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ON STRONG CONVERGENCE FOR A FORWARD-BACKWARD SPLITTING METHOD IN BANACH SPACES

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Abstract. We study strong convergence for a sum of a maximal monotone operator and a monotone and Lipschitz continuous mapping in a real Banach space in this paper. We propose a modified forward-backward splitting method and prove a new strong convergence theorem in a 2-uniformly convex and uniformly smooth Banach space. From this result, we also get a new result for variational inequality problems.

1. Introduction

Throughout this paper, let E be a real Banach space with norm $\|\cdot\|$, E^* its dual space, and for $x \in E$ and $x^* \in E^*$, let $\langle x, x^*\rangle$ be the value of x^* at x. And we denote by $\mathbb{N}$ the set of all positive integers. Let A and B be maximal monotone operators in $E \times E^*$ such that $A + B$ is maximal and $(A + B)^{-1}0$ is nonempty. Finding an element of $(A + B)^{-1}0$ contains a lot of important problems such as convex minimization problems, variational inequality problems, complementary problems and others. Lions and Mercier [17] and Passty [28] proposed the forward-backward splitting method as one of the methods of finding a point of $(A + B)^{-1}0$ in a real Hilbert space H as follows:

$$x_1 = x \in D(B), \quad x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n w_n)$$

for every $n \in \mathbb{N}$, where $D(B)$ is the domain of B, $D(A) \subset D(B)$, $w_n \in Bx_n$, $\{\lambda_n\} \subset (0,\infty)$ and $J^A_{\lambda_n}$ is the resolvent of A. After that, Gabay [12], Chen and Rockafellar [9], Moudafi and Thera [21] and Tseng [37] widely researched the splitting method. In a real Hilbert space, Many researchers [4, 10, 12, 22, 26, 27, 29, 33, 40] studied weak and strong convergence for a forward-backward splitting method and several modified forward-backward splitting methods by a maximal monotone operator A and an inverse-strongly-monotone mapping B, where $B : H \to H$ is called inverse-strongly-monotone [5, 11] if there exists $\alpha > 0$ such that $(x - y, Bx - By) \geq \alpha\|Bx - By\|^2$ for all $x, y \in H$; see [18, 42]. In a 2-uniformly convex and uniformly smooth Banach space, Kimura and the author [15] considered a modified forward-backward splitting method by the same A and B as above, and they proved strong convergence. Tseng [37] proposed the following forward-backward-forward splitting method by a maximal monotone operator $A \subset H \times H$ and a single valued monotone

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operator $B : H \longrightarrow H$:

$$\begin{cases} x_1 = x \in C, \\ y_n = J^A_{\lambda_n}(x_n - \lambda_n B x_n), \\ x_{n+1} = P_C(y_n - \lambda_n(By_n - Bx_n)), \end{cases}$$

for all $n \in \mathbb{N}$, where C is nonempty, closed and convex subset of H, P_C is the metric projection of H onto C, $A + B$ is maximal monotone and $F = C \cap (A + B)^{-1}0 \neq \emptyset$. He proved that if B is Lipschitz continuous on $C \cup D(A)$, $\{x_n\}$ converges weakly to a point of F under some conditions. This result is applicable to a monotone and Lipschitz continuous mapping which is more general than an inverse-strongly-monotone operator. Recently, Shehu [34] and the author [25] studied different modified forward-backward-forward splitting methods, respectively and proved strong convergence theorems in a 2-uniformly convex and uniformly smooth Banach space.

Malitsky and Tam [20] introduced the method, which requires only one forward evaluation per iteration instead of two, which is called forward-reflected-backward splitting method. The method for a maximal monotone operator $A \subset H \times H$ and a monotone and Lipschitz continuous mapping B of H into H with a constant $L > 0$ such that $(A + B)^{-1}0 \neq \emptyset$ is described as

$$\begin{cases} x_0, x_1 \in H, \\ x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n B x_n - \lambda_{n-1}(Bx_n - Bx_{n-1})), \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$. When $\{\lambda_n\} \subset [\varepsilon, (1 - 2\varepsilon)/(2L)]$ for some $\varepsilon > 0$, they proved $\{x_n\}$ converges weakly to an element of $(A + B)^{-1}0$.

In this paper, we consider strong convergence for the method of [20] in a real Banach space. We propose a modified forward-reflected-backward splitting method and prove strong convergence in a 2-uniformly convex and uniformly smooth Banach space E. Let C be a nonempty, closed and convex subset of E, $A \subset E \times E^*$ a maximal monotone operator and B a monotone and Lipschitz continuous mapping of C into E^* with $D(A) \subset C$ and $(A + B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_1, x_2 \in C, \\ x_{n+1} = J^A_{\lambda_n} J^{-1}(Jx_n - \lambda_n B x_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(Jx_n - Ju)) \end{cases}$$

for every $n \in \mathbb{N}$ with $n \geq 2$, where $u \in E$, J is the duality mapping of E, $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then we prove $\{x_n\}$ converges strongly to $\Pi_{(A+B)^{-1}0}u$ under some assumptions, where $\Pi_{(A+B)^{-1}0}$ is the generalized projection of E onto $(A + B)^{-1}0$. From this result, we obtain new strong convergence for a maximal monotone operator and a monotone Lipschitz continuous mapping and for variational inequality problems in a 2-uniformly convex and uniformly smooth Banach space and a real Hilbert space.

