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ON THE KKM THEORY OF ORDERED SPACES

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Dedicated to Professor Hidetoshi Komiya on his retirement

Abstract. Since Horvath and Llinares Ciscar in 1996 began to study maximal elements and fixed points for binary relations on topological ordered spaces, there have appeared many works related to the KKM theory on such spaces by several authors. Independently to these works, we began to study the KKM theory on abstract convex spaces from 2006. Our aim in the present paper is to extend the known results on topological ordered spaces to the corresponding ones on abstract convex spaces.

1. Introduction

In 1996 Horvath and Llinares Ciscar [6] began to study maximal elements and fixed points for binary relations on topological ordered spaces. Their topological ordered spaces are actually topological semilattices with path-connected intervals. Since then there have appeared many works on such spaces related to the KKM theory by several authors. For example, Llinares [9–11] in 1998-2002, Luo [14] in 2001, Sanchez, Llinares, and Subiza [34] in 2003, Lu [12] in 2009, Al-Homidan and Ansari [2] in 2011, Al-Homidan, Ansari, and Yao [3] in 2011, Altwaijry, Ounaies, and Chebbi [1] in 2018, and others. These authors studied various problems related to topological semilattices with path-connected intervals which are known to be particular generalized convex spaces or G-convex spaces due to ourselves.

In 1998, Llinares [9] published a paper on unified treatment of existence of maximal elements in binary relations in mc-spaces. His mc-spaces are very complicated and known to be particular to G-convex spaces later. He stated in [9] that the problem of looking for sufficient conditions that ensure the existence of maximal elements of a binary relation is one of the most important problems in economic theory.

Independently to such works, from 2006 we began to study the KKM theory on abstract convex spaces more general than G-convex spaces; see [23]. In fact, in the last decade, we have extended a large number of KKM theoretic results on particular types of KKM spaces to the corresponding ones on abstract convex spaces; for example, see [27, 28] and parksehie.com.

Our aim in the present survey is to extend the known results on topological ordered spaces in [6], mc-spaces in [8], and some other related results to the corresponding ones on abstract convex spaces.

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This article is organized as follows: Section 2 is for a short history of the birth of generalized convex space, mc-space, L-space, and abstract convex space. We introduce mutual relations of these spaces and some subclasses of abstract convex spaces. In Section 3, we are concerned with topological semilattices with path-connected intervals derived by Horvath and Llinares Ciscar [6]. Key results there are extended to some KKM theoretic results. Section 4 devotes to mc-spaces or L-spaces as in Llinares [9]. Key results there are extended by applying contemporary concepts.

Section 5 is mainly concerned with connected ordered spaces which are KKM spaces recognized by ourselves in 2007 [26]. We recall our study on connected ordered spaces from 1998 and that the study of abstract convex spaces is applicable to all spaces mentioned in preceding sections. In Section 6, we introduce several related papers with their abstracts and give some comments if necessary. Finally, Section 7 is for a short conclusion.

2. From G-convex spaces to abstract convex spaces

Multimaps are also called simply *maps*. Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

In 1993, we began to study the KKM theory on generalized convex spaces (simply, G-convex spaces) [32]. Since 1998, we have adopted the following new definition [18]:

Definition 2.1. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X and a map $\Gamma : \langle D\rangle \to 2^X \setminus \{\emptyset\}$ such that

(*) for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exist a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_J denotes the face of the n-simplex Δ_n corresponding to $J \in \langle A\rangle$.

Since then hundreds of articles by many authors have appeared on G-convex spaces in the old or new sense.

