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EXTENSION OF STRICTLY MONOTONIC FUNCTIONS AND UTILITY FUNCTIONS ON ORDER-SEPARABLE SPACES

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ABSTRACT. The classical Uryson-Titze theorem states that every continuous function defined on a closed subset of a normal topological space can be extended to the whole space. However, not every continuous and monotone function defined on a closed subset of a normally preordered space is extendable to the whole space. Nachbin found a necessary and sufficient condition for the existence of such an extension for nonstrictly monotone functions. This paper provides a necessary and sufficient condition for the extendability of the continuous strictly monotone functions defined on closed subsets of a normally preordered space with the separable preorder. Important examples of such spaces are the Euclidean spaces with the strict componentwise order. This extendability result implies the existence of a utility function for a separable continuous preorder of a normally preordered space. Applications to the extension of the strictly monotone preferences in Euclidean spaces are given.

1. Introduction and preliminaries

It is somewhat surprising that the problem of extending a continuous monotone function defined on a subset of a Euclidean space, or more generally on a subset of a preordered topological space, into the entire space has received scant attention.

Nachbin (1976) studied the problem of extending a continuous, monotone and bounded function defined on a closed subset of a normally preordered topological space. A normally preordered topological space, introduced by Nachbin (1976) is a topological space equipped with a preorder R, in which every pair of disjoint $R-$increasing and $R-$decreasing closed subsets can be separated by $R-$increasing and $R-$decreasing open sets. He discovered a property (henceforth, the *Nachbin property*) that is necessary and sufficient for the existence of an extension that satisfies the said properties Nachbin (1976, Theorem 2). Nachbin's extension theorem has found applications in diverse fields of mathematics, decision theory, computer science, and others.

We start by formulating a property that is equivalent to the Nachbin property and hence get a modification of Nachbin's extension theorem. This property readily adapts to the problem of extending strictly monotone functions.

A *preorder* $\succcurlyeq$ on a set X is a reflexive and transitive binary relation on this set. The strict part of preorder $\succcurlyeq$, denoted as $\succ$, is defined as: for $x, y \in X$, $x \succ y$ if $x \succcurlyeq y$ and not $y \succcurlyeq x$. If in addition, preorder $\succcurlyeq$ is antisymmetric (that is $x \succcurlyeq y$ and $y \succcurlyeq x$ imply $x = y$) then it is called an *order.* A real function f

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y if $x \succcurlyeq y$ and not $y \succcurlyeq x$. If in addition, preorder $\succcurlyeq$ is antisymmetric (that is $x \succcurlyeq y$ and $y \succcurlyeq x$ imply $x = y$) then it is called an *order.* A real function f defined on a subset D of the preordered set X is $\succcurlyeq$ *−increasing* if for any two points $x, y \in D$ such that $x \succcurlyeq y$, we have $f(x) \geqslant f(y)$. Function f is $\succcurlyeq$ *−strictly increasing* if it is $\succcurlyeq$ −increasing and for any two points $x, y \in D$ such that $x \succ y$, we have $f(x) > f(y)$. We will say *increasing* and *strictly increasing* in cases where no confusion should arise. *Decreasing* and *strictly decreasing* functions are defined in a dual way. A function is *strictly monotone* if it is strictly increasing or strictly decreasing. Throughout the paper we will discuss strictly increasing functions with the understanding that all the results, assumptions and arguments are modified in trivial ways for strictly decreasing functions. For two sets D, D' with $D \subset D'$ and a function $f : D \to R$ function $f' : D' \to R$ is called an *extension* of function f if $f'(x) = f(x)$ for all $x \in D$.