2. Preliminaries

We use $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ will symbolize weak convergence. We define the modulus of convexity δ_E

of E as follows: δ_E is a function of $[0,2]$ into $[0,1]$ such that

$$\delta_E(\varepsilon) = \inf\{1 - \|x+y\|/2 : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\}$$

for every $\varepsilon \in [0,2]$. For $p > 1$, we say that E is p-uniformly convex if there exists a constant $c > 0$ with $\delta_E(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$ and it is known that L_p space is p-uniformly convex if $p > 2$ and 2-uniformly convex if $1 < p \leq 2$, see [39]. E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$. It is obvious that a p-uniformly convex Banach space is uniformly convex. We say that E is strictly convex if $\|x+y\|/2 < 1$ for every $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We know that a uniformly convex Banach space is strictly convex and reflexive. The duality mapping $J : E \to 2^{E^*}$ of E is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}$$

for all $x \in E$. It is known that if E is strictly convex and reflexive, the duality mapping J of E is bijective and $J^{-1} : E^* \to 2^E$ is the duality mapping of E^*. We say that E is smooth if the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1}$$

exists for each $x, y \in S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$. E is said to be uniformly smooth if the limit (1) is attained uniformly for (x,y) in $S(E) \times S(E)$. We know that E is smooth if and only if the duality mapping J of E is single valued and if J is single valued, J is norm to weak* continuous. It is known that if E is uniformly smooth, J is uniformly continuous on bounded subsets of E, that is, for any bounded subset B of E and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in B$, $\|x - y\| < \delta$ implies $\|Jx - Jy\| < \varepsilon$; see [35, 36] for more details. Xu [39] proved the following; see also [41].

Theorem 2.1. *Let E be a smooth Banach space. Then, E is 2-uniformly convex if and only if there exists a constant $c > 0$ such that for all $x, y \in E$, $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, Jx \rangle + c\|y\|^2$ holds.*

Let E be a smooth Banach space and the function $\phi : E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for every $x, y \in E$. It is obvious that $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$ and $\phi(x,y) + \phi(z,u) = \phi(x,u) + \phi(z,y) - 2\langle z - x, Ju - Jy \rangle$ for each $x, y, z, u \in E$. We have the following result [14] by Theorem 2.1.

Theorem 2.2. *Let E be a 2-uniformly convex and smooth Banach space. Then, for every $x, y \in E$, $c\|x-y\|^2 \leq \phi(x,y)$ and $c\|x-y\|^2 \leq \langle x - y, Jx - Jy \rangle = \frac{1}{2}(\phi(x,y) + \phi(y,x))$ hold, where c is the constant in Theorem 2.1.*

Let C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E and $x \in E$. Then, there exists a unique point $x_0 \in C$ such that

$$\phi(x_0, x) = \inf_{y \in C} \phi(y, x).$$

We denote x_0 by $\Pi_C x$ and call Π_C the generalized projection of E onto C; see [1, 2, 13]. We have the following result [1, 2, 13] for the generalized projection.

Lemma 2.3. *Let C be a nonempty and convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, $\phi(x_0, x) = \inf_{y \in C} \phi(y, x)$ if and only if $\langle x_0 - z, Jx - Jx_0 \rangle \geq 0$ for all $z \in C$, or equivalently, $\phi(z, x) \geq \phi(z, x_0) + \phi(x_0, x)$ for each $z \in C$.*

An operator $A \subset E \times E^*$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for every $(x, x^*), (y, y^*) \in A$. We say that a monotone operator A is maximal if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operator A is maximal if and only if for $(u, u^*) \in E \times E^*$, $\langle x - u, x^* - u^* \rangle \geq 0$ for all $(x, x^*) \in A$ implies $(u, u^*) \in A$. Let $f : E \to (-\infty, \infty]$ be a proper and convex function. Then, the subdifferential ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^* \rangle, \ \forall y \in E\}$$

for each $x \in E$. Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Then we know that the subdifferential ∂f of f is a maximal monotone operator; see [30, 31]. The following was proved by Rockafellar [32]; see also [8].

Theorem 2.4. *Let E be a strictly convex, reflexive and smooth Banach space and $A \subset E \times E^*$ be a monotone operator. Then, A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$, where $R(J + rA)$ is the range of $J + rA$.*

Let E be a strictly convex, reflexive and smooth Banach space and $A \subset E \times E^*$ be a maximal monotone operator. By Theorem 2.4 and strict convexity of E, for any $x \in E$ and $r > 0$, there exists a unique element $x_r \in D(A)$ such that

$$J(x) \in J(x_r) + rAx_r.$$

We define J_r^A by $J_r^A x = x_r$ for each $x \in E$ and $r > 0$ and such J_r^A is called the resolvent of A; see [6, 36] for more details.

