $s=\min_{a\le t\le b} g^{''}(t)$, $S=\max_{a\le t\le b} g^{''}(t)$. Then for any $m,n\in\mathbb{N}\cup\{0\}$*

(1)

$$
\begin{aligned}
&L^{(1)}_{g,n}(a,b)-\frac{s}{2^{n+1}}\sum_{k=1}^{2^n}(a+(2k-1)h_{n+1})^2+\frac{s(b^3-a^3)}{6(b-a)}\le\frac{1}{b-a}\int_a^b g(t)dt\\
&\quad\le L^{(2)}_{g,m}(a,b)-\frac{s}{2^{m+2}}\left\{a^2+b^2+2\sum_{k=1}^{2^m-1}(a+kh_m)^2\right\}+\frac{s(b^3-a^3)}{6(b-a)}.
\end{aligned}
$$

(2)

$$L_{g,m}^{(2)}(a,b) - \frac{S}{2^{m+2}}(a^2+b^2+2\sum_{k=1}^{2^m-1}(a+kh_m)^2) + \frac{S(b^3-a^3)}{6(b-a)} \le \frac{1}{b-a}\int_a^b g(t)dt$$
$$\le L_{g,n}^{(1)}(a,b) - \frac{S}{2^{n+1}}\sum_{k=1}^{2^n}(a+(2k-1)h_{n+1})^2 + \frac{S(b^3-a^3)}{6(b-a)}.$$

Proof. (1) For given twice differentiable function g, we define an auxilliary function f by $f(t) = g(t) - \frac{st^2}{2}$. Since $f^{''}(t) = g^{''}(t) - s \ge 0$, we find out that f is a convex function on $[a,b]$. Therefore, applying the form of refined Hermite-Hadamard inequality given by (1.3), we obtain

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{g(a+(2k-1)h_{n+1}) - \frac{s}{2}(a+(2k-1)h_{n+1})^2\right\} \le \frac{1}{b-a}\int_a^b f(t)dt$$
$$\le \frac{1}{2^{m+1}}\left\{g(a) - \frac{sa^2}{2} + g(b) - \frac{sb^2}{2} + 2\sum_{k=1}^{2^m-1}\left\{g(a+kh_m) - \frac{s}{2}(a+kh_m)^2\right\}\right\}.$$

Then

$$L_{g,n}^{(1)}(a,b) - \frac{s}{2^{n+1}}\sum_{k=1}^{2^n}(a+(2k-1)h_{n+1})^2$$
$$\le \frac{1}{b-a}\int_a^b g(t)dt - \frac{s}{2(b-a)}\int_a^b t^2 dt$$
$$\le L_{g,m}^{(2)}(a,b) - \frac{s}{2^{m+2}}\left\{a^2+b^2+2\sum_{k=1}^{2^m-1}(a+kh_m)^2\right\}.$$

Hence we get the result.

(2) We define an auxilliary function f by $f(t) = \frac{St^2}{2} - g(t)$. Since $f^{''}(t) = S - g^{''}(t) \ge 0$, we find out that f is a convex function on $[a,b]$. Therefore, applying the form of refined Hermite-Hadamard inequality given by (1.3), we obtain

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{\frac{S}{2}(a+(2k-1)h_{n+1})^2 - g(a+(2k-1)h_{n+1})\right\} \le \frac{1}{b-a}\int_a^b f(t)dt$$
$$\le \frac{1}{2^{m+1}}\left\{\frac{Sa^2}{2} - g(a) + \frac{Sb^2}{2} - g(b) + 2\sum_{k=1}^{2^m-1}\left\{\frac{S}{2}(a+kh_m)^2 - g(a+kh_m)\right\}\right\}.$$

Then

$$\frac{S}{2^{n+1}}\sum_{k=1}^{2^n}(a+(2k-1)h_{n+1})^2 - L_{g,n}^{(1)}(a,b)$$
$$\le \frac{S}{2(b-a)}\int_a^b t^2 dt - \frac{1}{b-a}\int_a^b g(t)dt$$

$$\leq \frac{S}{2^{m+2}}\Big(a^2+b^2+2\sum_{k=1}^{2^m-1}(a+kh_m)^2\Big)-L^{(2)}_{g,m}(a,b).$$

Hence we get the result. ☐

5. Case of operator convex functions

If A, B are self-adjoint operators on a Hilbert space H with spectra in an interval J and f is an operator convex function on J, then the following two theorems hold.