Next we define two orders on R^n. For two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in R^n we write $x \geqslant y$ if $x_k \geqslant y_k$ for all $k = 1, \ldots, n$, $x \geq y$ if $x \geqslant y$ and $x \neq y$, and $x > y$ if $x_k > y_k$ for all $k = 1, \ldots, n$. We also write $x \leqslant y,\ x \leq y,\ x < y$ if $y \geqslant x,\ y \geq x,\ y > x$, respectively. For $x, y \in R^n$, set $x \geqslant_s y$ if $x = y$ or $x_k > y_k$ for all $k = 1, \cdots, n$. So, the strict part of preorder $\geqslant_s$ is $>$. It is easy to see that this preorder is separable[1]. Note that a function $f : R^n \to R$ is $\geqslant$-strictly increasing if $f(x) > f(y)$ whenever $x \geq y$, and f is $\geqslant_s$-strictly increasing if $f(x) > f(y)$ for $x, y \in R^n$ with $x_k > y_k$ for all $k = 1, \ldots, n$.

A set A in X is *decreasing* if $x \in A$ and $x \succcurlyeq y$ imply $y \in A$. An *increasing* set is defined dually. A set X equipped with the both topology τ and preorder (order) $\succcurlyeq$ is said to be *normally preordered (ordered) space* if, for any two closed disjoint subsets F_0 and F_1 of X, such that F_0 is decreasing and F_1 is increasing, there exist disjoint open sets U_0 and U_1 of X, such that U_0 is decreasing and contains F_0 and U_1 is increasing and contains F_1.

Let X be a set, τ a topology, and $\succcurlyeq$ a preorder on X. For a subset $A \subset X$ denote by $d(A) = \{x \in X : y \succcurlyeq x,\ \exists y \in A\}$ the *decreasing cover* of A, and $i(A) = \{x \in X : x \succcurlyeq y,\ \exists y \in A\}$ the *increasing cover* of A. The *decreasing closure*, denoted as $\mathcal{D}(A)$, of a set A in X is the smallest decreasing and closed set containing A. *The increasing closure* of A is defined dually and denoted as $\mathcal{I}(A)$. Further, we will assume that the preorder $\succcurlyeq$ of a normally preordered space $(X, \tau, \succcurlyeq)$ is continuous in the following sense: for every open set $V \subset X$ both the decreasing cover $d(V)$ and the increasing cover $i(V)$ are open sets. It is clear that both of the orders $\geqslant$ and $\geqslant_s$ are continuous. This type of continuity was introduced by McCartan (1971). Let $(X, \tau, \succcurlyeq)$ be a normally preordered space. For a $\succcurlyeq$ −increasing function $f : D \to R$, where $D \subset X$, and a real α, set

$$L_f(\alpha) = \{x \in D : f(x) \leqslant \alpha\} \text{ and } U_f(\alpha) = \{x \in D : f(x) \geqslant \alpha\}.$$

Now the *Nachbin property* reads as follows: for each $\alpha, \alpha' \in R$ such that $\alpha < \alpha'$

$$\mathcal{D}(L_f(\alpha)) \cap \mathcal{I}(U_f(\alpha')) = \emptyset. \tag{1.1}$$

[1]For a formal definition of the *separable preorder* see the next section.

The author (2010) introduced a condition that is equivalent to the Nachbin property (1.1). To formulate this condition we need some notations. For $x \in X$ denote by $\mathcal{V}_d^x$ and $\mathcal{V}_i^x$ the collections of open decreasing and open increasing sets containing x, respectively. For a continuous and increasing function $f : D \to R$, where D is an arbitrary set in X, and point $x \in X$ we set

$$m_f(x) = \inf_{V_d \in \mathcal{V}_d^x} \sup\{f(z) : z \in D \cap V_d\} \text{ and } M_f(x) = \sup_{V_i \in \mathcal{V}_i^x} \inf\{f(z) : z \in D \cap V_i\},$$

with the agreement that $m_f(x) = \inf\{f(z) : z \in D\}$ and $M_f(x) = \sup\{f(z) : z \in D\}$, if $D \cap V_d = \emptyset$ for some $V_d \in \mathcal{V}_d^x$ and $D \cap V_i = \emptyset$ for some $V_i \in \mathcal{V}_i^x$, respectively. Clearly, in the definitions of functions m_f and M_f collections V_d^x and V_i^x can be replaced with arbitrary open bases of V_d^x and V_i^x, respectively. Further we will omit the subindex f in the notations m_f and M_f in cases where it is clear which function is referred to.