Let C be a nonempty, closed and convex subset of E and A a single valued mapping of C into E^*. We consider the variational inequality problem [16] for A, that is, the problem of finding a point $z \in C$ such that

$$\langle x - z, Az \rangle \geq 0 \text{ for all } x \in C.$$

The set of all solutions of the variational inequality problem for A is denoted by $VI(C, A)$.

We say that a function $i : \mathbb{N} \to \mathbb{N}$ is eventually increasing if $\lim_{n \to \infty} i(n) = \infty$ and $i(n) \leq i(n+1)$ for every $n \in \mathbb{N}$. Mainge [19, Lemma 3.1] proved the following, see also [3].

Lemma 2.5. *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exist $n_0 \in \mathbb{N}$ and an eventually increasing function i such that $\Gamma_{i(n)} \leq \Gamma_{i(n)+1}$ and $\Gamma_n \leq \Gamma_{i(n)+1}$ for every $n \geq n_0$.*

3. A main result

Let C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth Banach space E, $A \subset E \times E^*$ a maximal monotone operator and B a mapping of C into E^* such that $F = (A+B)^{-1}0 \neq \emptyset$. Then, by the idea of [7], we consider the following condition (I) [25] for A, B and C: For a bounded sequence $\{u_n\} \subset C$ and $\{\lambda_n\} \subset (0, \infty)$ with $\inf_{n\in\mathbf{N}} \lambda_n > 0$, $\|u_n - J^A_{\lambda_n} J^{-1}(Ju_n - \lambda_n Bu_n)\| \to 0$ implies $\omega_w(\{u_n\}) \subset F$, where $\omega_w(\{u_n\})$ is the set of all weak cluster points of $\{u_n\}$. We have the following examples [25] for the condition (I).

Example 3.1. [25, Theorem 4.1] Let E be a strictly convex, reflexive and uniformly smooth Banach space, $A \subset E \times E^*$ a maximal monotone operator and B a monotone and Lipschitz continuous mapping of E into E^* such that $F = (A+B)^{-1}0 \neq \emptyset$. Then, A, B and E satisfy the condition (I).

Example 3.2. [25, Theorem 4.2] Let C be a nonempty, closed and convex subset of a strictly convex, reflexive and uniformly smooth Banach space E and B a monotone and Lipschitz continuous mapping of C into E^* such that $F = VI(C,B) \neq \emptyset$. Let i_C be the indicator function of C. Then, for $A = \partial i_C$ and B, we know that A is maximal monotone with $D(A) = C$, $J^A_\lambda x = \Pi_C x$ for all $\lambda > 0$ and $x \in E$ and $(A+B)^{-1}0 = VI(C,B)$. Further, A, B and C satisfy the condition (I).

We also get the following result [25, Lemma 3.1].

Lemma 3.3. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and smooth Banach space E, A a maximal monotone operator in $E \times E^*$, B a monotone and Lipschitz continuous mapping of C into E^* such that $D(A) \subset C$ and $F = (A+B)^{-1}0 \neq \emptyset$. Then, F is closed and convex.*

Now, we prove a new strong convergence theorem.

Theorem 3.4. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E, $A \subset E \times E^*$ a maximal monotone operator and B a monotone and Lipschitz continuous mapping of C into E^* with a Lipschitz constant $L > 0$ such that $D(A) \subset C$, $F = (A+B)^{-1}0 \neq \emptyset$ and A, B and C satisfy the condition (I). Let $u \in E$ and $\{x_n\}$ a sequence generated by*

$$\begin{cases} x_1, x_2 \in C, \\ x_{n+1} = J^A_{\lambda_n} J^{-1}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(Jx_n - Ju)) \end{cases}$$

for every $n \in \boldsymbol{N}$ with $n \geq 2$, where $0 < \inf_{n\in\boldsymbol{N}} \lambda_n \leq \sup_{n\in\boldsymbol{N}} \lambda_n < c/(2L)$, where c is the constant in Theorem 2.1 and $0 < \alpha_n \leq 1$ for all $n \in \boldsymbol{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_F u$.

Proof. From Lemma 3.3, F is nonempty, closed and convex and hence, Π_F is well defined. Let $z \in F$. We have $-Bz \in Az$ and

$$(1/\lambda_n)(Jx_n - Jx_{n+1}) - Bx_n - (\lambda_{n-1}/\lambda_n)(Bx_n - Bx_{n-1}) \\ -(\alpha_n/\lambda_n)(Jx_n - Ju) \in Ax_{n+1}$$

for every $n \geq 2$. Since A is monotone, we get

$$\langle x_{n+1} - z, (Jx_n - Jx_{n+1}) - \lambda_n(Bx_n - Bz) - \lambda_{n-1}(Bx_n - Bx_{n-1}) \\ -\alpha_n(Jx_n - Ju)\rangle \geq 0$$

and hence,

$$\begin{aligned} \phi(z, x_{n+1}) &\leq \phi(z, x_n) - \phi(x_{n+1}, x_n) - 2\lambda_n\langle x_{n+1} - z, Bx_n - Bz\rangle \\ &\quad - 2\lambda_{n-1}\langle x_{n+1} - z, Bx_n - Bx_{n-1}\rangle - 2\alpha_n\langle x_{n+1} - z, Jx_n - Ju\rangle \end{aligned}$$