Theorem 5.1. *Let $f(x)$ be an operator convex function on J. Then for any $m, n \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned}\frac{1}{2^n}\sum_{k=1}^{2^n} f\left(\Big(1-\frac{2k-1}{2^{n+1}}\Big)A+\frac{2k-1}{2^{n+1}}B\right) &\leq \int_0^1 f((1-t)A+tB)dt \\ &\leq \frac{1}{2^{m+1}}\left\{f(A)+f(B)+2\sum_{k=1}^{2^m-1} f\left(\Big(1-\frac{k}{2^m}\Big)A+\frac{k}{2^m}B\right)\right\}.\end{aligned} \tag{5.1}$$

Proof. It is clear that $g(t)=\langle f((1-t)A+tB)x,x\rangle$ is a real valued convex function on $[0,1]$. By putting $a=0, b=1$ in Theorem 1.1, we have

$$\begin{aligned}\frac{1}{2^n}\sum_{k=1}^{2^n} g(\frac{2k-1}{2^{n+1}}) &\leq \int_0^1 g(t)dt \\ &\leq \frac{1}{2^{m+1}}\left\{g(0)+g(1)+2\sum_{k=1}^{2^m-1} g\Big(\frac{k}{2^m}\Big)\right\}\end{aligned}$$

We have (5.1) by rewriting the above inequalty. ☐

Theorem 5.2. *Let $f(x)$ be an operator convex function on J. Then for any $v \in [0,1]$ and $m, n \in \mathbb{N} \cup \{0\}$,*

$$\begin{aligned}&\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{v\, f\left(\Big(1-\frac{(2k-1)v}{2^{n+1}}\Big)A+\frac{(2k-1)v}{2^{n+1}}B\right)\right. \\ &\qquad \left.+(1-v)f\left(\Big(1-v-\frac{(2k-1)(1-v)}{2^{n+1}}\Big)A+\Big(v+\frac{(2k-1)(1-v)}{2^{n+1}}\Big)B\right)\right\} \\ &\quad \leq \int_0^1 f((1-t)A+tB)dt \\ &\quad \leq \frac{1}{2^{m+1}}\{vf(A)+(1-v)f(B)+f((1-v)A+vB)\} \\ &\qquad +\frac{1}{2^m}\sum_{k=1}^{2^m-1}\left\{vf\left(\Big(1-\frac{kv}{2^m}\Big)A+\frac{kv}{2^m}B\right)\right. \\ &\qquad\qquad \left.+(1-v)f\left(\Big(1-v-\frac{k(1-v)}{2^m}\Big)A+\Big(v+\frac{k(1-v)}{2^m}\Big)B\right)\right\}\end{aligned} \tag{5.2}$$

Proof. Let $g(t) = \langle f((1-t)A + tB)x, x\rangle$. By putting $a = 0, b = 1$ in Theorem 1.2, we have

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{vg\Big(\frac{(2k-1)v}{2^{n+1}}\Big) + (1-v)g\Big(v + \frac{(2k-1)(1-v)}{2^{n+1}}\Big)\right\}$$
$$\leq \int_0^1 g(t)dt \leq \frac{1}{2^{m+1}}\{vg(0) + (1-v)g(1) + g(v)\}$$
$$+ \frac{1}{2^m}\sum_{k=1}^{2^m-1}\left\{vg\Big(\frac{kv}{2^m}\Big) + (1-v)g\Big(v + \frac{k(1-v)}{2^m}\Big)\right\}$$

We have (5.2) by rewriting the above inequality. □

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