In 1994, Llinares introduced mc-spaces in his Ph.D. Thesis [7] as follows:

Definition 2.2. A topological space X is an mc-space (or has an mc-structure) if for any non-empty finite subset of X, $A \subset X$, there exists an ordering on it, namely $A = \{a_0, a_1, \dots, a_n\}$, a family of elements $\{b_0, b_1, \dots, b_n\}$ $\subset X$, and a family of functions $P_i^A : X \times [0, 1] \to X$, such that for $i = 0, 1, \dots, n$,

1. $P_i^A(x, 0) = x$, $P_i^A(x, 1) = b_i$, for all $x \in X$.
2. The following function $G_A : [0, 1]^n \to X$ given by

$$G_A(t_0, t_1, \dots, t_{n-1}) = P_0^A(\dots(P_{n-1}^A(P_n^A(b_n, 1), t_{n-1}), \dots), t_0),$$

is a continuous function.

Since then Llinares published several articles on mc-spaces [8–10, 33].

Apparently motivated by G-convex spaces, later in 1998, Ben-El-Mechaiekh, Chebbi, Florenzano, and Llinares [4] introduced the L-spaces as follows:

Definition 2.3. An L-structure on a topological space E is given by a nonempty set-valued map $\Gamma : \langle E\rangle \to E$ verifying:

(**) For every $A \in \langle E\rangle$, say $A = \{x_0, x_1, \ldots, x_n\}$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, 1, \ldots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j,\ j \in J\})$.

The pair (E, Γ) is then called an L-space and $X \subset E$ is said to be L-convex if $\forall A \in \langle X\rangle,\ \Gamma(A) \subset X$.

Note that L-spaces (E, Γ) are particular to G-convex spaces $(X, D; \Gamma)$ for the case $E = X = D$. Later Llinares admitted that his mc-spaces are same to L-spaces; see [34].

Those L-space theorists repeatedly claimed the false statement in the last two decades that their L-spaces generalize our G-convex spaces; see our responses to them in [29–31].

In order to upgrade the KKM theory, in 2006-2010 [23–27], we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of various known types of particular spaces and adequate to establish the KKM theory.

Recall the following in [27–31]:

Definition 2.4. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, such that the Γ-*convex hull* of any subset $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A : A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to $D' \subset D$ if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, i.e., $\mathrm{co}_\Gamma D' \subset X$.

For the case to emphasize $E \supset D$, $(E, D; \Gamma)$ will be denoted by $(E \supset D; \Gamma)$; and if $E = D$, then $(E; \Gamma) = (E, E; \Gamma)$.

Definition 2.5. Let $(E, D; \Gamma)$ be an abstract convex space. If a multimap $G : D \multimap E$ satisfies

$$\Gamma_A \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \quad A \in \langle D\rangle,$$

then G is called a *KKM map*.

Definition 2.6. An abstract convex space $(E, D; \Gamma)$ is called a *partial KKM space* if, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property, and a *KKM space* if the same property also holds for any open-valued KKM map.

Definition 2.7. A triple $(X \supset D; \Gamma)$ is called a *Horvath space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of homotopically trivial subsets of X indexed by $A \in \langle D \rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D \rangle$.

This is more general than H-spaces or c-spaces [5].

Now we have the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

$$c\text{-space} \Longrightarrow \text{mc-space} \Longleftrightarrow \text{L-space} \Longrightarrow \text{G-convex space.}$$

The first three spaces are pairs of the form $(E; \Gamma)$. Moreover, we have

$$\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space}$$
$$\Longrightarrow \text{Horvath space} \Longrightarrow \text{G-convex space} \Longrightarrow \text{KKM space}$$
$$\Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space}$$

for triples $(E, D; \Gamma)$.

Recall that we could obtain a very general KKM type theorem for abstract convex spaces by adopting the intersectionally closed concept due to Luc et al. [13].

3. Path-connected semilattices

This section is mainly concerned with topological semilattices with path-connected intervals as in Horvath and Llinares Ciscar [6].

A semilattice, or, more exactly, a sup-semilattice, is a partially ordered set $(X, \leqslant)$ for which any pair (x, x') of elements has a l.u.b. $x \vee x'$. Then any finite subset A of X has a l.u.b., denoted by $\sup A$. In the case $x \leqslant x'$, the set $[x, x'] = \{y \in X : x \leqslant y \leqslant x'\}$ is called an order interval.