for all $n \geq 2$. By

$$\begin{aligned} \langle x_{n+1} - z, Bx_n - Bz\rangle &= \langle x_{n+1} - z, Bx_n - Bx_{n+1}\rangle + \langle x_{n+1} - z, Bx_{n+1} - Bz\rangle \\ &\geq \langle x_{n+1} - z, Bx_n - Bx_{n+1}\rangle, \end{aligned}$$

we obtain

$$\begin{aligned} &\phi(z, x_{n+1}) + 2\lambda_n\langle x_{n+1} - z, Bx_n - Bx_{n+1}\rangle + \phi(x_{n+1}, x_n) \\ &\quad \leq \phi(z, x_n) + 2\lambda_{n-1}\langle x_n - z, Bx_{n-1} - Bx_n\rangle \\ &\qquad - 2\lambda_{n-1}\langle x_{n+1} - x_n, Bx_n - Bx_{n-1}\rangle \\ &\qquad - 2\alpha_n\langle x_{n+1} - z, Jx_n - Ju\rangle \end{aligned} \tag{2}$$

for each $n \geq 2$. Let $M = \sup_{n\in\mathbb{N}} \lambda_n$. We have

$$\begin{aligned} &-2\lambda_{n-1}\langle x_{n+1} - x_n, Bx_n - Bx_{n-1}\rangle \\ &\quad \leq 2\lambda_{n-1}\|x_{n+1} - x_n\| \cdot \|Bx_n - Bx_{n-1}\| \\ &\quad \leq 2ML\|x_{n+1} - x_n\| \cdot \|x_n - x_{n-1}\| \\ &\quad \leq ML(\|x_{n+1} - x_n\|^2 + \|x_n - x_{n-1}\|^2) \end{aligned} \tag{3}$$

for every $n \geq 2$. Since $M < c/(2L)$, there exists $\varepsilon_1 \in (0, 1)$ with $(1 - 2\varepsilon_1)c > 2ML$. From Theorem 2.2,

$$\begin{aligned} \phi(x_{n+1}, x_n) &= (1 - \varepsilon_1)\phi(x_{n+1}, x_n) + \varepsilon_1\phi(x_{n+1}, x_n) \\ &\geq (1 - \varepsilon_1)c\|x_{n+1} - x_n\|^2 + \varepsilon_1\phi(x_{n+1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. By (2) and (3), we get

$$\begin{aligned} &\phi(z, x_{n+1}) + 2\lambda_n\langle x_{n+1} - z, Bx_n - Bx_{n+1}\rangle + ((1 - \varepsilon_1)c - ML)\|x_{n+1} - x_n\|^2 \\ &\qquad + \varepsilon_1\phi(x_{n+1}, x_n) \\ &\quad \leq \phi(z, x_n) + 2\lambda_{n-1}\langle x_n - z, Bx_{n-1} - Bx_n\rangle + ML\|x_n - x_{n-1}\|^2 \\ &\qquad - 2\alpha_n\langle x_{n+1} - z, Jx_n - Ju\rangle \end{aligned}$$

for each $n \geq 2$. From $c/2 < (1 - \varepsilon_1)c - ML$, we obtain

$$\begin{aligned} &\phi(z, x_{n+1}) + 2\lambda_n\langle x_{n+1} - z, Bx_n - Bx_{n+1}\rangle + (c/2)\|x_{n+1} - x_n\|^2 \\ &\qquad + \varepsilon_1\phi(x_{n+1}, x_n) + (c/2 - ML)\|x_n - x_{n-1}\|^2 \\ &\quad \leq \phi(z, x_n) + 2\lambda_{n-1}\langle x_n - z, Bx_{n-1} - Bx_n\rangle + (c/2)\|x_n - x_{n-1}\|^2 \\ &\qquad - 2\alpha_n\langle x_{n+1} - z, Jx_n - Ju\rangle \end{aligned} \tag{4}$$

for every $n \geq 2$. Since

$$2\langle x_{n+1} - z, Jx_n - Ju\rangle = -\phi(x_{n+1}, x_n) + \phi(x_{n+1}, u) + \phi(z, x_n) - \phi(z, u),$$

we have

$$\begin{aligned}
&\phi(z, x_{n+1}) + 2\lambda_n\langle x_{n+1} - z, Bx_n - Bx_{n+1}\rangle + (c/2)\|x_{n+1} - x_n\|^2 \\
&\qquad + (\varepsilon_1 - \alpha_n)\phi(x_{n+1}, x_n) + (c/2 - ML)\|x_n - x_{n-1}\|^2 \\
&\leq \phi(z, x_n) + 2\lambda_{n-1}\langle x_n - z, Bx_{n-1} - Bx_n\rangle + (c/2)\|x_n - x_{n-1}\|^2 \\
&\qquad - \alpha_n(\phi(x_{n+1}, u) + \phi(z, x_n) - \phi(z, u))
\end{aligned} \tag{5}$$