Proposition 1.1. *The Nachbin property is equivalent to the following property:*

$$m_f(x) \leqslant M_f(x) \quad \textit{for all} \quad x \in X. \tag{1.2}$$

Proof. Assume that the Nachbin property (1.1) is satisfied, but $m(x) > M(x)$ for some $x \in X$. Denote $\varepsilon = \frac{1}{4}[m(x) - M(x)]$. By Theorem 2 from Nachbin (1976, p. 36) there exists a continuous $\succcurlyeq$ −increasing extension, F, of function f. Let V^x be an open neighborhood of x such that

$$|F(y) - F(x)| < \varepsilon \text{ for } y \in V^x.$$

By the continuity of preorder $\succcurlyeq$, $i(V^x)$ and $d(V^x)$ are respectively, decreasing and increasing open neighborhoods of x. Since function F is $\succcurlyeq$ −increasing we have

$$F(z) > F(x) - \varepsilon \text{ for } z \in i(V^x) \text{ and } F(z) < F(x) + \varepsilon \text{ for } z \in d(V^x).$$

These inequalities imply that

$$M(x) \geqslant \inf\{f(z) : z \in D \cap i(V^x)\} \geqslant F(x) - \varepsilon,$$
$$m(x) \leqslant \sup\{f(z) : z \in D \cap d(V^x)\} \leqslant F(x) + \varepsilon.$$

So $F(x) - \varepsilon \leqslant M(x) < m(x) \leqslant F(x) + \varepsilon$. From this and the definition of ε

$$4\varepsilon = m(x) - M(x) < 2\varepsilon,$$

a contradiction. So the Nachbin property implies (1.2).

Assume that the Nachbin property (1.1) is violated, that is, there exist reals α, α' with $\alpha < \alpha'$ such that

$$\mathcal{D}(L_f(\alpha)) \cap \mathcal{I}(U_f(\alpha')) \neq \emptyset.$$

Let x belong to this intersection. Then $m(x) \geqslant \alpha'$ and $\alpha \geqslant M(x)$, which imply $m(x) > M(x)$. This contradicts property (1.2). □

The equivalence of the properties (1.1) and (1.2) allows us to formulate the following version of Nachbin's extension theorem:

Theorem 1.2. *Let X be a normally preordered space with a continuous preorder, and let $D \subset X$ be a closed set. Let $f : D \to R$ be a continuous and increasing function. Then f has an extension into X if and only if property (1.2) holds.*

Remark. Nachbin's extension theorem assumes the boundedness of function f. However, the following simple observation removes this assumption. Since for an increasing homeomorphism $\phi : R \to (0,1)$, $\phi \circ m_f = m_{\phi \circ f}$ and $\phi \circ M_f = M_{\phi \circ f}$, condition (1.2) is satisfied if and only if it is satisfied for function $\phi \circ f$. Now if function $F : X \to R$ is an extension of function $\phi \circ f$ then obviously function $\phi^{-1} \circ F$ is an extension of function f.

Now we give some examples illustrating property (1.2) and its strengthening (see property (2.1) below).

Examples 1-3. Set $D_+ = \{(x_1, x_2) \in R^2 \mid x_1 x_2 = -1,\ x_2 > 0\}$, $D_- = -D_+$, and $D = D_+ \cup D_-$. Define functions $f_i : D \to R$ as $\frac{x_2}{1+x_2}$ on D_+ and $i - 2 + \frac{x_2}{1-x_2}$ on D_- for $i = 1, 2, 3$. We consider R^2 with the order $\geqslant$ defined above. Obviously each of functions f_i $(i = 1, 2, 3)$ is strictly increasing with respect to order $\geqslant$.