Now assume that $(X, \leqslant)$ is a semilattice and $A \subset X$ is a non-empty finite subset. The set $\Delta(A) = \bigcup_{a \in A} [a, \sup A]$ is well-defined and it has the following properties:

(a) $A \subset \Delta(A)$,

(b) if $A \subset A'$, then $\Delta(A) \subset \Delta(A')$.

We say that a subset $E \subset X$ is convex if, for any nonempty finite subset $A \subset E$, we have $\Delta(A) \subset E$.

For details on topological semilattices, see Horvath-Llinares [6], where six non-trivial examples of topological semilattices are given.

Lemma 3.1 ([6]). *Let X be a path-connected topological semilattice with an element $\bar{x} \in X$ such that $x \leqslant \bar{x}$ for each $x \in X$. Then X is homotopically trivial.*

Theorem 3.2 ([6]). *Let X be a topological semilattice with path-connected intervals and $\{R_i : i = 0, \ldots, n\}$ a family of closed [resp. open] subsets of X. If there exist points $x_0, \ldots, x_n$ of X such that, for any family $\{i_0, \ldots, i_k\}$ of indices, $\Delta(\{x_{i_0}, \ldots, x_{i_k}\}) \subset \bigcup_{j=0}^{k} R_{i_j}$, then the set $\bigcap_{i=0}^{n} R_i$ is not empty.*

The following result is contained in the proof of Theorem 3.2 and noted by [2, Lemma 2.1]:

Lemma 3.3. *Let X be a topological semilattice with path-connected intervals. Then, for any $J \subset \{0, 1, \dots, n\}$ and any $\{x_j : j \in J\} \subset X$, there exists a continuous function $f : \Delta_n \to X$ such that $f(\Delta_J) \subset \Delta(\{x_j : j \in J\})$.*

This shows that $(X; \Delta)$ is an L-space.

More precisely, from Theorem 3.2, we obtained the following [17, Lemma 2]:

Lemma 3.4. *Any topological semilattice $(X, \leqslant)$ with path-connected intervals is a G-convex space. Moreover, let D be a nonempty subset of X and $\Gamma : \langle D \rangle \multimap X$ a map such that*

$$\Gamma_A = \Gamma(A) := \bigcup_{a \in A} [a, \sup A] \quad \text{for } A \in \langle D \rangle.$$

Then $(X, D; \Gamma)$ is a G-convex space.

Now we have the following diagram for subclasses of abstract convex spaces $(E, D; \Gamma)$:

$$\text{topological semilattice with path-connected intervals} \Longrightarrow \text{mc-space}$$

$$\Longleftrightarrow \text{L-space} \Longrightarrow \text{G-convex space} \Longrightarrow \text{KKM space}$$

The first three spaces are pairs of the form $(E; \Gamma)$.

The following extension of the 1961 KKM lemma of Ky Fan is a trivial consequence of definition of partial KKM spaces:

Theorem 3.5. *Let $(X, D; \Gamma)$ be a partial KKM space and $G : D \multimap X$ be a closed-valued KKM map such that $G(x_0)$ is compact for some $x_0 \in D$. Then $\bigcap_{x \in D} G(x) \neq \emptyset$.*

Now we have an order theoretical version of the well-known KKM theorem:

Theorem 3.6. *Let X be a topological semilattice with path-connected intervals, D a nonempty subset of X, and $R : D \multimap X$ a map such that*

(1) *for each $x \in D$, $R(x)$ is closed [resp. open] in X; and*
(2) *for each $N \in \langle D \rangle$, $\Gamma_N \subset R(N)$.*

Then $\{R(x) : x \in D\}$ has the finite intersection property.

This is clear since $(X, D; \Gamma)$ is a G-convex space and hence a KKM space.