for all $n \geq 2$. Let $a_n = \phi(z, x_n) + 2\lambda_{n-1}\langle x_n - z, Bx_{n-1} - Bx_n\rangle + (c/2)\|x_n - x_{n-1}\|^2$ $(\forall n \geq 2)$ for $z \in F$. From Theorem 2.2 and $2ML < c$,

$$\begin{aligned}
a_n &\geq c\|x_n - z\|^2 - c\|x_n - z\| \cdot \|x_{n-1} - x_n\| + (c/2)\|x_{n-1} - x_n\|^2 \\
&= (c/2)(\|x_{n-1} - x_n\| - \|x_n - z\|)^2 + (c/2)\|x_n - z\|^2 \\
&\geq (c/2)\|x_n - z\|^2 \geq 0 \quad (\forall n \geq 2).
\end{aligned} \tag{6}$$

(i) We show that $\{x_n\}$ is bounded. If $\{a_n\}$ decreases at infinity, it is obvious that $\{x_n\}$ is bounded by (6). Suppose that $\{a_n\}$ does not decrease at infinity. From Lemma 2.5, there exist $n_1 \in \mathbb{N}$ with $n_1 \geq 2$ and an eventually increasing function j such that $a_{j(n)} \leq a_{j(n)+1}$ and $a_n \leq a_{j(n)+1}$ for each $n \geq n_1$. By $a_{j(n)} \leq a_{j(n)+1}$ $(\forall n \geq n_1)$, $ML < c/2$ and $\alpha_n \to 0$ in (5), we obtain

$$a_{j(n)} \leq a_{j(n)+1} \leq a_{j(n)} - \alpha_{j(n)}(\phi(x_{j(n)+1}, u) + \phi(z, x_{j(n)}) - \phi(z, u))$$

for sufficiently large $n \in \mathbb{N}$. By $\alpha_{j(n)} > 0$, we have

$$\phi(x_{j(n)+1}, u) + \phi(z, x_{j(n)}) \leq \phi(z, u)$$

which implies $\{x_{j(n)}\}$ and $\{x_{j(n)+1}\}$ are bounded. From (6), we get

$$\begin{aligned}
&(c/2)\|x_n - z\|^2 \leq a_n \leq a_{j(n)+1} \\
&\quad = \phi(z, x_{j(n)+1}) + 2\lambda_{j(n)}\langle x_{j(n)+1} - z, Bx_{j(n)} - Bx_{j(n)+1}\rangle \\
&\qquad + (c/2)\|x_{j(n)+1} - x_{j(n)}\|^2 \\
&\quad \leq \phi(z, x_{j(n)+1}) + 2ML\|x_{j(n)+1} - z\| \cdot \|x_{j(n)} - x_{j(n)+1}\| \\
&\qquad + (c/2)\|x_{j(n)+1} - x_{j(n)}\|^2
\end{aligned}$$

for all $n \geq n_1$. So, it holds that $\{x_n\}$ is bounded.

(ii) We show that $x_n \to \Pi_F u$. Assume that $\{a_n\}$ with $z = \Pi_F u$ decreases at infinity. So, there exists $\lim_{n\to\infty} a_n$. By $\varepsilon_1 > 0$, $ML < c/2$, the boundedness of $\{x_n\}$ and $\alpha_n \to 0$ in (4), $\phi(x_{n+1}, x_n) \to 0$ holds. From Theorem 2.2, we obtain

$$\|x_{n+1} - x_n\| \to 0. \tag{7}$$

Let $y_n = J^A_{\lambda_n} J^{-1}(Jx_n - \lambda_n Bx_n)$. We have $(1/\lambda_n)(Jx_n - Jy_n - \lambda_n Bx_n) \in Ay_n$. Since $(1/\lambda_n)(Jx_n - Jx_{n+1} - \lambda_n Bx_n) - (\lambda_{n-1}/\lambda_n)(Bx_n - Bx_{n-1}) - (\alpha_n/\lambda_n)(Jx_n - Ju) \in Ax_{n+1}$ and A is monotone, we get

$$\langle x_{n+1} - y_n, (Jy_n - Jx_{n+1}) - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(Jx_n - Ju)\rangle \geq 0$$

for each $n \in \mathbb{N}$. So, we obtain

$$-\lambda_{n-1}\langle x_{n+1} - y_n, Bx_n - Bx_{n-1}\rangle - \alpha_n\langle x_{n+1} - y_n, Jx_n - Ju\rangle \geq \langle x_{n+1} - y_n, Jx_{n+1} - Jy_n\rangle$$

and hence,

$$ML\|x_{n+1} - y_n\| \cdot \|x_n - x_{n-1}\| + \alpha_n\|x_{n+1} - y_n\| \cdot \|Jx_n - Ju\| \geq \langle x_{n+1} - y_n, Jx_{n+1} - Jy_n\rangle$$

for every $n \in \mathbb{N}$. By Theorem 2.2,

$$ML\|x_{n+1} - y_n\| \cdot \|x_n - x_{n-1}\| + \alpha_n\|x_{n+1} - y_n\| \cdot \|Jx_n - Ju\| \geq c\|x_{n+1} - y_n\|^2$$

which implies

$$ML\|x_n - x_{n-1}\| + \alpha_n\|Jx_n - Ju\| \geq c\|x_{n+1} - y_n\|$$

for all $n \in \mathbb{N}$. It follows from (7) and $\alpha_n \to 0$ that $\|x_{n+1} - y_n\| \to 0$ and hence,

$$\|y_n - x_n\| \to 0. \tag{8}$$

By the condition (I), $\omega_w(x_n) \subset F$ holds. Next, we show that

$$l = \limsup_{n\to\infty}\langle \Pi_F u - x_{n+1}, Jx_n - Ju\rangle \geq 0. \tag{9}$$