It is easy to see that function f_1 has no increasing extension, f_2 has an increasing extension but not a strictly increasing one, and f_3 has a strictly increasing extension into R^2. We have $m_{f_1}(0) = m_{f_2}(0) = m_{f_3}(0) = 0$ and $M_{f_i}(0) = i - 2$ for $i = 1, 2, 3$. Hence $m_{f_1}(0) > M_{f_1}(0)$, $m_{f_2}(0) = M_{f_2}(0)$, and $m_{f_3}(0) < M_{f_3}(0)$. Thus property (1.2) is violated for function f_1 and satisfied for functions f_2 and f_3 at point $(x, y) = (0, 0)$.

2. Extension of strictly increasing functions

Let $(X, \succcurlyeq)$ be a preordered set. For $a, b \in X$ such that $a \prec b$ *open order interval* with the endpoints a, b is defined as $(a, b) = \{z \in X\ : a \prec z \prec b\}$.

A preorder $\succcurlyeq$ on a set X is *separable* if

(a) there is a countable set S in X such that for any pair of points $x, y \in X$ with $x \prec y$ and such that the order interval (x, y) is nonempty there is $s \in S$ such that $x \prec s \prec y$[2]

(b) there is a finite or countable subset I of the set of all pairs $x, y \in X$ with $(x, y) = \emptyset$, such that for each empty interval (x, y) there is $(x', y') \in I$ with $x \sim x'$ and $y \sim y'$.

A set $S \subset X$ satisfying property (a) is called *countable order dense* set. The separability of a preorder defined here is equivalent to each of the Debreu and Jaffray separabilities (see Bridges and Mehta (1995), Definition 1.4.3, Debreu (1964),

[2]If there is no nonempty interval in $(X, \succcurlyeq)$ then this assumption reduces to X being an infinite set.

Jaffray (1975), Herden (1989)). This will be proved in the Appendix. There we will also bring an example showing that Birkhoff (1948) separability is not equivalent to the above separabilities. Note that the equivalence of the latter two separability properties to each other is shown in Proposition 1.4.4 of Bridges and Mehta (1995).

Let $(X, \tau, \succcurlyeq)$ be a normally preordered space. For function $f : D \to R$, where $D \subset X$, define functions

$$\bar{m}_f(x) = \sup\{f(y) : \ y \in D, \ x \succcurlyeq y\} \ \text{ and } \ \bar{M}_f(x) = \inf\{f(y) : \ y \in D, \ y \succcurlyeq x\}$$

on X. If set $\{y \in D : \ x \succcurlyeq y\}$ is empty we assume $\bar{m}_f(x) = -\infty$ and if set $\{y \in D : \ y \succcurlyeq x\}$ is empty we assume $\bar{M}_f(x) = \infty$. Note that like functions m_f and M_f these functions are increasing. As for functions m_f and M_f we will omit the subindex f in the notations $\bar{m}_f$ and $\bar{M}_f$ in cases where it is clear which function is referred to.

Obviously $M(x) \leqslant \bar{M}(x)$ and $m(x) \geqslant \bar{m}(x)$ for all $x \in X$. Note that if f is a $\succcurlyeq$ −strictly increasing function then

$$\bar{M}(y) = f(y) > f(x) = \bar{m}(x) \ \text{ for } \ x, y \in D, \ y \succ x.$$

Theorem 2.1. *Let $(X, \tau, \succcurlyeq)$ be a normally preordered space with the separable and continuous preorder $\succcurlyeq$. Let $D \subset X$ be a nonempty, closed set, and $f : D \to R$ a continuous, strictly increasing function. Then, there exists a continuous, strictly increasing function $F : X \to R$ such that $F(x) = f(x)$ for $x \in D$ if and only if f satisfies the following conditions:*

$$(2.1a) \qquad m(x) \leqslant M(x) \textit{ for all } x \in X,$$
$$(2.1b) \qquad \bar{m}(x) < \bar{M}(y) \textit{ for all } x \in X, \ y \in X \textit{ with } y \succ x.$$