Theorem 3.6 is a restatement of [6, Theorems 1 and 1′]. Moreover, [6, Theorem 2] can be restated as follows:

Theorem 3.7. *Let X be a topological semilattice with path-connected intervals, D a nonempty subset of X, and $R : D \multimap X$ a map such. that*

(1) *for each $x \in D$, $R(x)$ is closed in X;*
(2) *for each $N \in \langle D \rangle$, $\Gamma_N \subset R(N)$; and*

(3) *$R(x_0)$ is compact for some $x_0 \in D$.*

Then $\bigcap_{x\in D} R(x) \neq \emptyset$.

The following existence result of continuous selections in [6, Theorem 3] also follows from our result in [17]:

Theorem 3.8. *Let K be a compact topological space, X a topological semilattice with path-connected intervals, and $R : K \multimap X$ a map such that*

(1) *for each $y \in K$, if $x_1, x_2 \in R(y)$ then $[x_1, x_1 \vee x_2] \subset R(y)$; and*

(2) *$K = \bigcup\{\operatorname{Int} R^{-1}(x) : x \in X\}$.*

Then there exist a simplex Δ_n and two continuous functions $g : \Delta_n \to X$ and $h : K \to \Delta_n$ such that $(gh)(y) \in R(y)$ for all $y \in K$.

Note that [6, Corollary 1] is a simple consequence of Theorem 3.8 or a corollary to Theorem 3.5.

The following is [6, Theorem 2]:

Corollary 3.9 ([6])**.** *Let X be a topological semilattice with path-connected intervals, $X_0 \subset X$ a non-empty subset of X, and $R \subset X_0 \times X$ a binary relation such that*

(i) *For each $x \in X_0$, the set $R(x)$ is not empty and closed in $R(X_0)$.*

(ii) *There exists $x_0 \in X_0$ such that $R(x_0)$ is compact.*

(iii) *For any non-empty finite subset $A \subset X_0$:*

$$\bigcup_{x\in A} [x, \sup A] \subset \bigcup_{x\in A} R(x).$$

Then the set $\bigcap_{x\in A} R(x)$ is not empty.

The following is [14, Theorem 2]:

Corollary 3.10 ([14])**.** *Let X be a topological semilattice with path-connected intervals, let $X_0 \subset X$ be a nonempty subset of X, and let $R \subset X_0 \times X$ be a binary relation such that*

(1) *$G : X_0 \multimap X$ is transfer closed valued, where $G(x) = \{y \in X : (x, y) \in R\}$ for each $x \in X_0$.*

(2) *There exists $x_0 \in X_0$ such that the set $\operatorname{cl} G(x_0)$ is compact.*

(3) *For any nonempty finite subset $A \subset X_0$, $\bigcup_{x\in A}[x, \sup A] \subset \bigcup_{x\in A} G(x)$.*

Then the set $\bigcap_{x\in X_0} G(x)$ is not empty.

The following is the *Fan-Browder fixed point property* in [27, (V)]:

Theorem 3.11. *Let $(E, D; \Gamma)$ be a KKM space and $S : E \multimap D$, $T : E \multimap E$ be maps satisfying*

(1) *for each $x \in E$, $\operatorname{co}_\Gamma S(x) \subset T(x)$;*

(2) *$S^{-1}(z)$ is open [resp. closed] for each $z \in D$; and*

(3) *$E = \bigcup_{z\in M} S^{-1}(z)$ for some $M \in \langle D\rangle$.*

Then T has a fixed point $x_0 \in E$; that is, $x_0 \in T(x_0)$.

The following is [6, Corollary 1]:

Corollary 3.12 ([6]). *Let X be a compact topological semilattice with path-connected intervals and $R \subset X \times X$ a binary relation such that:*

(i) $X = \bigcup_{x \in X} \operatorname{int} R^{-1}(x)$.

(ii) *For each $x \in X$, the set $R(x)$ is not empty and, if $x_1,\ x_2 \in R(x)$, then $[x_1,\ x_1 \vee x_2] \subset R(x)$.*

Then there exists $\hat{x} \in X$ such that $\hat{x} \in R(\hat{x})$.