Assume that $l < 0$. There exists $n_2 \in \mathbb{N}$ with $n_2 \geq 2$ such that $\langle \Pi_F u - x_{n+1}, Jx_n - Ju\rangle \leq (l/2)$ for each $n \geq n_2$. By $\varepsilon_1 > 0$ and $ML < c/2$ in (4) with $z = \Pi_F u$,

$$-l\alpha_n \leq 2\alpha_n\langle x_{n+1} - \Pi_F u, Jx_n - Ju\rangle \leq a_n - a_{n+1}$$

for every $n \geq n_2$ and hence

$$\sum_{n=n_2}^{\infty}(-l)\alpha_n \leq a_{n_2} < \infty.$$

From $\sum_{n=1}^{\infty}\alpha_n = \infty$, this is a contradiction. So, we have (9). Next, we have

$$\begin{aligned}
&\langle \Pi_F u - x_{n+1}, Jx_n - Ju\rangle \\
&= \langle \Pi_F u - x_{n+1}, Jx_n - Jx_{n+1}\rangle + \langle \Pi_F u - x_{n+1}, Jx_{n+1} - J\Pi_F u\rangle \\
&\quad + \langle \Pi_F u - x_{n+1}, J\Pi_F u - Ju\rangle \\
&= \langle \Pi_F u - x_{n+1}, Jx_n - Jx_{n+1}\rangle + \langle \Pi_F u - x_{n+1}, Jx_{n+1} - J\Pi_F u\rangle \\
&\quad + \langle \Pi_F u - x_n, J\Pi_F u - Ju\rangle + \langle x_n - x_{n+1}, J\Pi_F u - Ju\rangle \\
&\leq \|\Pi_F u - x_{n+1}\| \cdot \|Jx_n - Jx_{n+1}\| - (1/2)\phi(\Pi_F u, x_{n+1}) \\
&\quad + \langle \Pi_F u - x_n, J\Pi_F u - Ju\rangle + \|x_{n+1} - x_n\| \cdot \|J\Pi_F u - Ju\|
\end{aligned}$$

for every $n \in \mathbb{N}$. Since J is uniformly continuous on bounded subsets of E and (7), we get

$$\|Jx_n - Jx_{n+1}\| \to 0. \tag{10}$$

So, we obtain

$$0 \leq \limsup_{n\to\infty}\langle \Pi_F u - x_{n+1}, Jx_n - Ju\rangle$$

$$\leq \quad -(1/2) \liminf_{n\to\infty} \phi(\Pi_F u, x_{n+1}) + \limsup_{n\to\infty} \langle \Pi_F u - x_n, J\Pi_F u - Ju\rangle.$$

And there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w \in F$ and

$$\limsup_{n\to\infty} \langle \Pi_F u - x_n, J\Pi_F u - Ju\rangle = \lim_{j\to\infty} \langle \Pi_F u - x_{n_j}, J\Pi_F u - Ju\rangle.$$

By Lemma 2.3,

$$\lim_{j\to\infty} \langle \Pi_F u - x_{n_j}, J\Pi_F u - Ju\rangle = \langle \Pi_F u - w, J\Pi_F u - Ju\rangle \leq 0$$

holds. Hence, we have

$$\liminf_{n\to\infty} \phi(\Pi_F u, x_{n+1}) = 0.$$

Since $\lim_{n\to\infty} a_n$ exists, (7) and

$$\begin{aligned} &|a_{n+1} - \phi(\Pi_F u, x_{n+1})| \\ \leq\ & 2\lambda_n |\langle x_{n+1} - \Pi_F u, Bx_n - Bx_{n+1}\rangle| + (c/2)\|x_{n+1} - x_n\|^2 \\ \leq\ & 2ML\|x_{n+1} - \Pi_F u\| \cdot \|x_n - x_{n+1}\| + (c/2)\|x_{n+1} - x_n\|^2, \end{aligned}$$

there exists $\lim_{n\to\infty} \phi(\Pi_F u, x_{n+1})$. Therefore, $\{x_n\}$ converges strongly to $\Pi_F u$ from Theorem 2.2.