Proof. The necessity of condition (2.1a) follows from Theorem 1.2, and the necessity of condition (2.1b) is obvious. Prove sufficiency. Let $f : D \to R$ be a continuous, strictly increasing function. As in the above Remark, since for an increasing homeomorphism $\phi : R \to (0, 1)$, $\phi \circ \bar{m}_f = \bar{m}_{\phi \circ f}$ and $\phi \circ \bar{M}_f = \bar{M}_{\phi \circ f}$, conditions (2.1) are satisfied for function f if and only if it is satisfied for function $\phi \circ f$. Therefore we may assume that $f(D) \subset (0, 1)$.

Further we use the modification of Nachbin's extension theorem, Theorem 1.2, to show that for any two points $a, b \in X$ such that $b \succ a$ there exists an increasing extension $f_{ab} : D \cup \{a, b\} \to (0, 1)$ of function f such that $f(a) < f(b)$. The case $a \in D$ and $b \in D$ is obvious. Assume $a \in D$ and $b \notin D$. Then, by condition (2.1b), $\bar{m}(a) < \bar{M}(b)$. Set $f_{ab}(b) = \frac{\bar{m}(b)+\bar{M}(b)}{2}$. From the definitions of functions $\bar{m}$ and $\bar{M}$ it is easy to see that f_{ab} is a strictly increasing function. The case $a \notin D$ and $b \in D$ is treated similarly. Assume now $a \notin D$ and $b \notin D$. We set $f_{ab}(a) = \frac{2}{3}\bar{m}(a) + \frac{1}{3}\bar{M}(a)$ and $f_{ab}(b) = \frac{1}{3}\bar{m}(b) + \frac{2}{3}\bar{M}(b)$. Since functions $\bar{m}$ and $\bar{M}$ are increasing and satisfy conditions (2.1) we have $f_{ab}(a) < f_{ab}(b)$ and that function f_{ab} is strictly increasing. It is easy to see that for all the above considered cases function f_{ab} satisfies condition (2.1a). As X is normally preordered and preorder $\succcurlyeq$ is continuous, by Theorem 1.2

there exists an increasing extension of f_{ab} into X. We denote this extension also as f_{ab}.

Next, as in Peleg (1970), we use the separability of preorder $\succcurlyeq$ to show the existence of a countable family of increasing functions $f_k = f_{a_k b_k}$ as above whose weighted sum is a continuos, strictly increasing extension of f. Let S be a countable order dense set in X. It is easily seen that there exists a countable superset S' of S such that for any pair of points $x, y \in X$ with $x \succ y$, either there exists $s \in S'$ with $x \succ s \succ y$ or there exist $s_1, s_2 \in S'$ with $x \sim s_1$ and $y \sim s_2$. Denote $N = \{1, 2, \ldots\}$ and write set $\{(s, s') \in S' \times S' \ : \ s' \succ s\}$ as a sequence $\{(a_k, b_k),\ k \in N_0\}$, where $N_0 \subset N$. As shown above for each $k \in N_0$ there exists an increasing extension $f_k = f_{a_k b_k} : X \to (0, 1)$ of function f, such that $f_k(b_k) > f_k(a_k)$. Define

$$F = \sum_{k \in N_0} \frac{1}{2^k} f_k.$$

Clearly, function F is continuous, increasing, and $F(x) = f(x)$ for $x \in D$. It remains to show that F is strictly increasing. Let $a, b \in X$ and $b \succ a$. Let $(a_l, b_l) \in S'$ be such that $b \succcurlyeq b_l \succ a_l \succcurlyeq a$. This implies $f_l(b) \geqslant f_l(b_l) > f_l(a_l) \geqslant f_l(a)$. This and $f_k(b) \geqslant f_k(a)$ for all $k \in N_0$ imply that $F(b) > F(a)$. □

The following example demonstrates the importance of order separability assumption in this theorem.