In Section 3 of [6], by applying the preceding results, the authors established the existence of a greatest or maximal element for a preference relation. Moreover, in Section 4 of [6]. the authors added further results on continuous selections and maximal elements. For these results we may apply our methods on abstract convex space theory.

4. L-spaces or mc-spaces

This section is mainly concerned with mc-spaces or L-spaces as in Llinares [9] in 1998. There Llinares aimed twofold: On the one hand, to present an existence result that covers both ways (convexity and acyclicity) of analyzing the existence of maximal elements in non-transitive binary relations. On the other hand, to introduce mc-spaces.

Nowadays it is known that mc-spaces are the same to L-spaces and their usefulness is replaced by G-convex spaces, or more generally, by KKM spaces as shown in the preceding section.

The following is a simple consequence of Theorem 3.11:

Theorem 4.1. *Let $(E;\Gamma)$ be a compact partial KKM space and $F : E \multimap E$ a multimap with open inverse images and non-empty Γ-convex values. Then F has a fixed point.*

Corollary 4.2 ([15]). *Let K be a nonempty compact Δ-convex subset of a topological semilattice with path connected intervals M, $F : K \multimap K$ with nonempty Δ-convex values, and $F^{-1}(y) \subset K$ be open for any $y \in K$. Then F has a fixed point.*

More early we obtained the following in 1997 [17]:

Theorem 4.3. *Let $(E;\Gamma)$ be a compact G-convex space and $F : E \multimap E$ a multimap with open inverse images and non-empty Γ-convex values. Then F has a continuous selection and a fixed point.*

More detailed results than this are given in [20] in 1999. The following consequence of Theorem 4.3 is [9, Theorem 1] in 1998:

Corollary 4.4 ([9]). *Let X be a compact Hausdorff topological mc-space and $\Gamma : X \multimap X$ a correspondence with open inverse images and non-empty mc-set values. Then Γ has a continuous selection and a fixed point.*

This generalizes the well-known results of Browder in 1967 and Yannelis-Prabhakar in 1983. In order to prove this Corollary, its author spent 4 pages, which shows the ineffectiveness of the definition of mc-spaces.

The following is an equivalent form of Theorem 4.1:

Theorem 4.5. *Let $(E;\Gamma)$ be a compact partial KKM space and $F : E \multimap E$ a multimap with open inverse images and Γ-convex values. Suppose F has no fixed point. Then the set $\{x^* \in E : F(x^*) = \emptyset\}$ of maximal elements is non-empty and compact.*

Corollary 4.6. *Let $(E;\Gamma)$ be a compact L-space and $F : E \multimap E$ a multimap with open inverse images and L-convex values. Suppose F has no fixed point. Then the set $\{x^* \in E : F(x^*) = \emptyset\}$ of maximal elements is non-empty and compact.*

Let us compare this with the following [9, Theorem 3]:

Theorem 4.7 ([9]). *Let Y be a compact Hausdorff topological mc-space and let U be a binary relation defined on X, verifying the continuity condition (T) and the convexity condition (C). Then the set of maximal elements, $\{x^* : U(x^*) = \emptyset\}$ is non-empty and compact.*

Continuity condition (T): If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \operatorname{int} U^{-1}(x')$.

Convexity condition (C): Let X be an mc-space, and let $U(\cdot)$ be a correspondence (upper contour sets). Then, $\forall x \in X$ and $\forall A \subset X$, A finite, $A \cap U(x) \neq \emptyset$ it is verified $x \notin G_{A|U(x)}([0,1])^m)$, (where $m = \operatorname{card}(A \cap U(x))$.

This convexity condition seems to be not practical, and hence Corollary 4.6 is a better form of Theorem 4.7. In [9], Llinares showed that the results of Sonnenschein in 1971 and Walker in 1977 follows from Theorem 4.7. However, the results also follow from our Corollary 4.6 which is a better form of Theorem 4.7.

5. Connected ordered spaces

This section is mainly concerned with connected ordered spaces which is a KKM space recognized by ourselves in 2007 [26]. Our study on connected ordered space begins in 1998.