Suppose that $\{a_n\}$ with $z = \Pi_F u$ is not decreasing at infinity. By Lemma 2.5, there exist $n_3 \in \mathbb{N}$ and an eventually increasing function i such that $i(n_3) \geq 2$, $a_{i(n)} \leq a_{i(n)+1}$ and $a_n \leq a_{i(n)+1}$ for every $n \geq n_3$. From (5) with $z = \Pi_F u$ and $a_{i(n)} \leq a_{i(n)+1}$, we get

$$\begin{aligned} &a_{i(n)} + (\varepsilon_1 - \alpha_{i(n)})\phi(x_{i(n)+1}, x_{i(n)}) + (c/2 - ML)\|x_{i(n)} - x_{i(n)-1}\|^2 \\ \leq\ & a_{i(n)+1} + (\varepsilon_1 - \alpha_{i(n)})\phi(x_{i(n)+1}, x_{i(n)}) + (c/2 - ML)\|x_{i(n)} - x_{i(n)-1}\|^2 \\ \leq\ & a_{i(n)} - \alpha_{i(n)}(\phi(x_{i(n)+1}, u) + \phi(\Pi_F u, x_{i(n)}) - \phi(\Pi_F u, u)) \end{aligned}$$

which implies

$$\begin{aligned} &(\varepsilon_1 - \alpha_{i(n)})\phi(x_{i(n)+1}, x_{i(n)}) + (c/2 - ML)\|x_{i(n)} - x_{i(n)-1}\|^2 \\ \leq\ & -\alpha_{i(n)}(\phi(x_{i(n)+1}, u) + \phi(\Pi_F u, x_{i(n)}) - \phi(\Pi_F u, u)) \end{aligned}$$

for all $n \geq n_3$. Since $\varepsilon_1 > 0$ and $ML < c/2$, $\{x_n\}$ is bounded and $\alpha_{i(n)} \to 0$, we obtain $\|x_{i(n)} - x_{i(n)-1}\| \to 0$ and $\phi(x_{i(n)+1}, x_{i(n)}) \to 0$ and hence,

$$\|x_{i(n)+1} - x_{i(n)}\| \to 0 \tag{11}$$

by Theorem 2.2. As in the proof of (8), we have

$$\|y_{i(n)} - x_{i(n)}\| \to 0.$$

From the condition (I), $\omega_w(\{x_{i(n)+1}\}) = \omega_w(\{x_{i(n)}\}) \subset F$ holds. Since (4) with $z = \Pi_F u$, $\varepsilon_1 > 0$, $ML < c/2$ and $a_{i(n)} \leq a_{i(n)+1}$,

$$a_{i(n)} \leq a_{i(n)+1} \leq a_{i(n)} - 2\alpha_{i(n)}\langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Ju\rangle$$

which implies

$$\langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Ju\rangle \leq 0$$

for every $n \geq n_3$ by $\alpha_{i(n)} > 0$. We get

$$\begin{aligned}
&\langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Ju\rangle \\
&\quad = \langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Jx_{i(n)+1}\rangle + \langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)+1} - J\Pi_F u\rangle \\
&\qquad + \langle x_{i(n)+1} - \Pi_F u, J\Pi_F u - Ju\rangle \\
&\quad \geq \langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Jx_{i(n)+1}\rangle \\
&\qquad + (1/2)\phi(\Pi_F u, x_{i(n)+1}) + \langle x_{i(n)+1} - \Pi_F u, J\Pi_F u - Ju\rangle
\end{aligned}$$

and hence,

$$\begin{aligned}
&\phi(\Pi_F u, x_{i(n)+1}) \\
&\quad \leq -2\langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Jx_{i(n)+1}\rangle - 2\langle x_{i(n)+1} - \Pi_F u, J\Pi_F u - Ju\rangle
\end{aligned}$$

for each $n \geq n_3$. So,

$$\begin{aligned}
&\limsup_{n\to\infty} \phi(\Pi_F u, x_{i(n)+1}) \\
&\quad \leq -2\liminf_{n\to\infty}\langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Jx_{i(n)+1}\rangle \\
&\qquad - 2\liminf_{n\to\infty}\langle x_{i(n)+1} - \Pi_F u, J\Pi_F u - Ju\rangle.
\end{aligned} \tag{12}$$

There exists a subsequence $\{x_{i(n_k)+1}\}$ of $\{x_{i(n)+1}\}$ such that $x_{i(n_k)+1} \rightharpoonup p \in F$ and

$$\begin{aligned}
\liminf_{n\to\infty}\langle x_{i(n)+1} - \Pi_F u, J\Pi_F u - Ju\rangle &= \lim_{k\to\infty}\langle x_{i(n_k)+1} - \Pi_F u, J\Pi_F u - Ju\rangle \\
&= \langle p - \Pi_F u, J\Pi_F u - Ju\rangle.
\end{aligned}$$

By Lemma 2.3 and $p \in F$, we obtain $\langle p - \Pi_F u, J\Pi_F u - Ju\rangle \geq 0$ and hence,

$$\liminf_{n\to\infty}\langle x_{i(n)+1} - \Pi_F u, J\Pi_F u - Ju\rangle \geq 0. \tag{13}$$

Since J is uniformly continuous on bounded subsets of E and (11), $\|Jx_{i(n)+1} - Jx_{i(n)}\| \to 0$ holds. So, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\langle x_{i(n)+1} - \Pi_F u, Jx_{i(n)} - Jx_{i(n)+1}\rangle \\
&\quad \geq -\limsup_{n\to\infty}\|x_{i(n)+1} - \Pi_F u\| \cdot \|Jx_{i(n)} - Jx_{i(n)+1}\| = 0.
\end{aligned}$$