Example 4. Let $X = [0, 1]^2$ with the lexicographic order $\succcurlyeq$. That is for $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X, $x \succcurlyeq y$ iff $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. We consider X with the $\succcurlyeq$ $-$interval topology. Clearly X is a normally ordered space. Since there are uncountable infinity of pairwise disjoint intervals in X but not in real with its usual order there is no strictly increasing real-valued function defined on X. Therefore, for no strictly increasing function defined on a closed subset of X there exists a strictly increasing extension into X.

Theorem 2.1 implies the existence of a utility function for the separable continuous preorders on a topological space (X, τ) such that $(X, \tau, \geqslant)$ is a normally topological space.

Corollary 2.2. *Let $(X, \tau, \geqslant)$ be a normally preordered space with a separable continuous preorder $\geqslant$. Then there exists a continuous, $\geqslant$ $-$strictly increasing function (utility) on X.*

Proof. Let $x_0 \in X$. Set $D = \{x_0\}$ and let $f(x_0) = 0$. Obviously f is a continuous strictly increasing function on D which satisfies assumptions (2.1) of Theorem 2.1. □

Corollary 2.2 implies Corollary 4.1.3 of Bridges and Mehta (1995) on the existence of a utility function that is derived from Peleg's utility existence theorem. Indeed, Corollary 4.1.3 assumes the topological space X is equipped with a connected strict partial order such that the decreasing and increasing closures, $(\leftarrow, x)$ and $(x, \rightarrow)$, are open for each point x in X. Moreover, it assumes the existence of a countable Cantor order dense set in X. First, it is easily seen that openness of the decreasing and increasing closures of points in X implies that the space in Corollary 4.1.3 is a normally preordered space and that the preorder $\succcurlyeq$, which is the reflexive closure of strict partial order in Corollary 4.1.3, is (McCartan) continuous. Secondly, the existence of a countable Cantor order dense set in X implies that the preorder $\succcurlyeq$ is separable according to the definition given in this paper.

In Bridges and Mehta (1995) the classical Eilenberg (1941) and Debreu (1964) theorems are derived from Corollary 4.1.3. Therefore, they are also consequences of Corollary 2.2.

Note that Peleg Theorem (1970) assumes a stronger separability property than the one assumed in Theorem 2.1 and Corollary 2.2. However, he does not assume that the space $(X, \tau, \geqslant)$ is normally preordered.

Classical examples of normally ordered spaces are Euclidean spaces ordered with a componentwise order, that is with either of the orders $\geqslant_s$ or $\geqslant$. It is easily seen that R^n with either of orders $\geqslant_s$ and $\geqslant$ is a normally ordered space. Obviously $(R^n, \geqslant_s)$ is order separable but $(R^n, \geqslant)$ is not.

Example 5. Function f_2 from Example 2 is $\geqslant -$strictly increasing. As noted in Example 2 it has no $\geqslant -$strictly increasing extension. However, conditions (2.1) are satisfied for f_2. So, conditions (2.1) are not sufficient for extendability in the case of nonseparable orders.

Denote by e the vector in R^n all of whose components are 1, and $e_k\,(k = 1, \dots, n)$ the vector in R^n whose k-th component is 1 and all other components are 0.