In 1998 [19], we studied an arc X or, more generally, a connected ordered space X with two end points. If a multimap $F : X \multimap X$ has connected graph, then F has a fixed point. We deduced several consequences from this fixed point theorem and a generalization of the Bolzano intermediate value theorem.

In 2000 [21], we gave some new examples of G-convex spaces and, simultaneously, show that some abstract convexities of other authors are simple particular examples of our G-convexity. Especially, mc-spaces due to LLinares is shown to be G-convex.

Moreover, in 2001 [22], a more detailed version of [19] was given. The following is our main result of [22]:

Theorem 5.1. *Let X be a connected ordered space with two end points and $F : X \multimap X$ a multimap. If F has connected graph, then F has a fixed point $x \in X$; that is, $x \in F(x)$.*

After we introduced abstract convex spaces in 2006, we noticed the following in 2007 [26]:

A connected ordered space $(X, \leq)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty subset $D \subset X$ by defining

$$\Gamma_A := [\min A, \ \max A] = \{x \in X : \min A \leq x \leq \max A\}$$

for each $A \in \langle D \rangle$.

Moreover, it is a KKM space; see [26, Theorem 5(i)].

Definition 5.1. A linearly ordered set $(X, \leq)$ is called an *ordered space* if it has the order topology whose subbase consists of all sets of the form $\{x \in X \mid x < s\}$ and $\{x \in X \mid x > s\}$ for $s \in X$.

Note that an ordered space X is connected iff it is Dedekind complete (that is, every subset of X having an upper bound has a supremum) and whenever $x < y$ in X, then $x < z < y$ for some z in X. For details, see Willard [36].

A connected ordered space $(X, \leq)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty subset $D \subset X$ by defining $\Gamma_A := [\min A, \ \max A] = \{x \in X \mid \min A \leq x \leq \max A\}$ for each $A \in \langle D \rangle$.

Examples. We give some examples of connected ordered spaces.

(1) Any nonempty interval of the real line $\mathbb{R}$.

(2) Connected $[0, 1]$-spaces; that is, connected spaces admitting a continuous bijection onto the unit interval; see [16, 33].

(3) The following is a connected ordered space:

$$X = \{(0, 0)\} \cup \{(x, y) \mid x \in (0, 1] \ \text{ and } \ y = \sin 1/x\} \subset \mathbb{R}^2.$$

(4) A generalized arc; that is, a continuum which has exactly two non-cut points. For example, the extended long line L^* constructed from the ordinal space $[0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology; see [35].

Recall that any KKM space satisfies at least twenty six KKM theoretic properties as shown in [27] (which needs minor corrections), [28] and others. Any KKM space has or satisfies, for example, matching property, section property, Fan-Browder fixed point property, maximal elements, minimax inequality, variational inequality, von Neumann-Sion minimax theorem, von Neumann-Fan intersection theorem, Nash-Fan type equilibrium theorem, and so on.

Since connected ordered spaces are KKM spaces as well as topological semilattices with path-connected intervals, they satisfy all of the KKM theoretic results in [27, 28] and many literature.

6. Comments on some related works

In this section we deal with some recent articles related to topological ordered spaces, topological semilattice spaces, and others appeared in this article. We introduce them in the chronological order and add some comments if necessary.

Luo [14] (2001)

Abstract: In this paper, we obtain a generalized KKM theorem, a generalized Fan-Browder fixed theorem, and an existence theorem of Nash equilibria in topological ordered spaces.

Comments: This was based on excellent observations in that time. However, nowadays by simply noting that topological ordered spaces are partial KKM spaces, much more than given results can be derived by applying [27, 28].