From (12) and (13), we get $\limsup_{n\to\infty}\phi(\Pi_F u, x_{i(n)+1}) = 0$ which implies

$$\lim_{n\to\infty}\phi(\Pi_F u, x_{i(n)+1}) = 0.$$

Since (11) and

$$\begin{aligned}
&|a_{i(n)+1} - \phi(\Pi_F u, x_{i(n)+1})| \\
&\leq 2ML\|x_{i(n)+1} - \Pi_F u\| \cdot \|x_{i(n)} - x_{i(n)+1}\| + (c/2)\|x_{i(n)} - x_{i(n)+1}\|^2,
\end{aligned}$$

we get $\lim_{n\to\infty} a_{i(n)+1} = 0$. By $a_n \leq a_{i(n)+1}$ for every $n \geq n_3$, we obtain $\lim_{n\to\infty} a_n = 0$. By (6), $\{x_n\}$ converges strongly to $\Pi_F u$. □

4. Deduced results

At first, we get a new strong convergence theorem for a sum of maximal monotone operators by Example 3.1 and Theorem 3.4.

Theorem 4.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, A a maximal monotone operator in $E \times E^*$, B a monotone and Lipschitz continuous mapping of E into E^* with a Lipschitz constant $L > 0$ such that $F = (A+B)^{-1}0 \neq \emptyset$. Let $u \in E$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, x_2 \in E, \\ x_{n+1} = J^A_{\lambda_n} J^{-1}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(Jx_n - Ju)) \end{cases}$$

for every $n \in \boldsymbol{N}$ with $n \geq 2$, where $0 < \inf_{n\in\boldsymbol{N}} \lambda_n \leq \sup_{n\in\boldsymbol{N}} \lambda_n < c/(2L)$ where c is the constant in Theorem 2.1 and $0 < \alpha_n \leq 1$ for all $n \in \boldsymbol{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_F u$.

Tufa and Zegeye [38] and the author [25] proved the strong convergence theorems of variational inequality problems for a monotone and Lipschitz continuous mapping in a 2-uniformly convex and uniformly smooth Banach space, respectively (see also [23, 24]). From Example 3.2 and Theorem 3.4, we have a new result which is different from those.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let B be a monotone and Lipschitz continuous mapping of C into E^* with a Lipschitz constant $L > 0$ such that $VI(C, B) \neq \emptyset$. Let $u \in E$ and $\{x_n\}$ a sequence generated by*

$$\begin{cases} x_1, x_2 \in C, \\ x_{n+1} = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(Jx_n - Ju)) \end{cases}$$

for every $n \in \boldsymbol{N}$ with $n \geq 2$, where $0 < \inf_{n\in\boldsymbol{N}} \lambda_n \leq \sup_{n\in\boldsymbol{N}} \lambda_n < c/(2L)$ where c is the constant in Theorem 2.1 and $0 < \alpha_n \leq 1$ for all $n \in \boldsymbol{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_{VI(C,B)} u$.

In a real Hilbert space H, we have $c = 1$ in Theorem 2.1, $J = J^{-1} = I$, where I is the identity mapping and $\Pi_C = P_C$ for every nonempty, closed and convex subset C of H, where P_C is the metric projection of C onto H. So, we get new results in a real Hilbert space by Theorems 4.1 and 4.2.

Theorem 4.3. *Let A be a maximal monotone operator in $H \times H$ and B a monotone and Lipschitz continuous mapping of H into H with a Lipschitz constant $L > 0$ such that $F = (A + B)^{-1}0 \neq \emptyset$. Let $u \in H$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1, x_2 \in H, \\ x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(x_n - u)) \end{cases}$$

for every $n \in \boldsymbol{N}$ with $n \geq 2$, where $0 < \inf_{n\in\boldsymbol{N}} \lambda_n \leq \sup_{n\in\boldsymbol{N}} \lambda_n < 1/(2L)$ and $0 < \alpha_n \leq 1$ for all $n \in \boldsymbol{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_F u$

Theorem 4.4. *Let C be a nonempty, closed and convex subset of H and B a monotone and Lipschitz continuous mapping of C into H with a Lipschitz constant $L > 0$ such that $VI(C, B) \neq \emptyset$. Let $u \in H$ and $\{x_n\}$ a sequence generated by*

$$\begin{cases} x_1, x_2 \in C, \\ x_{n+1} = P_C(x_n - \lambda_n Bx_n - \lambda_{n-1}(Bx_n - Bx_{n-1}) - \alpha_n(x_n - u)) \end{cases}$$

for every $n \in \boldsymbol{N}$ with $n \geq 2$, where $0 < \inf_{n \in \boldsymbol{N}} \lambda_n \leq \sup_{n \in \boldsymbol{N}} \lambda_n < 1/(2L)$ and $0 < \alpha_n \leq 1$ for all $n \in \boldsymbol{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{VI(C,B)}u$.

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