It is easy to see that for both the orders $\geqslant$ and $\geqslant_s$ sets

$$V_r^1 = \{z \in R^n \ : \ z < x + re\} \ \text{ and } \ V_r^2 = \{z \in R^n \ : \ z > x - re\} \ \ (r > 0)$$

are the bases of open decreasing and open increasing sets, respectively, that contain x. So the definitions of functions m_f and M_f are simplified as follows

$$m_f(x) = \inf_{r>0} \sup\{f(z) \ : \ z \in D \cap V_r^1\} \ \text{ and } \ M_f(x) = \sup_{r>0} \inf\{f(z) \ : \ z \in D \cap V_r^2\}.$$

Corollary 2.3. *Let $D \subset R^n$ be a nonempty, closed set and $f : D \to R$ be a continuous, $\geqslant_s -$strictly increasing function. Then there exists a continuous, strictly $\geqslant_s -$increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$ if and only if*

$$(2.2a) \qquad m(x) \leqslant M(x) \ \text{ for all } \ x \in R^n,$$

$$(2.2b) \qquad \bar{m}(x) < \bar{M}(y) \ \text{ for all } \ x, y \in R^n \ \text{ with } \ y > x.$$

Examples 2 and 3 above show that if the preorder is nonseparable, then conditions (2.2) are not sufficient for the existence of an extension.

Examples 2 and 3 revisited. As we observed above function f_2 from Example 2 has no $\geqslant$ −strictly increasing extension into R^2. Note that f_2 satisfies the assumption of Corollary 2.2. Therefore, it has a $\geqslant_s$ −strictly increasing extension. For example, function F defined on R^2 as $F(x_1, x_2) = \frac{x_2}{1+x_2}$ when $x_2 \geq 0$, and $F(x_1, x_2) = \frac{x_2}{1-x_2}$ when $x_2 < 0$ is such an extension.

Clearly, function f_1 from Example 3 satisfies assumption (2.2b) of Corollary 2.2. However, $m_{f_1}(0) = 0$ and $M_{f_1}(0) = -1$, and so assumption (2.2a), the 'Nachbin's condition', is violated. So, this example shows that assumption (2.2b) alone is not sufficient for the existence of a $\geqslant_s$ −increasing extension.

For some closed domains in R^n the assumptions of Theorem 2.1 are satisfied for an arbitrary continuous, $\geqslant_s$ −strictly increasing function.

Corollary 2.4. *Let $D \subset R^n$ be a nonempty, compact set, and $f : D \to R$ a continuous, $\geqslant_s$ −strictly increasing function. Then there exists a continuous, $\geqslant_s$ −strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$.*

Proof. As sets

$$L(x, D) = \{z \in D \,:\, z \leqslant x\} \text{ and } U(x, D) = \{z \in D \,:\, z \geqslant x\}$$

are compact (possibly empty) and function f is continuous it is easy to see that both conditions in (2.2) are satisfied. Corollary 2.3 applies. □

We conclude this section with the following corollary of Theorem 2.1.

Corollary 2.5. *Let $Z = \{0, \pm 1, \pm 2, \cdots\}$, $D \subset Z^n$, and $f : D \to R$ be a continuous, $\geqslant_s$ −strictly increasing function. Then, there exists a continuous, $\geqslant_s$ −strictly increasing function $F : R^n \to R$ such that $F(x) = f(x)$ for $x \in D$.*

Proof. Note that for every point x in Z^n set $\{z \in D \,:\, z \leqslant x\}$ has a finite number of maximal elements and that set $\{z \in D \,:\, z \geqslant x\}$ has a finite number of minimal elements. This observation makes it easy to show that the assumptions of Theorem 2.1 are satisfied. □

3. Applications to the extension of strictly increasing preferences

Here we consider the question of extension of $\geqslant_s$ −strictly monotonic preferences defined on closed subsets of R^n.

Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be two preorders on a subset D of R^n. We say $\succcurlyeq_2$ is $\succcurlyeq_1$ −increasing if $x \succ_1 y$ implies $x \succ_2 y$. A complete preorder on D is called a preference.