Lu [12] (2009)

Abstract: Section theorem has become an important role in social economy and mathematics field. The solutions of several problems, for example, optimization problem, fixed point problem, non-cooperative game problem, complementarity problem, and variational inequality problem, can be formulated as special cases of section theorem. So it is necessary to study section problem. In the setting of topological ordered spaces, the main purpose of this paper is to prove a section theorem, and next, as its applications, a weighted Nash equilibrium existence theorem and a Pareto equilibrium existence theorem for multi-objective games are obtained in topological ordered spaces. Our results improve and unify the corresponding results in the recently existing literatures.

Comments: Not only topological ordered spaces, for other types of abstract convex spaces, the results other than in [26, 27] can be possible.

Al-Homidan and Ansari [2] (2011)

Abstract: In this paper, we establish fixed point theorems for a family of multi-valued maps defined on the product space of topological semilattice spaces. By using our fixed point theorems, we derive a result on the nonempty intersection of sets without convex structure and equilibrium existence theorems for generalized abstract economies with two constraint correspondences. We present some special cases of our results which generalize several known results in the literature. We consider systems of generalized vector quasi-equilibrium problems and their special cases. As an application of our equilibrium existence theorems, we establish some existence results for solutions of systems of generalized vector quasi-equilibrium problems and their special cases. The results of this paper improve and extend several results in the literature.

Comments: Not only topological semilattice spaces, other types of KKM spaces can be deduced by similar methods. The authors adopt artificial and useless terminology like compactly open (closed), compact closure, compact interior, transfer compactly open valued, compactly local intersection property, etc. The authors obtained fixed point theorems for a family of multimaps defined on the product space of topological semilattice spaces. Such theorems can be proved by the KKM theory on abstract convex spaces more simple way with shorter proofs.

Al-Homidan, Ansari, and Yao [3] (2011)

Abstract: In this article, we establish a collectively fixed point theorem and a maximal element theorem for a family of multimaps in the setting of topological semilattice spaces. As an application of our maximal element theorem, we prove the existence of solutions of generalized abstract economies with two constraint correspondences. We consider the system of (vector) quasi-equilibrium problems (in short, (S(V)QEP)) and system of generalized vector quasi-equilibrium problems (in short, (SGVQEP)). We first derive the existence result for a solution of (SQEP) and then by using this result, we prove the existence of a solution of system of a generalized implicit quasi-equilibrium problems. By using existence result for a solution of (SQEP) and weighted sum method, we derive an existence result for solutions of (SVQEP). By using our maximal element theorem, we also establish some existence results for the solutions of (SGVQEP). Some applications of our results to constrained Nash equilibrium problem for vector-valued functions with infinite number of players and to semi-infinite problems are also given.

Comments: Not only topological semilattice spaces, other types of spaces can be considered.

The authors presented the following Browder-type collectively fixed point theorem [3, Theorem 3.1] in the setting of topological semilattice spaces with almost two page proof:

Theorem 6.1 ([3]). *For each $i \in I$, let $S_i, T_i : K \multimap K_i$ be multimaps such that the following conditions are satisfied:*

(i) *For each $i \in I$ and for all $x \in K$, $\mathrm{CO}_\Delta(S_i(x)) \subset T_i(x)$.*
(ii) *For each $i \in I$, $K = \bigcup_{y_i \in K_i} \mathrm{cint}_K S_i^{-1}(y_i)$, where $S_i^{-1}(y_i) = \{x \in K : y_i \in S_i(x)\}$.*
(iii) *For each $i \in I$ and for all $M_i \in \langle K_i \rangle$, there exists a compact Δ-convex subset $L_{M_i} \subset K_i$ containing M_i such that for all $x \in K \setminus D$ and for each $i \in I$, there exists $\tilde{y}_i \in L_{M_i}$ such that $x \in \mathrm{cint}\, S_i^{-1}(\tilde{y}_i)$.*

Then there exists $\bar{x} \in K$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

Since topological semilattice spaces are G-convex spaces, Theorem 6.1 can be extended as follows:

Theorem 6.2. *For each $i \in I$, let $(X_i; \Gamma_i)$ be an abstract convex space such that their product $(X; \Gamma) = (\prod_{i \in I} X_i; \prod_{i \in I} \Gamma_i)$ is a partial KKM space, let K be a nonempty compact subset of X, and let $S_i, T_i : X \multimap X_i$ be multimaps such that the following conditions are satisfied:*

(i) *For each $i \in I$ and for all $x \in X$, $\mathrm{co}_{\Gamma_i}(S_i(x)) \subset T_i(x)$.*
(ii) *For each $i \in I$, $X = \bigcup_{y_i \in X_i} \mathrm{int}_X S_i^{-1}(y_i)$.*
(iii) *For each $i \in I$ and for any $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset $L_{N_i} \subset X_i$ containing N_i such that for all $x \in X \setminus K$ and for each $i \in I$, there exists $\tilde{y}_i \in L_{M_i}$ such that $x \in \mathrm{int}\, S_i^{-1}(\tilde{y}_i)$.*

Then there exists $\bar{x} \in K$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

The proof will be given in a later work of ourselves.

H. Kim [7] (2013)

Abstract: Topological semilattices with path-connected intervals are special abstract convex spaces. In this paper, we obtain generalized KKM type theorems and their analytic formulations, maximal element theorems and collectively fixed point theorems on abstract convex spaces. We also apply them to topological semilattices with path-connected intervals, and obtain generalized forms of the results of Horvath and Ciscar, Luo, and Al-Homidan et al.

Altwaijry, Ounaies, and Chebbi [1] (2018)

Abstract: In this paper, we give a uniform approach for generalized convexity by using the concept of L-convexity defined by Ben El-Mechaiekh et al. [4]. We prove that the generalized notion of L-space contains well-known generalized convex spaces defined in the literature in topological vector spaces as well as several generalized convexity structures defined on metric spaces. In this context, we give a generalized version of the Fan-Knaster-Kuratowski-Mazurkiewicz Principle (FKKM Principle) in L-spaces and a Browder-Fan type theorem about the existence of fixed points for open lower section set-valued maps defined in an L-space. As an application, we prove the existence of equilibria for an abstract economy with an infinite number of agents.

Comments: In this paper the authors defined: Let $(E, \leq)$ be an order topological space with intervals that are arc connected and $\Gamma : \langle E \rangle \multimap E$ defined by $\Gamma(A) = \bigcup_{a \in A} [a, \sup A]$. Then, they say that (E, Γ) is called an HL-space.

In this paper, the authors repeated the twenty year old routine false statement of the L-space theorists that L-spaces generalize G-convex spaces; see our ten year old works [28, 29].

The contents of this paper is entirely obsolete. The authors repeat basic facts on their L-spaces and give well-known examples of G-convex spaces as examples of L-spaces. Other results in this paper are already well-known in more general forms.

7. Conclusion

Since 2006, we have been establishing the Grand KKM Theory on abstract convex spaces $(E, D; \Gamma)$. In this article. we found that almost all results on topological ordered spaces, topological semilattice spaces, and others can be extended to certain types of abstract convex spaces. As we closing this article, we would like to express several things that will help future readers.

Many authors mentioned in this article adopted some artificial terminology like compactly open, compactly closed, compact interior, compact closure, transfer compactly open valued, etc. Such terminology were not practical and useless as shown by ourselves several times since 2000; see [22, 29–31]. In fact, they can be eliminated by switching the original topology of the underlying space to its compactly generated extension.

For a topological space $(X, \mathcal{T})$, the compactly generated extension (or the k-extension) $\mathcal{T}_k$ of the original topology $\mathcal{T}$ is a new topology of X finer than $\mathcal{T}$ such that $\mathcal{T}_k$ is the collection of all compactly open [resp. compactly closed] subsets of $(X, \mathcal{T})$.

We emphasize that the study of abstract convex spaces is applicable to not only for all spaces mentioned in preceding sections but for a large number of examples of partial KKM spaces.

In this paper, we found some authors who restrict themselves within L-spaces for two decades without checking other authors' related works. Some advices to them were given in [32-34].

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