Proposition 3.1. *Let D be a closed subset of R^n and $\succcurlyeq$ a continuous , $\geqslant_s$ −strictly increasing preference on D. Then, preference $\succcurlyeq$ is extendable into R^n if and only if $\succcurlyeq$ satisfies the following condition: for each $w, x, y, z \in D$ with $w \succcurlyeq x$, $x > y$, $y \succcurlyeq z$ we have $w \succ z$.*

Proof. Necessity is obvious. Show the sufficiency. Let $\succcurlyeq$ be a continuous, $\geqslant_s$ −strictly increasing preference on D satisfying the assumption of the proposition. Let $f : D \to R$ be an arbitrary continuous representation of $\succcurlyeq$ that exists by the Debreu Theorem (1964). As $\succcurlyeq$ is $\geqslant_s$ −strictly increasing, function f is $\geqslant_s$ −strictly increasing. As noted above R^n with its usual topology and preference $\geqslant_s$ is a normally preodered space and $\geqslant_s$ is continuous and separable. Now it is easy to check that the assumptions (2.1) of Theorem 2.1 follow from the assumption of the proposition. Hence by Theorem 2.1 there exists a $\geqslant_s$ −strictly increasing extension F of function f. The preference relation represented by function F is the required one. □

Proposition 3.2. *Let D be a compact subset of R^n. Then, every continuous, $\geqslant_s$ −strictly increasing preference on D is extendable into R^n.*

Proof is a direct consequence of Corollary 2.4. □

4. Appendix: Equivalence of separabilities

Following Bridges and Mehta (1996) we bring the definitions of Debreu, Jaffray, and Birkhoff separabilites of preorders.

Let $(X, \succcurlyeq)$ be a preordered set.

The preorder $\succcurlyeq$ is *Debreu separable* if there exists a countable subset Z of X such that if $x \succ y$, then there exists $z \in Z$ with $x \succcurlyeq z \succcurlyeq y$.

The preorder $\succcurlyeq$ is *Jaffray separable* if there exists a countable subset Z of X such that if $x \succ y$, then there exist $z_1, z_2 \in Z$ with $x \succcurlyeq z_1 \succ z_2 \succcurlyeq y$.

The preorder is $\succcurlyeq$ is *Birkhoff separable* if there exists a countable subset Z of X such that for all x, y in $X \setminus Z$ with $x \succ y$ there exists $z \in Z$ with $x \succ z \succ y$.

Proposition 4.1. *The separability of a preorder as defined in this paper is equivalent to each of the Debreu and Jaffray separabilities.*

Proof. As the last two separabilities are equivalent to each other by Bridges and Mehta (1995, Proposition 1.1.4) it suffices to show that the separability as defined here is equivalent to Jaffray separability.

Jaffray separability implies that there is a countable subset Z of X such that for every interval (x, y) in X there are points $z_1, z_2 \in Z$ such that $x \sim z_1$ and $y \sim z_2$. This implies property (b) in the definition given here. Jaffray separability also implies that for every nonempty interval (x, y) there is a point $z \in Z$ such that $z \in (x, y)$. As Z is a countable set this implies property (b) in the definition given here.

Now let a preorder be separable as defined here. Then setting Z to be the union of set S in property (a) and of the set of all endpoints of a countable set of intervals as in property (b) one can easily check that Z is Jaffray dense in X. □

The following simple example shows that Debreu and Jaffray separabilities are not equivalent to Birkhoff separability.

Example. Set $X_0 = \{(t, 0) : t \in R\}$ and $X_1 = \{(t, 1) : t \in R\}$, and $X = X_0 \cup X_1$, and let preorder $\succcurlyeq$ on X be defined as

$$\succcurlyeq = (X_0^2 \cup X_1^2) \cup (X_1 \times X_0).$$

Obviously $x \sim y$ for all $x, y \in X_0$, for all $x, y \in X_1$, and $x \succ y$ for all $x \in X_1,\ y \in X_0$. Now it is easy to see that

$$Z = \{(\frac{k}{k+1}, 1) :\ k = 1, 2, ...\} \cup \{(0, 0)\}$$

is a countable Jaffray dense set in X. On the other hand X has continuum of different pairs $x, y \in X$ with $(x, y) = \emptyset$, which implies that X is not Birkhoff separable.

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